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Chern-Simons Holonomies and the Appearance of Quantum Groups

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Abstract

The monodromy representation defined by the quantum holonomies and realized on the physical state space of the Chern-Simons model is investigated. It is shown that this representation naturally extends to a representation of the braid group related to a particular solution of the quantum Yang-Baxter equation. The presence of a hidden quantum group symmetry is established.

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The three-dimensional Chern-Simons (C-S) model is extensively studied nowadays in connection with conformal field theory (CFT) [1,2,3] and link invariants [1,4,5,6]. In this paper we investigate a related subject, namely, the representation of the braid group B_n realized on the physical state space of the quantized C-S theory. We first apply the canonical formalism to construct the wave functions of the states associated with non-Abelian point-like sources (punctures) on a fixed time plane. Afterwards we report the expressions of the monodromy matrices $\{B\}$, defined by the transport of a puncture along a closed path in the punctured plane R^2_{n-1} . We show that the B -matrices give rise to a representation ϱ of the fundamental group $\pi_1(R^2_{n-1})$ which naturally extends to a representation φ of B_n . Recently, Kohno [7] has proven that φ in turn is equivalent to a well known B_n -representation θ related to a particular solution of the quantum Yang-Baxter equation. Therefore, the representation θ provides a bridge between the C-S model and quantum groups (QG). It turns out that the physical states associated with the fixed time cuts of Wilson line operators are QG-singlets. In this sense, the quantum group appears as a sort of *hidden* symmetry of the three-dimensional generally-covariant C-S field theory.

We start by briefly recalling the construction of the physical states of the C-S model in presence of Wilson lines. The C-S action in a $(1+2)$ -decomposition of R^3 reads*

$$S_{CS} = \frac{k}{4\pi} \int d^3x \left(A_2^a \dot{A}_1^a + A_0^a F_{12}^a \right), \quad (1)$$

where $F_{12}^a = \partial_1 A_2^a - \partial_2 A_1^a - f^{abc} A_1^b A_2^c$ and f^{abc} are the structure constants of a simple group G . The time-components A_0^a appear in (1) without time derivatives and can be viewed as Lagrange multipliers enforcing the Gauss law, which in the absence of sources reads

$$F_{12}^a = 0. \quad (2)$$

The quantization is performed in the usual way: we impose the canonical commutation relations

$$[A_1^a(\vec{x}), A_2^b(\vec{y})] = i\delta^{ab} \frac{4\pi}{k} \delta^2(\vec{x} - \vec{y}). \quad (3)$$

and select the physical states by means of the Gauss constraint. As well known,

$$G(\Lambda) = -\frac{k}{4\pi} \int d^2x F_{12}^a(\vec{x}) \Lambda^a(\vec{x}), \quad (4)$$

* In our notations $x^\mu = (x^0, \vec{x}) = (x^0, x^i)$ with $i = 1, 2$. We shall use also the two-dimensional antisymmetric tensors ϵ_{ij} and ϵ^{ij} defined by $\epsilon_{12} = \epsilon^{21} = 1$, $\epsilon_{21} = \epsilon^{12} = -1$.

where $\Lambda^a(\vec{x})$ has compact support, generates the residual local gauge transformations. In what follows we use the standard Schrödinger representation of (3). It is convenient [8] to introduce the field variable $U(\vec{x})$ defined by

$$T^a A_1^a(\vec{x}) = -i U^{-1}(\vec{x}) \partial_1 U(\vec{x}), \quad (5)$$

where T^a are the generators of the defining representation of G and their normalization is given by $\text{Tr} T^a T^b = \delta^{ab}/2$. The wave function Ψ_0 of the vacuum takes a simple form in terms of $U(\vec{x})$ [8], namely

$$\Psi_0 = \exp \left[-i \frac{k}{12\pi} \int_D d^3y \epsilon^{\alpha\beta\gamma} \text{Tr} (U^{-1} \partial_\alpha U U^{-1} \partial_\beta U U^{-1} \partial_\gamma U) - i \frac{k}{4\pi} \int d^2x \text{Tr} (U^{-1} \partial_1 U U^{-1} \partial_2 U) \right], \quad (6)$$

where the integral of the Wess-Zumino three form is performed on a three dimensional manifold D whose boundary is R^2 . The requirement of single valuedness of Ψ_0 implies that k takes integer values (our normalization refers to the case $G = SU(N)$). One can easily verify that Ψ_0 satisfies the Gauss law (2), i.e. $G(\Lambda)\Psi_0 = 0$.

We would like to describe now the states corresponding to the case when there are some distinguished points (punctures) on the plane R^2 with a definite non-trivial representation of the local gauge group assigned to each puncture. Such states appear as intermediate states at fixed time $t = t_0$ in computing the expectation values $\langle W(L) \rangle$ of Wilson line operators. The punctures arise as intersection points of the link L with the plane $t = t_0$ [1]. The corresponding wave functions can be constructed as follows. Since the matrix valued variable $U(\vec{x})$ takes values in G , we can define $U(\vec{x}, R)$ to be the representative of $U(\vec{x})$ in the R -representation of G . Consider then the wave function

$$\Psi(\vec{y}, R)_{IJ} = U(\vec{y}, R)_{IJ} \Psi_0, \quad (7)$$

where the indices I and J label the matrix elements of $U(\vec{y}, R)$. By making use of eqs.(3,4) one finds

$$(iG(\Lambda)\Psi(\vec{y}, R))_{IJ} = i\Psi(\vec{y}, R)_{IK} R_{KJ}^a \Lambda^a(\vec{y}). \quad (8)$$

Eq.(8) shows that $\Psi(\vec{y}, R)_{IJ}$ describes a state associated with a single puncture located in $\vec{y}$ and in the R -representation of G . The different vectors of the R -representation

on which $G^a(\vec{y})$ act, are labelled by the second index J of $\Psi[\vec{y}, R]_{IJ}$. Analogously, the wave function

$$\Psi[\vec{y}, R]^{IJ} = U^{-1}(\vec{y}, R)_{IJ} \Psi_0 \quad (9)$$

also describes a puncture in $\vec{y}$ associated with the R -representation of the group G . Differently from the wave function (7), however, the vectors of the R -representation associated with $G^a(\vec{y})$ are labelled in this case by the first index I of $\Psi[\vec{y}, R]^{IJ}$. Clearly, multiplying Ψ_0 by several matrix elements of the variable U (and/or of its inverse U^{-1}) in different points and in different representations, one obtains a wave function of the type

$$\Psi[\vec{y}_1, R_1]_{IJ}[\vec{y}_2, R_2]^{KL} \dots = U(\vec{y}_1, R_1)_{IJ} U^{-1}(\vec{y}_2, R_2)_{KL} \dots \Psi_0, \quad (10)$$

which describes several punctures (located in $\vec{y}_1, \vec{y}_2, \dots$) associated with the representations $R_1, R_2, \dots$ of G respectively.

Observe that the first lower index and the second upper index of the wave function (10) are free; they label the states of the charges "at infinity" and $G(\Lambda)$ does not act on them. We call these indices q -indices. As we shall see later, the q -indices give rise to the degrees of freedom on which the "quantum deformation" G_q of G is acting. In the wave functions corresponding to Wilson lines defined in a finite region, all q -indices are contracted. Let us consider indeed the wave function $\Psi(\Gamma)$,

$$\Psi_{IJ}(\Gamma) = W_{IJ}(\Gamma) \Psi_0, \quad (11)$$

obtained by applying the quantum holonomy operator

$$W(\Gamma) = P \exp \left(i \int_{\Gamma} A_i^a(\vec{x}) R^a dx^i \right) \quad (12)$$

on the vacuum. The path-ordered integral in eq.(12) is performed along an oriented non-intersecting smooth path Γ in R^2 connecting the point $\vec{y}_1$ with $\vec{y}_2$. Since $G^a(\vec{x})$ generate gauge transformations, it follows that

$$(iG^a(\vec{x}) \Psi(\Gamma))_{IJ} = -i\delta(\vec{x} - \vec{y}_1) R_{IK}^a \Psi_{KJ}(\Gamma) + i\delta(\vec{x} - \vec{y}_2) \Psi_{IK}(\Gamma) R_{KJ}^a. \quad (13)$$

Therefore, $\Psi(\Gamma)$ describes two punctures in $\vec{y}_1$ and $\vec{y}_2$ associated with the same representation R of G . Eq.(13) is exactly the operator Gauss law constraint in presence of (in this case two) punctures. As usual, we call *physical* the states satisfying the operator Gauss law.

In deriving the explicit form of $\Psi(\Gamma)$, some care is needed. In order to preserve general covariance, a framing procedure has to be introduced [1,6]. For Wilson lines defined on a plane, one can adopt the so-called constant framing [9]. In constant framing one finds (see for details [9])

$$\Psi_{IJ}(\Gamma) = \Psi[\vec{y}_1, R]^{IK}[\vec{y}_2, R]_{KJ}, \quad (14)$$

which shows that in wave functions corresponding to Wilson lines, the q -indices are indeed contracted.

We want to study now the monodromy properties of the physical states. A similar problem has been investigated by different methods in [10,11]. Let us consider for example the wave function Ψ associated with a four-puncture configuration,

$$\Psi = \Psi[\vec{y}_1, R_1]^{IM}[\vec{y}_2, R_2]_{MJ}[\vec{y}_3, R_3]^{KN}[\vec{y}_4, R_4]_{NL}. \quad (15)$$

The parallel transport, say, of the puncture $\vec{y}_2$ along a loop C defines a monodromy on the four-puncture physical states. This operation gives the state

$$\begin{aligned} B\Psi &= U^{-1}(\vec{y}_1, R_1)_{IM} U(\vec{y}_2, R_1)_{MQ} \left(P \exp i \int_C dx^i A_i^a(\vec{x}) R_1^a \right)_{QJ} \\ &\quad \times U^{-1}(\vec{y}_3, R_3)_{KN} U(\vec{y}_4, R_3)_{NL} \Psi_0. \end{aligned} \quad (16)$$

For a clockwise oriented C enclosing the puncture $\vec{y}_4$ only, see Fig.1, one gets [9] (in constant framing)

$$B\Psi = B_{M,N}^{S,Q} \Psi[\vec{y}_1, R_1]^{IM}[\vec{y}_2, R_1]_{SJ}[\vec{y}_3, R_2]^{KN}[\vec{y}_4, R_2]_{QL}, \quad (17)$$

where $B_{M,N}^{S,Q}$ are the matrix elements of

$$B = \exp \left(i \frac{4\pi}{k} R_1^a \otimes R_2^a \right). \quad (18)$$

The matrix B acts on the tensor product of the two representations R_1 and R_2 of the group G , whose vectors are labelled by the q -indices S and Q in the expression $U(\vec{y}_2, R_1)_{SJ} U(\vec{y}_4, R_2)_{QL}$. Moreover, all the indices of B are contracted with q -indices. Eqs.(17,18) can easily be generalized to the case of several punctures. Apart from some phase ambiguities related to the framing, i.e. to the "self-interaction" of the quantum holonomy operator [8,9], the results (17,18) are invariant under smooth deformations

of the contour C on the punctured plane. For a loop enclosing several punctures the associated monodromy is the ordered product of B -matrices of the type (18).

Summarising, the monodromy properties of the physical states of the C-S model are described by a set of matrices $\{B\}$ of the type (18), which act on the q -indices of the physical states. In order to study the representation of the braid group B_n , it is sufficient to concentrate on $\{B\}$.

It is useful to introduce now a few definitions. Let $R_n^2 = R^2 \setminus \{\bar{y}_1, \bar{y}_2, \dots, \bar{y}_n\}$ be the punctured plane, $X_n = \{\bar{y}_1, \bar{y}_2, \dots, \bar{y}_n : \bar{y}_p \neq \bar{y}_q \text{ for } p \neq q\}$ and $Y_n = X_n/S_n$, S_n being the permutation group of n elements. The fundamental groups $P_n = \pi_1(X_n)$ and $B_n = \pi_1(Y_n)$ are called the pure braid group and the Artin braid group respectively.

Our strategy in deriving the representation of B_n is the following. First of all we show that an appropriate subset $\mathcal{M} \subset \{B\}$ generates a representation ϱ of $\pi_1(R_{n-1}^2)$. Afterwards we introduce a flat connection $\hat{A}$ on R_{n-1}^2 , whose holonomies define a representation χ of $\pi_1(R_{n-1}^2)$ which is equivalent to ϱ . The advantage of introducing $\hat{A}$ is that the representation χ can naturally be extended to a representation of P_n , which in the case of equivalent punctures gives rise to a representation φ of B_n . We apply at this point a theorem of Kohn [7], which provides a precise relation between φ and a quantum deformation of the Lie algebra of the group G . This final step establishes the relation between the C-S model and quantum groups.

In order to construct ϱ , we consider the state

$$\Psi[\bar{x}_1, R_1]^{i_1 K_1} [\bar{y}_1, R_1]^{j_1 K_1} \dots [\bar{x}_n, R_n]^{i_n K_n} [\bar{y}_n, R_n]^{j_n K_n} \quad (19)$$

and concentrate on the set $\mathcal{M}$ of B -matrices obtained by moving, say $\bar{y}_q$, along loops which do not enclose any of the punctures $\{\bar{x}_1, \dots, \bar{x}_n\}$. $\mathcal{M}$ gives a representation of $\pi_1(R_{n-1}^2)$. Denote by $\{e_p : p = 1, \dots, q-1, q+1, \dots, n\}$ the generators of $\pi_1(R_{n-1}^2)$; e_p is associated with a clockwise oriented loop winding once around $\bar{y}_p$ and is represented by

$$\varrho(e_p) = \exp \left(i \frac{4\pi}{k} R_p^a \otimes R_q^a \right) \quad (20)$$

Let us turn now to the construction of the representation χ . We consider the punctured plane R_n^2 ; to each puncture $\bar{y}_p$ there is associated a finite dimensional linear space H_p on which a representation R_p of G is acting. The "induced" Gauss constraint reads [1]

$$F_{12}(\bar{x}) = -\frac{4\pi}{k} \sum_{p=1}^n \hat{R}_p^a \delta(\bar{x} - \bar{y}_p) \quad (21)$$

where $\hat{R}_p^a$ acts on $H = H_1 \otimes \dots \otimes H_n$ as $1 \otimes \dots \otimes R_p^a \otimes \dots \otimes 1$. Eq.(21) can be interpreted as a constraint induced on H by the Gauss law (13). Clearly (21) has no c -number solutions, but one can easily show [9] that

$$\hat{A}_i^a(\bar{x}) = \frac{2}{k} (\epsilon_{ij} + i\delta_{ij}) \sum_{p=1}^n \hat{R}_p^a \partial^j \ln |\bar{x} - \bar{y}_p| \quad (22)$$

is a matrix-valued solution of (21). When the puncture $\bar{y}_q$ is moved along a closed contour $C \subset R_{n-1}^2$, the connection $\hat{A}_{(q)}$ which is relevant in computing the associated holonomy is obtained from eq.(22) by eliminating the self-interaction, i.e.

$$\hat{A}_{(q)}(\bar{x}) = \hat{A}_{(q)}(\bar{x}) - \frac{2}{k} (\epsilon_{ij} + i\delta_{ij}) \sum_{p \neq q} \hat{R}_p^a \hat{R}_q^a \partial^j \ln |\bar{x} - \bar{y}_p| \quad (23)$$

The relative curvature reads

$$F_{(q)}(\bar{x}) = -\frac{4\pi}{k} \sum_{p \neq q} \hat{R}_p^a \hat{R}_q^a \delta(\bar{x} - \bar{y}_p) \quad (24)$$

which shows that $\hat{A}_{(q)}$ is a flat connection on R_{n-1}^2 . Therefore the holonomy

$$\chi(C) = P \exp \left(i \oint_C \hat{A}_{(q)} dx^i \right) \quad (25)$$

defines a representation χ of $\pi_1(R_{n-1}^2)$. The point now is that ϱ and χ are equivalent. Consider indeed the connection

$$\begin{aligned} \hat{A}_{(q)1}^{n_0}(\bar{x}) &= 0 \\ \hat{A}_{(q)2}^{n_0}(\bar{x}) &= \frac{4\pi}{k} \sum_{p \neq q} \hat{R}_p^a \hat{R}_q^a \delta(x^2 - y_p^2) \theta(y_p^1 - x^1) \end{aligned} \quad (26)$$

It is easily seen that the curvature of $\hat{A}_{(q)}^{n_0}$ equals $F_{(q)}$ and that the relative holonomy $\chi^{n_0}(C)$ coincides with $\varrho(C)$. One can show also that there exists a "gauge transformation" Ω_0 acting in H and interpolating between $\hat{A}_{(q)}$ and $\hat{A}_{(q)}^{n_0}$. Therefore

$$\varrho(C) = \chi^{n_0}(C) = \Omega_0^{-1}(\bar{y}_q) \chi(C) \Omega_0(\bar{y}_q) \quad (27)$$

which establishes the equivalence between χ and ϱ .

We would like to compute now the holonomies associated with the simultaneous displacements of several punctures. In this case all the "interactions" between the point-like sources have to be taken into account; each puncture is moving in the potential

(23) produced by the remaining $n - 1$ punctures. In complex coordinates eq.(23) can be rewritten in the form

$$\hat{A}_q(z)dz = \frac{2}{k} \sum_{p \neq q} \Omega_{pq} d \ln(z - z_p), \quad (28)$$

where $\Omega_{pq} = \hat{R}_p^* \hat{R}_q^*$. Moreover it is easy to verify that

$$\begin{aligned} [\Omega_{pq}, \Omega_{pr} + \Omega_{qr}] &= [\Omega_{pq} + \Omega_{pr}, \Omega_{qr}] = 0, \quad p < q < r, \\ [\Omega_{pq}, \Omega_{rs}] &= 0 \quad \text{for distinct } p, q, r \text{ and } s, \end{aligned} \quad (29)$$

which are known as infinitesimal pure braid relations. Let us consider now the superposition

$$\omega = \frac{1}{2} \sum_{q=1}^n \hat{A}_q(z_q) dz_q = \frac{2}{k} \sum_{1 \leq p < q \leq n} \Omega_{pq} d \ln(z_p - z_q), \quad (30)$$

which is precisely the Kohno connection. This connection appears also in the Knizhnik-Zamolodchikov equation [12] for the Wess-Zumino-Witten (WZW) model. The relation between the WZW level ℓ and k is given by $k = \ell + c_0$, where c_0 is the quadratic Casimir operator in the adjoint representation. The one-form (30) represents a connection of the trivial vector bundle over $X_n = \{(z_1, \dots, z_n) \in C^2 : z_p \neq z_q \text{ if } p \neq q\}$ with fibre H . As a consequence of eqs.(29), ω is a flat connection. Let γ be an element of the fundamental group $\pi_1(X_n)$ and let

$$\chi(\gamma) = P \exp \left(- \oint_{\gamma} \omega \right) \quad (31)$$

be the associated holonomy. Eq.(31) generalizes eq.(25) to the case in which several punctures are moving simultaneously. By construction $\chi(\gamma)$ acts on H and defines a linear representation of P_n in H . In the particular case when all the $\{R_p\}$ -representations coincide, ω is symmetric under permutations of the coordinates $\{z_p\}$ and induces a connection over Y_n . Therefore in this case χ extends actually to a representation φ of the braid group B_n on H . This representation, called the monodromy representation of B_n , has been investigated by Kohno [7] who has shown that φ is related to the B_n -representation defined by the universal R -matrix [13] associated with the Lie algebra of G . More precisely, for the fundamental representation of $SU(N)$ the following theorem holds:

i) Considering $\frac{1}{k}$ as a parameter, the braid group representation φ is equivalent to

$$\theta(\sigma_i) = q^{\nu} [1 \otimes \dots \otimes T(q) \otimes \dots \otimes 1], \quad (32)$$

where $\nu = \frac{1-N}{2}$, $T(q)$ acts on the i -th and $i+1$ -th factors of H by

$$T(q) = \sum_{\alpha} E_{\alpha\alpha} \otimes E_{\alpha\alpha} + \sqrt{q} \sum_{\alpha \neq \beta} E_{\alpha\beta} \otimes E_{\beta\alpha} + (1-q) \sum_{\alpha > \beta} E_{\alpha\alpha} \otimes E_{\beta\beta} \quad (33)$$

and $q = \exp(-i\frac{2\pi}{k})$;

ii) θ commutes with the diagonal action of $SU(N)_q$ on H and defines actually a Hecke algebra representation of B_n , known as Pimsner-Popa-Temperley-Lieb representation.

From point i) of the theorem it follows that there exists an invertible linear mapping

$$V : H \rightarrow H \text{ such that} \quad \varphi = V^{-1} \theta V. \quad (34)$$

Since φ is an extension of χ , eq.(34) relates χ with θ as well. The diagonal action of $SU(N)_q$ on H is defined by the standard co-product in the quantum algebra. From eq.(34) it is clear that φ commutes with the V -conjugate action of the $SU(N)_q$ representation commuting with θ . For generic q , one can show [14] that $\theta(\sigma_i)$ generate actually the centralizer of the quantum group. Finally, the above theorem holds [7,14] also for the fundamental representations of all non-exceptional simple Lie groups and, with slight modifications, for an arbitrary irreducible finite dimensional representation of $SU(2)$.

The theorem of Kohno establishes the role of quantum groups in the C-S theory. As well known any link can be obtained as the closure of a suitable braid element. The trefoil T is, for example, the closure of $\alpha \in B_3$ shown in Fig.2. The intermediate physical states contributing to $\langle W(T) \rangle$ are determined by intersecting T with the planes $t = \text{const}$. At $t = 0$ one can choose a standard physical state

$$\Phi_0 = \delta_{j_1}^{i_1} \delta_{j_2}^{i_2} \delta_{j_3}^{i_3} \Psi[\vec{x}_1]^{K_1, j_1} [\vec{x}_2]^{K_2, j_2} [\vec{y}_1]^{K_3, j_3} [\vec{y}_2]^{K_4, j_4} [\vec{y}_3]^{K_5, j_5} |j_1, j_2, j_3\rangle, \quad (35)$$

We are interested in the state Φ_1 at $t = 1$. In order to use the θ representation, it is necessary to perform a change of basis in the q -indices. Two steps are involved in this operation. First, one has to relate the B -matrices with the χ representation (31). In this case, the intertwining operator $\tilde{\Omega}$ is the natural extension in X_n of the operator Ω_0

defined previously. Then we have to use eq.(34) to connect φ with θ . The combined intertwining operator $\mathcal{U} = \tilde{\Omega} V$ only depends on the deformation parameter q . We introduce then the wave functions

$$\Psi_{L_1 L_2 L_3; S_1 S_2 S_3}^{K_1 K_2 K_3; Q_1 Q_2 Q_3} = (\mathcal{U}^{-1})_{J_1 J_2 J_3}^{Q_1 Q_2 Q_3} \mathcal{U}_{S_1 S_2 S_3}^{J_1 J_2 J_3} \Psi[\vec{x}_1]^{K_1 J_1} [\vec{x}_2]^{K_2 J_2} [\vec{x}_3]^{K_3 J_3} [\vec{y}_1]_{I_1 L_1} [\vec{y}_2]_{I_2 L_2} [\vec{y}_3]_{I_3 L_3} \quad (36)$$

The indices $Q_1 Q_2 Q_3$ and $S_1 S_2 S_3$, which originate from the q -indices, represent quantum group (QG) indices in the standard basis and eq.(35) can be rewritten as

$$\Phi_0 = \delta_{Q_1}^{S_1} \delta_{Q_2}^{S_2} \delta_{Q_3}^{S_3} \Psi_{L_1 L_2 L_3; S_1 S_2 S_3}^{K_1 K_2 K_3; Q_1 Q_2 Q_3} \quad (37)$$

The physical state Φ_1 at $t = 1$ is obtained from Φ_0 by replacing the δ -factors in (37) by the monodromy $\theta(\alpha)$, i.e.

$$\Phi_1 = \theta(\alpha)_{Q_1 Q_2 Q_3}^{S_1 S_2 S_3} \Psi_{L_1 L_2 L_3; S_1 S_2 S_3}^{K_1 K_2 K_3; Q_1 Q_2 Q_3} \quad (38)$$

Observe now that θ acts on the QG-indices and that these indices appear contracted in the physical states. Therefore, any invertible linear transformation $\mathcal{F}$ acting on the QG-indices (the lower and upper QG-indices transform by $\mathcal{F}$ and $\mathcal{F}^{-1}$ respectively) and commuting with θ represents a hidden symmetry of the quantized C-S theory. The theorem of Kohno states that $G_q \subset \{\mathcal{F}\}$. The QG-indices appear quite naturally in the above construction and are associated with the group indices of the charges "at infinity". The quantum group symmetry has a peculiar realization: any physical state obtained by cutting a Wilson line (defined in a finite region) is a G_q -singlet.

Quantum group symmetries play an important role in integrable systems [13,15] and are expected to emerge also in certain conformal field theories [16]. The C-S theory represents a remarkable example of a continuous three-dimensional generally covariant field theory, admitting a hidden quantum group symmetry.

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FIGURE CAPTIONS

Fig.1 Parallel transport of the puncture 2 along the loop C .

Fig.2 The trefoil as a closure of the braid $\alpha \in B_3$.

Fig. 1

Fig. 2