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COHERENT BEAM-BEAM INTERACTION LIMIT
IN ASYMMETRIC RING COLLIDERS

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ABSTRACT

We study the effect of the beam-beam interaction on the coherent dipole oscillations of rigid bunches circulating in asymmetric e^+e^- (two-ring) colliders, with circumference ratio K_+/K_- , where K_+ bunches of e^+ beam collide with K_- bunches of e^- beam. Unstable regions appear densely in the tune diagram, when one or both of K_+ and K_- are large. This effect poses a severe restriction on the choice of K_+ and K_- .

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1 Introduction

Recently, several projects for the construction of B-factories[1] proposed rings with different energies for e^+ and e^- beams because it was recognized[2] that such a configuration has advantages for studying CP violating processes in B_0 decay. The two beams circulate in different rings with different circumferences and collide at an interaction point (IP). So far, such asymmetric colliders have not been studied in detail from the point of view of beam-beam interactions. The aim of this paper is to investigate coherent beam-beam modes[3,4] for this case, based on Ref[5]. This phenomenon is not only a useful probe of the beam-beam interaction but also gives an absolute limit of the colliding ring performance.

2 The Kinematics

We will consider a pair of asymmetric rings as follows. The ring for the $e^\pm$ beam has $K_\pm$ bunches, which are spaced equally and contain $N_\pm$ particles each. The relativistic Lorentz factor ($\gamma_\pm$), the value of the beta functions at the IP ($\beta_{x,y}^\pm$), the beam sizes at the IP ($\sigma_{x,y}^\pm$) and the tunes ($\nu_{x,y}^\pm$) may be different for the two beams. The ratio of circumferences of both rings is K_+/K_- . Without loss of generality, we can assume that $K_+ \leq K_-$ and that K_+ and K_- have no common factor: it would appear when we discuss the luminosity but it would be irrelevant for the dynamics of the coherent beam-beam modes. While an e^+ bunch completes K_- turns in its ring, the e^- bunch completes K_+ turns in its ring. This period is called ‘super period’. At the IP, the e^+ and e^- bunches are coupled by the beam-beam force.

We label the bunches as $n = 1, 2, \dots, K_+$ for e^+ beam and $m = 1, 2, \dots, K_-$ for e^- beam. The collision of one particular set of bunches (n, m) occurs once in every super period. For the i -th bunch, the barycentre and its slope are denoted by $\bar{z}_i$ and $\bar{z}'_i$, respectively, where i is either n or m and z refers to either horizontal (x) or vertical (y) coordinates. As shown in Ref.[5], when a pair of bunches collides, the deflections can be written as

$$\delta \bar{z}'_\pm = -\frac{4\pi}{\beta_z^\pm} \Xi_z^\pm (\bar{z}_\pm - \bar{z}_\mp) \quad (1)$$

where Ξ_z is the effective beam-beam parameter

$$\Xi_z^\pm = \frac{r_e N_\mp \beta_z^\pm}{2\pi \gamma_\pm \Sigma_z (\Sigma_x + \Sigma_y)} \quad (2)$$

Here r_e is the classical electron radius and Σ_z is the effective beam size

$$\Sigma_z^\pm = \sqrt{(\sigma_z^+)^2 + (\sigma_z^-)^2}$$

When two bunches have the same beam sizes, Ξ is half the conventional beam-beam strength parameter ξ . In the above, we assume that all bunches have a Gaussian density distribution with fixed beam sizes, i.e. the rigid-Gaussian model[5].

3 The Dynamics

As canonical variables, we choose

$$\vec{W} = \begin{pmatrix} Z \\ P \end{pmatrix} = \sqrt{N\gamma} \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix} \begin{pmatrix} \bar{z} \\ \bar{z}' \end{pmatrix} \quad (3)$$

where $N\gamma$ plays the role of the bunch mass and α and β are Twiss parameters at the IP. By this choice, the kick can be written in an explicitly symplectic form. We construct a $2(K_+ + K_-)$ vector $\mathcal{W} = (\vec{W}_1^+, \dots, \vec{W}_n^+ \dots, \vec{W}_{K_+}^+, \vec{W}_1^-, \dots, \vec{W}_m^-, \dots, \vec{W}_{K_-}^-)^t$, where each $\vec{W}_i^\pm$ is that defined in Eq.(3) for the i -th bunch. The evolution of $\mathcal{W}$ in a super period is described by multiplying it from the left with a matrix C :

$$C = OR_{(K_+, K_-)} \cdots OR_{(n, m)} \cdots OR_{(1, 1)}$$

Here the 2×2 block diagonal matrix O represents the betatron oscillation in the arc:

$$O = \text{diag} \left[\overbrace{U(\mu_+/K_+), \dots, U(\mu_+/K_+)}^{K_+}, \overbrace{U(\mu_-/K_-), \dots, U(\mu_-/K_-)}^{K_-} \right]$$

where

$$U(A) = \begin{pmatrix} \cos A & \sin A \\ -\sin A & \cos A \end{pmatrix}$$

and $\mu_\pm = 2\pi\nu_\pm$. The matrices $R_{(n, m)}$ represent the kicks due to the collision between the n -th and the m -th bunches: In the subspace of $(\vec{W}_n^+, \vec{W}_m^-)^t$, $R_{(n, m)}$ can be written as

$$R_{(n, m)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -4\pi\Xi_+ & 1 & 4\pi\sqrt{\Xi_+\Xi_-} & 0 \\ 0 & 0 & 1 & 0 \\ 4\pi\sqrt{\Xi_+\Xi_-} & 0 & -4\pi\Xi_- & 1 \end{pmatrix}$$

In the rest of the space, $R_{(n, m)}$ is the unit matrix.

The perturbed tunes of the system can be obtained from the eigenvalues of C . There are $K_+ + K_-$ independent pairs of eigenvalues. The members of each pair are either complex conjugate or real and reciprocal. The system is unstable if at least one of the eigenvalues becomes greater than unity in absolute value. The eigenvectors are modes of coherent dipole oscillation.

4 Eigenvectors

It is difficult to obtain exact eigenvalues analytically. Instead, let us consider approximate eigenvectors and eigenvalues. When the currents, i.e. Ξ 's, are small enough, we can regard the beam-beam kicks as if they act continuously. For convenience, we rescale the canonical variables as

$$P_\pm \rightarrow P_\pm / \sqrt{\tilde{\nu}_\pm}, \quad \text{and} \quad Z_\pm \rightarrow Z_\pm \sqrt{\tilde{\nu}_\pm}.$$

The Hamiltonian of the system is then $H = T_+ + T_- + H_i$, where

$$\begin{aligned} T_{\pm} &= \frac{1}{2} \sum_{k=1}^{K_{\pm}} (P_k^{\pm})^2 \\ H_i &\simeq \frac{1}{2} \sum_{\pm} \sum_{k=1}^{K_{\pm}} (\tilde{\nu}_{\pm} Z_k^{\pm})^2 + \sum \left(\sqrt{\tilde{\nu}_+ \Xi_+} Z_n^+ - \sqrt{\tilde{\nu}_- \Xi_-} Z_m^- \right)^2 \\ &\equiv \frac{1}{2} \sum_{(i,j)=(1,1)}^{K_++K_-} V_{ij} Z_i Z_j \end{aligned}$$

Here $\tilde{\nu}_{\pm} \equiv K_{\mp} \nu_{\pm}$ is the ‘super tune’, which corresponds to the number of betatron oscillations of each bunch per one super period and the Z_i form a $(K_+ + K_-)$ vector, the order being half of that of W . Since V defined here is in symmetric quadratic form and the kinetic term $(T_+ + T_-)$ has $(K_+ + K_-)$ -fold degeneracy, we can perform an orthogonal transformation to V without changing the kinetic term.

It is easy to find the eigenvectors and eigenvalues of V , which give approximate eigenvectors and ‘eigen tunes’ squared of C . The vectors

$$(a_1, a_2, \dots, a_{K_+}, 0, 0, \dots, 0)^t, \quad \text{with} \quad \sum a_n = 0$$

and

$$(0, 0, \dots, 0, b_1, b_2, \dots, b_{K_-})^t, \quad \text{with} \quad \sum b_m = 0$$

form $K_+ - 1$ and $K_- - 1$ independent vectors with super tunes $\tilde{\nu}_+ + K_- \Xi_+$ and $\tilde{\nu}_- + K_+ \Xi_-$, and correspond to relative motions within e^+ and e^- bunches, respectively. Two vectors of the form

$$\left(\overbrace{A, A, \dots, A}^{K_+}, \overbrace{B, B, \dots, B}^{K_-} \right)^t$$

are also eigenvectors, where A and B are the nonzero solutions of

$$\begin{pmatrix} \tilde{\nu}_+^2 + 2\tilde{\nu}_+ K_- \Xi_+ & -2K_- \sqrt{\tilde{\nu}_+ \tilde{\nu}_- \Xi_+ \Xi_-} \\ -2K_+ \sqrt{\tilde{\nu}_+ \tilde{\nu}_- \Xi_+ \Xi_-} & \tilde{\nu}_-^2 + 2\tilde{\nu}_- K_+ \Xi_- \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \lambda \begin{pmatrix} A \\ B \end{pmatrix}$$

with eigenvalue λ . The super tunes of eigenmodes are given by

$$\tilde{\nu}_{\omega}^2 = \frac{1}{2}(\tilde{\nu}_+^2 + \tilde{\nu}_-^2) + K_- \tilde{\nu}_+ \Xi_+ + K_+ \tilde{\nu}_- \Xi_- \pm \frac{1}{2} \sqrt{Q}, \quad (4)$$

$$Q = (\tilde{\nu}_+^2 - \tilde{\nu}_-^2)^2 + 4(\tilde{\nu}_+^2 - \tilde{\nu}_-^2)(K_- \tilde{\nu}_+ \Xi_+ - K_+ \tilde{\nu}_- \Xi_-) + 4(K_- \tilde{\nu}_+ \Xi_+ + K_+ \tilde{\nu}_- \Xi_-)^2$$

Let us call modes with $\pm$ in Eq.(4) $\omega_{\pm}$ mode. These eigenvectors correspond to the superpositions of barycentres of all e^+ bunches and all e^- bunches. When Q becomes negative, the $\omega_{\pm}$ mode becomes unstable. This occurs at the sum resonance:

$$\tilde{\nu}_+ + \tilde{\nu}_- = K_- \nu_+ + K_+ \nu_- \lesssim \text{integer} \quad (5)$$

The symbol $\lesssim$ indicates that the instability occurs just below the resonant tune. In general, when we fix $\nu_{\mp}$ and increase $\nu_{\pm}$ starting from some integer, we meet the sum resonance $K_{\mp}$

times while we go from that integer to the next. When $K_{\pm}$ is large, the resonant region can cover almost all range of $\nu_{\mp}$. In the particular case where $\tilde{\nu}_+ = \tilde{\nu}_-$, Eq.(4) implies that one of the super tunes is not changed and the other becomes

$$\tilde{\nu} \longrightarrow \tilde{\nu} + K_- \Xi_+ + K_+ \Xi_-.$$

This is the largest shift of the super tunes. To obtain the tunes of the $e^{\pm}$ rings, we should divide $\tilde{\nu}$'s by $K_{\mp}$. Thus, when $K_+ \ll K_-$, the tune shift of the e^- ring can be large. Also the half-integer resonances $2\nu_+ \lesssim \text{integer}$ or $2\nu_- \lesssim \text{integer}$ lead to an instability.

5 Numerical Results

The precise eigenvalues of C are calculated with a computer code ABMODE (Asymmetric Beam-beam Modes), derived from the BBMODE code[7]. In Fig. 1, we show the maximum stable value of Ξ as a function of $\nu_{\pm}$, for the case of $(K_+, K_-) = (2, 3)$. We assume that $\Xi_+ = \Xi_- \equiv \Xi$, which seems reasonable for maximizing the luminosity. The dips in Ξ due to the sum and half-integer resonances are clearly visible. From symmetry considerations, it is sufficient to show only $0 \leq \nu_+ \leq 1$ and $0 \leq \nu_- \leq 0.5$.

In Fig. 2, we show the growth rate of the instabilities, defined as the largest absolute eigenvalue Λ of C , for $(K_+, K_-) = (4, 5)$. The area in the tune diagram where the growth rate is larger than unity is called instability region. The rest is the safe region. The instability region increases as $K_{\pm}$ increases, as shown in Table 1. It is roughly proportional

	1	2	3	4	5	6	7
1	32.5	34.9	41.9	43.2	44.0	48.6	51.9
2	-	-	48.4	-	55.3	-	61.6
3	-	-	-	57.1	61.7	-	67.9
4	-	-	-	-	66.6	-	73.6
5	-	-	-	-	-	74.8	78.2
6	-	-	-	-	-	-	82.6

Table 1: Instability region (%) in the tune diagram for different set of (K_+, K_-) in case of $\Xi_{\pm} = 0.03$.

to the density of sum resonances, i.e. $\sqrt{K_+^2 + K_-^2}$, apart from the contribution of the half-integer resonance lines. The area of the instability region gives a rough measure of the difficulty in such rings.

6 Discussion

In the design of B-factories, there are two ways for avoiding the difficulties associated with the sum resonances in the case of $K_+ < K_-$. One may either avoid running into the sum resonances by keeping enough space in tune diagram, or one may limit the growth rate on

Figure 1: The largest Ξ as a function of (ν_+, ν_-) for $(K_+, K_-) = (2, 3)$. The right axis is $1 - \nu_+$ while the left one is ν_- . The plateau corresponds to $\Xi \geq 0.03$.

the sum resonances such that they can be compensated by a feed-back system. In the first case, the usual tolerances for the setting up and operation of the rings will lead to a required tune region. Its size can be expressed in terms of the diameter of the largest circle between neighbouring beam-beam resonances which is $1/\sqrt{K_+^2 + K_-^2}$, and therefore yields an upper limit on $K_{\pm}$. It appears that

$$\sqrt{K_+^2 + K_-^2} \lesssim 7$$

would be safe. In the second case, a smaller value of Ξ would have to be adopted in the design, such that the growth rates on the resonances, which are roughly proportional to Ξ , remain small enough. If the gain of the feed-back system is known, the maximum tolerable growth rate, and therefore the maximum tolerable value of Ξ , can be calculated.

Figure 2: The growth rate Λ as a function of (ν_+, ν_-) for $(K_+, K_-) = (4, 5)$ and $\Xi = 0.03$. The right and left axes are ν_+ and ν_- , respectively. The base is at $\Lambda = 1$, the plateau corresponds to $\Lambda \geq 2$.

Since in most B-factory proposals the circulating current is already rather high, a drop in luminosity in proportion to the drop in Ξ seems inevitable.

The difficulties described in this paper can be avoided to a very large extent by choosing $(K_+, K_-) = (1, 1)$, i.e. by making the circumferences of the two rings identical. In this case, the beam-beam effect couples each bunch in one ring to only one bunch in the other ring, even if there is a large – but equal – number of bunches in both rings, and there is only one sum resonance, at $\nu_+ + \nu_- \lesssim \text{integer}$, which is easily avoided by an appropriate choice of the tunes $\nu_{\pm}$.

Our linear treatment yielded the maximum values of Ξ for which small coherent bunch oscillation modes are stable and the growth rates when these modes are unstable. It has the shortcoming that the equilibrium beam sizes in the presence of radiation damping cannot be calculated. We did not study in detail to what extent a powerful feed-back system might be used to counteract the growth of the coherent oscillations on the resonances. For the sake of self-consistency, we did not take into account the recently proposed correction factors[6] to the rigid-Gaussian model, which will make the problem more serious. A more detailed and extended investigation will be published elsewhere.

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