Calabi-Yau manifold of four generations

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We analyze in detail a recently proposed four-generation superstring model, based on an algebraic variety in $CP_4 \times CP_4$. The group of honest symmetries G_H of the manifold is identified and the transformation properties of the light fields under G_H are given. We also discuss the flux breaking of E_6 symmetry, identify the candidates for matter parity, and analyze supersymmetry breaking. For the Pati-Salam version of the theory we isolate the light fields of the theory and discuss the issues of the naturally light Higgs field(s) and the resulting fermionic mass matrices.

I. INTRODUCTION

It seems that the most promising route towards unification of gravity and other forces in nature lies in space-time being ten dimensional. This, what probably should be a heretic statement, has today become almost a dogma.¹ The central question then is: how do we descend to our four-dimensional world? Nobody knows. One of the popular candidate pictures, though, is that the extra six dimensions are compactified into a complex and Ricci-flat, so-called Calabi-Yau manifold with $SU(3)$ holonomy.² It is not known, unfortunately, whether the number of such topologically distinct manifolds is finite or not and already thousands of such spaces have been found to exist as algebraic varieties in the products of CP_N spaces.³

Now, the main motivation for Calabi-Yau spaces lies in the belief in asymptotic freedom at energies below the Planck mass and $N=1$ supersymmetry propagating all the way into the TeV regime of weak-interaction physics.⁴ This immediately forbids more than four generations of quarks and leptons, and simplifies the situation dramatically. For example, only one of the thousands of these manifolds can be constructed as the three-generation space, a remarkable result.³ This is the so-called Tian-Yau manifold,⁵ which has been extensively studied by now.⁶⁻⁸ Tian-Yau manifold is based on the following algebraic variety in $CP_3 \times CP_3$:

$$\begin{pmatrix} 3 & || & 3 & 0 & 1 \\ 3 & || & 0 & 3 & 1 \end{pmatrix}. \quad (1.1)$$

The notation⁹ simply implies cubic polynomials in each of the CP_3 spaces and a mixed polynomial linear in the coordinates of both CP_3 's. Since its Euler characteristic $\chi_E = -18$ and the number of generations N_g is known to be^{1,4}

$$N_g = \frac{1}{2} |\chi_E| \quad (1.2)$$

one then divides the above space by the Z_3 symmetry in order to obtain a multiply connected manifold with three generations.⁵ It is characterized⁶ by $h_{11}=6$, $h_{21}=9$, and $\chi_E = -6$. The nontrivial embedding of Z_3 in E_6 allows, through the so-called flux breaking,¹⁰ for the intermediate $SU(3)_L \times SU(3)_R \times SU(3)_C$ symmetry which is then broken spontaneously by the Higgs field in 27 of E_6 (Ref. 6).

What about the situation involving four generations? Well, we have known for some time of two explicit examples (see below) and the systematic search for these manifolds indicates not too large a number.¹¹ However, at this point no complete analysis has been performed and we simply do not know how many of the candidate spaces could be actually constructed. Recall, the candidate manifolds are those which could allow dividing discrete symmetry free of fixed points, but only an explicit construction reveals whether or not this is actually achieved.

The existing candidates are $(4||5)$ and $(7||2222)$ manifolds,¹² both characterized by $h_{11}=1$, which leads to the serious problem in understanding the lightness of the neutrino.¹³ These manifolds are not realistic. This fact has recently prompted us to search for the explicitly constructed four-generation manifold with $h_{11} \neq 1$. The manifold we have constructed¹⁴ is in the complete analogy with the Tian-Yau manifold. It is an algebraic variety K_0 in $CP_4 \times CP_4$ (Ref. 14):

$$\begin{pmatrix} 4 & || & 2 & 2 & 0 & 0 & 1 \\ 4 & || & 0 & 0 & 2 & 2 & 1 \end{pmatrix} \quad (1.3)$$

with $\chi_E(K_0) = -32$. To obtain a four-generation manifold K , we divide K_0 by a fixed-point-free $Z_2 \times Z_2$ symmetry. We find

$$(i) \quad h_{11}(K) = 6, \quad h_{21}(K) = 10,$$

$$\chi_E(K) = -8. \quad (1.4)$$

(ii) The flux breaking allows the intermediate symmetry to be E_6 , $SU(2) \times SU(6)$, $SO(5) \times U(1) \times U(1)$, $SU(4)_C \times SU(2) \times SU(2) \times U(1)$, $SU(3)_C \times SU(3) \times U(1) \times U(1)$, and $SU(4)_C \times SU(2) \times U(1) \times U(1)$. We concentrate on the Pati-Salam $SU(4)_C \times SU(2)_L \times SU(2)_R \times U(1)$ version of the theory.

(iii) The manifold has $Z_2(A) \times Z_2(B) \times Z_2(S)$ honest symmetry, under which we classify the transformation properties of h_{21} and h_{11} fields and in turn quarks and leptons. None of these symmetries is an R symmetry. The model can also possess a Z_5 cyclic pseudosymmetry, which is again not an R symmetry.

(iv) The manifold has a naturally inbuilt mechanism for the suppression of proton decay in the form of three different candidate discrete symmetries, so-called *matter parities*.

(v) The quartic superpotential may not be flat at the intermediate scale of symmetry breaking. We discuss ways out of this impasse.

The rest of our paper is devoted to the detailed analysis of this manifold. We follow the pedagogical and systematic procedure of Greene, Kirklin, Miron, and Ross¹⁵ but give enough details to allow the nonexpert to go through our exposé. In the next section we present the manifold and compute its Hodge numbers and the Euler characteristic. We then demonstrate that the $Z_2 \times Z_2$ dividing symmetry is freely acting, but leave the nontrivial and tedious part of the transversality of our polynomial constraints for the Appendix.

Section II concerns the flux breaking and it is there where the physical fields are identified. The reader who is interested in the physics only can actually start from this section, together with Tables I and II below. Their transformation properties under the honest symmetry are computed in Sec. IV. In Sec. V we discuss the physical issues for the Pati-Salam version of the theory, such as the proton decay, supersymmetry breaking, the lightness of the Higgs field, and the problems associated with the fermionic mass matrices. Finally, in Sec. VI we give concluding remarks and offer some thoughts regarding further phenomenological consequences whose study we leave for the future publication.

II. A FOUR-GENERATION MANIFOLD: THE CONSTRUCTION

A. The simply connected manifold K_0

As we said in the Introduction, the manifold is based on an algebraic variety K_0 in $CP_4 \times CP_4$ with the most general polynomials

$$\begin{aligned} P_1 &= \frac{1}{2} \sum_{i=0}^4 x_i^2 = 0, \\ P_2 &= \frac{1}{2} \sum_{i=0}^4 a_i x_i^2 = 0, \\ P_3 &= \sum_{i,j=0}^4 c_{ij} c_i y_j = 0, \end{aligned} \quad (2.1)$$

$$P_4 = \frac{1}{2} \sum_{i=0}^4 b_i y_i^2 = 0,$$

$$P_5 = \frac{1}{2} \sum_{i=0}^4 y_i^2 = 0,$$

where x_i and y_i are the homogeneous coordinates on the two CP_4 's and a_i, b_i, c_{ij} are in general complex parameters (of course, you can always choose $a_0 = b_0 = c_{00} = 1$). We have used the freedom in linear redefinitions of the coordinates on CP_4 's to simultaneously diagonalize the quadratics P_1, P_2 and P_3, P_4 . Of course, (2.1) will not make sense for just any choice of these parameters; in order for it to define a smooth manifold the above polynomials must be transverse,¹⁶ i.e., $dP_1 \wedge dP_2 \wedge dP_3 \wedge dP_4 \wedge dP_5$ is not allowed to vanish on K_0 . The tedious, but important computation which establishes the transversality is given in the Appendix; it leads to a number of involved constraints on a_i, b_i , and c_{ij} . We now turn our attention to topological properties of K_0 .

Hodge numbers of K_0

K_0 is a Calabi-Yau manifold with $SU(3)$ holonomy. This implies $h_{20} = h_{10} = 0$, $h_{30} = 1$ and the only nontrivial Hodge numbers are given by h_{21} and h_{11} (Refs. 1 and 2). They simply count chiral fermionic zero modes on the manifold, with h_{21} counting 27 's of E_6 and h_{11} counting the $\overline{27}$'s. This guarantees $h_{21} - h_{11}$ light families which cannot be paired off above M_W scale; hence the number of generations is¹

$$N_g = h_{21} - h_{11} = -\frac{1}{2} \chi_E. \quad (2.2)$$

(i) h_{21} and the deformations of the polynomials. It is well known by now, that, except in some strange instances,¹⁷ h_{21} is given by small deformations of K_0 , i.e., small deformations of the polynomials. In other words, to each closed, nonexact (2,1) form corresponds an independent deformation of P_i . Independent deformations are simply those which are unrelated by linear analytic coordinate transformations. This is neatly summarized by Candelas:¹⁸ a deformation vanishes effectively (i.e., it corresponds to an exact form) if it is proportional to any of P_i or their derivatives:

$$z_i \frac{\partial P^\alpha}{\partial z_j} + \sum_\beta f_\beta^\alpha P^\beta \simeq 0, \quad (2.3)$$

where z_i are either of the two CP_4 's homogeneous coordinate x_i and y_i , $\alpha = 1, \dots, 5$ and f_β^α are arbitrary parameters. Let us see how this works on our manifold. First, there are altogether 85 possible deformations of polynomials P_i ; they are of the form: $x_i x_j, y_i y_j, x_i y_j$. Now, let $f_\beta^\alpha = 0$ in (2.3); in vector notation $z_i = x_i$ in (2.3) gives

$$\begin{pmatrix} x_i x_j \\ x_i a_j x_j \\ \sum_k x_i c_{jk} y_k \\ 0 \\ 0 \end{pmatrix} \simeq 0. \quad (2.4)$$

This means that the deformations of say P_1 (and similarly P_5) are determined by the deformations of P_2 , P_3 , and P_4 . Antisymmetrizing (2.4) gives in turn

$$\begin{pmatrix} 0 \\ (a_j - a_i)x_i x_j \\ \sum_k (x_i c_{jk} - x_j c_{ik})y_k \\ 0 \\ 0 \end{pmatrix} \simeq 0, \quad (2.5)$$

which tells us similarly that the off-diagonal deformations of P_2 (and also P_4) are not independent of the others. On the other hand, $f_\beta^\alpha \neq 0$ amounts to say that we can use both $\sum_i x_i^2 = 0$ and $\sum_i a_i x_i^2 = 0$ for (now only diagonal) deformations of P_2 (and P_4). But since

$$\sum_i a_i x_i \frac{\partial \mathbf{P}}{\partial x_i} = \begin{pmatrix} 0 \\ \sum_i a_i^2 x_i^2 \\ \sum_{i,k} a_i x_i c_{ik} y_k \\ 0 \\ 0 \end{pmatrix} \simeq 0 \quad (2.6)$$

one more diagonal deformation of P_2 (and P_4) can be traded for P_3 deformations.

In summary, we are left at the end with two diagonal deformations of P_2 and P_4 each and $25 - 1$ (use $P_3 = 0$) $= 24$ independent deformations of P_3 . This shows that

$$h_{21}(K_0) = 28. \quad (2.7)$$

The choice of the deformations we found convenient will be displayed later, when we discuss their transformation properties under the dividing symmetry.

(ii) h_{11} and the Lefschetz hyperplane theorem. Since the Euler characteristic of this manifold is independently known, we could simply use $h_{11} = h_{21} + \frac{1}{2}\chi_E$ from (2.2). However, in order to verify the deformation method and our computations we will evaluate h_{11} directly, using the Lefschetz hyperplane theorem. This theorem asserts that $h_{11}(K_0)$ is simply the sum of the corresponding Hodge numbers of the submanifolds obtained by ignoring the mixed polynomial(s).¹⁹ In our case this reads

$$h_{11}(K_0) = 2h_{11}(U), \quad (2.8)$$

where $U = (4||22)$. From $\chi_E(U) = 8$, $h_{11}(U) = 6$ and so

$$h_{11}(K_0) = 12. \quad (2.9)$$

B. The four-generation manifold K

Since $\chi_E(K_0) = -32$, we need a four-element dividing group to obtain a four-generation manifold. It is easy to see that the choice c_{ij} diagonal, i.e.,

$$\mathbf{P} = \begin{pmatrix} \sum_{i=0}^4 x_i^2 \\ \sum_{i=0}^4 a_i x_i^2 \\ \sum_{i=0}^4 c_i x_i y_i \\ \sum_{i=0}^4 a_i y_i^2 \\ \sum_{i=0}^4 y_i^2 \end{pmatrix}, \quad (2.10)$$

gives rise to a Z_2^5 symmetry to K_0 (see Sec. IV).

Notice that we have now assumed the swapping symmetry $x \leftrightarrow y$. This calls for a $Z_2 \times Z_2$ dividing symmetry, which up to reshuffling of the coordinates, is uniquely given by

$$\begin{aligned} g_1 &= \text{diag}(1, 1, 1, -1, -1), \\ g_2 &= \text{diag}(1, 1, -1, 1, -1). \end{aligned} \quad (2.11)$$

For K to be a smooth manifold, $Z_2 \times Z_2$ must act freely.¹ Let us see how this works for g_1 , in which case the fixed-point equation is

$$\begin{aligned} (x_0, x_1, x_2, x_3, x_4) &\rightarrow (x_0, x_1, x_2, -x_3, -x_4) \\ &= \lambda(x_0, x_1, x_2, x_3, x_4), \\ (y_0, y_1, y_2, y_3, y_4) &\rightarrow (y_0, y_1, y_2, -y_3, -y_4) \\ &= \lambda'(y_0, y_1, y_2, y_3, y_4), \end{aligned} \quad (2.12)$$

where λ and λ' are nonzero complex numbers. Equation (2.12) implies $\lambda = -1$ or $\lambda = 1$, of which the first case obviously gives no solution. The $\lambda = 1$ case requires a little care. It implies $x_3 = x_4 = y_3 = y_4 = 0$ and so from (2.10) we get

$$\begin{aligned} x_0^2 + x_1^2 + x_2^2 &= 0, \quad y_0^2 + y_1^2 + y_2^2 = 0, \\ a_0 x_0^2 + a_1 x_1^2 + a_2 x_2^2 &= 0, \\ a_0 y_0^2 + a_1 y_1^2 + a_2 y_2^2 &= 0, \\ c_0 x_0 y_0 + c_1 x_1 y_1 + c_2 x_2 y_2 &= 0. \end{aligned} \quad (2.13)$$

Solving for the x_1 , x_2 , y_1 , and y_2 from the first four equations implies

$$\begin{aligned} c_0 x_0 y_0 + c_1 x_1 y_1 + c_2 x_2 y_2 \\ = \left[\frac{1}{a_2 - a_1} [c_0(a_1 - a_2) \pm c_1(a_2 - a_0) \right. \\ \left. \pm c_2(a_0 - a_1)] \right] x_0 y_0 = 0. \end{aligned} \quad (2.14)$$

Assuming that the expression in brackets does not vanish (see also the Appendix), $x_0 = 0$ and (2.13) has no solution. In exactly the same number one can check that g_2 and $g_1 g_2$ have no fixed points.

The fact that G_F acts freely means that K is a smooth

manifold whose Euler characteristic is determined through $\chi_E(K_0)$ (Ref. 1)

$$\chi_E(K) = \frac{\chi_E(K_0)}{\text{No. of elements of } G_F} = \frac{-32}{4} = -8. \quad (2.15)$$

In other words, K , as promised, is a *four-generation manifold*. This will be even more manifest when we display its Hodge numbers below.

Hodge numbers of K

(i) h_{21} . As we said in Sec. II A we choose the 28 independent deformations of K_0 to be 2 diagonal for each P_2 and P_4 and 24 for P_3 . They are listed, together with their transformation properties under G_F , in Table I. As you can see, ten of them are invariant, which implies

$$h_{21}(K) = 10. \quad (2.16)$$

(ii) h_{11} . Computing $h_{11}(K)$ is somewhat more subtle and requires using the technique of Lefschetz fixed-point theorem.²⁰ Since we lack the explicit knowledge of h_{11} fields, in order to find their transformation properties we

TABLE I. The physical states from 27 of E_6 in terms of deformations for the $SU(4)_C \times SU(2)_L \times SU(2)_R \times U(1)$ intermediate symmetry and their transformation properties.

Field	Deformation	g_1	g_2	A	B	S
$(n, g, g^c)_1$	$x_0^2 + y_0^2$	1	1	1	1	1
$(n, g, g^c)_2$	$x_1^2 + y_1^2$	1	1	1	1	1
$(n, g, g^c)_3$	$x_1 y_1$	1	1	1	1	1
$(n, g, g^c)_4$	$x_2 y_2$	1	1	1	1	1
$(n, g, g^c)_5$	$x_3 y_3$	1	1	1	1	1
$(n, g, g^c)_6$	$x_4 y_4$	1	1	1	1	1
$(n, g, g^c)_7$	$x_0^2 - y_0^2$	1	1	1	1	-1
$(n, g, g^c)_8$	$x_1^2 - y_1^2$	1	1	1	1	-1
$(n, g, g^c)_9$	$x_1 y_0 + x_0 y_1$	1	1	-1	-1	1
$(n, g, g^c)_{10}$	$x_1 y_0 - x_0 y_1$	1	1	-1	-1	-1
$(q, l)_1$	$x_3 y_4 + x_4 y_3$	-1	1	1	1	1
$(q, l)_2$	$x_1 y_2 + x_2 y_1$	-1	1	1	-1	1
$(q, l)_3$	$x_0 y_2 + x_2 y_0$	-1	1	-1	1	1
$(q, l)_4$	$x_3 y_4 - x_4 y_3$	-1	1	1	1	-1
$(q, l)_5$	$x_1 y_2 - x_2 y_1$	-1	1	1	-1	-1
$(q, l)_6$	$x_0 y_2 - x_2 y_0$	-1	1	-1	1	-1
$(q^c, l^c)_1$	$x_2 y_4 + x_4 y_2$	1	-1	1	1	1
$(q^c, l^c)_2$	$x_1 y_3 + x_3 y_1$	1	-1	1	-1	1
$(q^c, l^c)_3$	$x_0 y_3 + x_3 y_0$	1	-1	-1	1	1
$(q^c, l^c)_4$	$x_2 y_4 - x_4 y_2$	1	-1	1	1	-1
$(q^c, l^c)_5$	$x_1 y_3 - x_3 y_1$	1	-1	1	-1	-1
$(q^c, l^c)_6$	$x_0 y_3 - x_3 y_0$	1	-1	-1	1	-1
H_1	$x_2 y_3 + x_3 y_2$	-1	-1	1	1	1
H_2	$x_1 y_4 + x_4 y_1$	-1	-1	1	-1	1
H_3	$x_0 y_4 + x_4 y_0$	-1	-1	-1	1	1
H_4	$x_2 y_3 - x_3 y_2$	-1	-1	1	1	-1
H_5	$x_1 y_4 - x_4 y_1$	-1	-1	1	-1	-1
H_6	$x_0 y_4 - x_4 y_0$	-1	-1	-1	1	-1

need to reconstruct the character table of $H_{11}(g_i)$ (Ref. 20) under G_F . This is achieved through the above theorem, which asserts that, for any g ,

$$\sum_{p,q=0}^3 (-1)^{p+q} \text{Tr} H_{pq}(g) = \chi \text{ (fixed points of } g), \quad (2.17)$$

where the right-hand side is the Euler characteristic for the fixed-point submanifold of g , and $H_{pq}(g)$ is the matrix representation of g acting on h_{pq} fields. Now, $g_i \in G_F$ act freely and (2.17) gives

$$\text{Tr} H_{11}(g_i) = \text{Tr} H_{21}(g_i), \quad (2.18)$$

where we used the fact that h_{00} and h_{30} are invariant under g_i . From Table I, (2.18) readily gives

$$\begin{aligned} \text{Tr} H_{11}(g_1) &= \text{Tr} H_{11}(g_2) = \text{Tr} H_{11}(g_1 g_2) = 4, \\ \text{Tr} H_{11}(1) &= 12. \end{aligned} \quad (2.19)$$

The number of elements c_i in each of the representations R_i (see Table I) is simply given by²¹

$$c_i = \frac{1}{4} \left[\sum_{\alpha} \text{Tr} H_{11}(g_{\alpha}) \times g_{\alpha}(R_i) \right], \quad (2.20)$$

where the sum is over all the group elements of G_F (recall that G_F is Abelian so all of its representations are one dimensional); $g_{\alpha}(R_i)$ is the value of g_{α} in the representation R_i (Table I). From (2.19) and (2.20),

$$c_1 = 6, \quad c_2 = c_3 = c_4 = 2. \quad (2.21)$$

Since all $g_i = 1$ ($g_i \in G_F$) in R_1 , there are six invariant h_{11} fields on K_0 , i.e.,

$$h_{11}(K) = 6. \quad (2.22)$$

From $\chi_E(K) = 2[h_{11}(K) - h_{21}(K)] = -8$, this is yet another demonstration that K contains four light generations of quarks and leptons (actually, the full 27's of E_6 —see the next section). This manifold is a natural extension of Tian-Yau manifold; the additional generation is obtained through the increase of h_{21} , while h_{11} 's are the same. The important difference and the additional motivation for our work lies in their intermediate symmetries obtained through the flux breaking. In the next section we discuss this at length and finally identify the quark and lepton fields.

III. FLUX BREAKING

It should be clear from its construction that K is a multiply connected manifold,¹ with $\pi_1(K) = G_F$. This important feature is responsible for the badly needed breaking of E_6 gauge symmetry, through the so-called flux breaking. In a sense, this justifies the otherwise *ad hoc* division by G_F in constructing a manifold with $N_g = 4$.

Recall that in superstring compactification we lack the usually present adjoint Higgs field, which is used at the first stage of symmetry breaking. Instead, the same effect is achieved via nontrivial embedding of G_F into E_6 :

$$g_i \rightarrow U_{g_i} \in E_6. \quad (3.1)$$

We use the Dynkin-diagram techniques; in the notation of Slansky²² (his Table 20) this amounts to

$$U_{gi} = e^{i\lambda_i \cdot \mathbf{H}}, \quad (3.2)$$

where $\mathbf{H}$ covers Cartan's subalgebra and $\lambda_i = (-c_i, c_i, a_i, b_i, c_i, 0)$. For the $Z_2 \times Z_2$ group a_i, b_i , and c_i take only values 0 and π .

There are three distinct possibilities:

(i) $U_{gi} = 1$, which implies unbroken E_6 symmetry. This is not realistic.

(ii) $U_{g1} = 1, U_{g2} \neq 1$ (or vice versa), in which case the straightforward analysis reveals the following possibilities for the intermediate symmetry:

$$SU(6) \times SU(2), \quad SO(10) \times U(1). \quad (3.3)$$

(iii) Both $U_{gi} \neq 0$. The additional possibilities now are

$$\begin{aligned} &SU(5) \times U(1) \times U(1), \\ &SU(4)_C \times SU(2)_L \times SU(2)_R \times U(1), \\ &SU(4)_C \times SU(2)_L \times U(1) \times U(1), \\ &SU(3)_C \times SU(3)_L \times U(1) \times U(1). \end{aligned} \quad (3.4)$$

We should stress that due to the presence of extra $U(1)$ symmetries, both in $SU(5)$ and in $SO(10)$ examples one has *flipped* possibilities of charge assignments²³ in addition to the standard scenarios. Also, in $SU(6) \times SU(2)$ cases, one can isoflip $SU(2)_L$ outside $SU(6)$ (Ref. 24). This is gratifying, since flipping is necessary in order to complete the symmetry breaking down to $SU(3)_C \times SU(2)_L \times U(1)$. Namely, recall, e.g., that in the standard Georgi-Glashow model both ν^c and n (see below) are $SU(5)$ singlets, so that through $\langle \nu^c \rangle$ and $\langle n \rangle$ $SU(5)$ symmetry remains unbroken.

The reader can easily reproduce our results in (3.3) and (3.4). For the sake of clarity, though, we give one such choice for U_{gi} , which leads to the Pati-Salam²⁵ version of the theory, which we now follow. So, take

$$a_1 = b_1 = c_1 = \pi, \quad a_2 = \pi, \quad b_2 = c_2 = 0. \quad (3.5)$$

We now identify the light fields of the theory for this case. Under $SU(4)_C \times SU(2)_L \times SU(2)_R \times U(1)$, **27** of E_6 decomposes in the following manner:

$$\begin{aligned} \mathbf{27} = & [q, l = (4_C, 2_L, 1_R)_1] + [q^c, l^c = (\bar{4}_C, 1_L, \bar{2}_R)_1] \\ & + [H = (1_C, 2_L, 2_R)_{-2}] \\ & + [g, g^c = (6_C, 1_L, 1_R)_{-2}] \\ & + [n = (1_C, 1_L, 1_R)_4]. \end{aligned} \quad (3.6)$$

In (3.6) the subscript index is the $U(1)$ charge.

Now to each **27** of E_6 (prior to flux breaking) there corresponds, as we said, a $(2,1)$ form and to each $(2,1)$ form an independent deformation of the defining polynomials. The fields that will remain light, i.e., those that survive the passage to K must satisfy

$$\psi(g_i x) = U_{gi} \psi(x). \quad (3.7)$$

In the sense of the above one-to-one correspondence between physical fields and the deformations, we can write, from (3.5),

$$\left. \begin{aligned} H &\simeq \frac{1}{2}(1-g_1)\frac{1}{2}(1-g_2) \\ q, l &\simeq \frac{1}{2}(1-g_1)\frac{1}{2}(1+g_2) \\ q^c, l^c &\simeq \frac{1}{2}(1+g_1)\frac{1}{2}(1-g_2) \\ n &\simeq \frac{1}{2}(1+g_1)\frac{1}{2}(1+g_2) \\ g, g^c &\simeq \frac{1}{2}(1+g_1)\frac{1}{2}(1+g_2) \end{aligned} \right\} \times \text{deformation}. \quad (3.8)$$

This simply means that n is a singlet under both $SU(2)_L$ and $SU(2)_R$; q a singlet under $SU(2)_R$, etc. Similarly, antiparticles are the same projections on $(1,1)$ forms.

From Table I we see that the light spectrum of $(2,1)$ forms consists of (let f stand for q, l, q^c, l^c)

$$h_{21}: 10(n, g, g^c) + 6(f, H). \quad (3.9)$$

From (2.21) we find similarly that the light antiparticles on K are

$$h_{11}: 6(\bar{n}, \bar{g}, \bar{g}^c) + 2(\bar{f}, \bar{H}). \quad (3.10)$$

The light spectrum of the theory contains

$$\begin{aligned} &4(n, g, g^c, f, H=27) \\ &+ 6(n, \bar{n}, g, \bar{g}, g^c, \bar{g}^c) + 2(f, \bar{f}, H, \bar{H}). \end{aligned} \quad (3.11)$$

In short, besides the four **27**'s guaranteed by the index theorem, there are additional light particles and mirror particles whose fate will depend on the next stage of Higgs-induced symmetry breaking.

IV. HONEST SYMMETRIES

We now discuss the "honest" symmetries of our four-generation manifold K , i.e., the symmetries by K_0 that survive the division by G_F and the flux breaking. We keep the general coefficients a_i and c_i in defining Eq. (2.10) of K_0 manifold embedded in $CP_4 \times CP_4$. Actually, it seems that no specific choice of these parameters can increase the honest symmetry, as we discuss in Sec. V.

As it is, (2.10) implies altogether Z_2^5 symmetry on K_0 :

(i) $Z_2(A)$:

$$\begin{aligned} x_i &\rightarrow A_{ij} x_j, \quad y_i \rightarrow A_{ij} y_j, \\ A_{ij} &= \text{diag}(-1, 1, 1, 1, 1), \end{aligned}$$

(ii) $Z_2(B)$: $B = \text{diag}(1, -1, 1, 1, 1)$,

(iii) $Z_2(C)$: $C = \text{diag}(1, 1, -1, 1, 1)$, (4.1)

(iv) $Z_2(D)$: $D = \text{diag}(1, 1, 1, -1, 1)$,

(v) $Z_2(S)$: $S(x_i) = y_i, \quad S(y_i) = x_i$.

Notice that $x_4 \rightarrow -x_4$ symmetry is not independent; it is generated by $-ABCD$.

Since all these generators commute with each other and with G_F , they all survive down to K . Since g_1 and g_2

depend on the generators (4.1) through $g_1 = ABC$, $g_2 = ABD$ on $CP_4 \times CP_4$, we define the honest symmetry of K as

$$G_H = \frac{Z_2(A) \times Z_2(B) \times Z_2(C) \times Z_2(D) \times Z_2(S)}{G_F = Z_2(g_1) \times Z_2(g_2)} \quad (4.2)$$

$G_H = Z_2^3$ is generated by A , B , and S . The trivial embedding in E_6 $U_A = U_B = U_S = 1$ keeps G_H unbroken even after the flux breaking (of course, this requires $U_C = U_{g_1}, U_D = U_{g_2}$).

We now give the transformation properties of physical fields under A and B generators (the behavior under swapping is obvious).

(i) h_{21} . It is straightforward to reproduce the transformation properties of (2,1) fields in Table I.

(ii) h_{11} . We could use again Lefschetz fixed-point theorems as given in (2.17); it would suffice to compute χ (fixed points of A) and χ (fixed points of B), since we already know $\text{Tr}H_{21}(A)$, $\text{Tr}H_{21}(B)$. However, this way we would miss the transformation properties of the fields under S . For this reason, and in order to have an independent check, one can employ the hyperplane version of the fixed-point theorem. In simplest terms, this just means that we can ignore the mixed polynomial P_3 of (2.10) and use (2.17) as applied on each individual space $U \equiv (4||22)$. In other words,

$$\text{Tr}H_{11}(g) \text{ on } K = 2[\text{Tr}H_{11}(g) \text{ on } U], \quad (4.3)$$

where g is the transformation of interest (g stands for A, B here); and $\text{Tr}H_{11}(g)$ on U is determined from

$$\sum_{p,q=0}^2 (-1)^{p+q} \text{Tr}H_{p,q}(g) \text{ on } U = \chi(\text{fixed points of } g \text{ on } U). \quad (4.4)$$

As before, the right-hand side is the Euler characteristic of the fixed submanifold of g . Since U is not a Calabi-Yau manifold [$c_1(U) \neq 0$] and $\chi_E(U) > 0$, $h_{20}(U) = h_{10}(U) = 0$ and (4.4) reads

$$[\text{Tr}H_{11}(g) \text{ on } U] + 2 = \chi(\text{fixed points of } g \text{ on } U). \quad (4.5)$$

Now, we would like to classify our physical fields in (3.11), so g will stand actually for the projections, such as $\frac{1}{2}(1+g_1)\frac{1}{2}(1+g_2)A$, etc. It is then straightforward to reproduce the result in Table II, which provides the transformation properties of 27 fields.

For the sake of eventual computation of Yukawa couplings it is important to know whether or not honest symmetries are R symmetries. Namely, the physics of a supersymmetric theory depends on the absolute value of the superpotential W only and so under a symmetry transformation f W can in general pick up a phase (when it does, the symmetry is called an R symmetry). There is, fortunately, a geometric criterion for this, since W transforms in precisely the same manner as the unique holomorphic (3,0) form Ω on a Calabi-Yau manifold. To see whether Ω picks up a phase under f , one can employ

TABLE II. The same as Table I, only for $\overline{27}$.

Field	g_1	g_2	A	B	S
$(\bar{n}, \bar{g}, \bar{g}^c)_1$	1	1	1	1	1
$(\bar{n}, \bar{g}, \bar{g}^c)_2$	1	1	1	-1	1
$(\bar{n}, \bar{g}, \bar{g}^c)_3$	1	1	-1	1	1
$(\bar{n}, \bar{g}, \bar{g}^c)_4$	1	1	1	1	-1
$(\bar{n}, \bar{g}, \bar{b}^c)_5$	1	1	1	-1	-1
$(\bar{n}, \bar{g}, \bar{g}^c)_6$	1	1	-1	1	-1
$(\bar{q}, \bar{l})_1$	-1	1	-1	-1	1
$(\bar{q}, \bar{l})_2$	-1	1	-1	-1	-1
$(\bar{q}^c, \bar{l}^c)_1$	1	-1	-1	-1	1
$(\bar{q}^c, \bar{l}^c)_2$	1	-1	-1	-1	-1
$\bar{H}_1$	-1	-1	-1	-1	1
$\bar{H}_2$	-1	-1	-1	-1	-1

Lefschetz fixed-point theorem (2.17), which on a Calabi-Yau manifold simplifies into

$$\chi(\text{fixed points}) = 2[1 + \text{Tr}H_{11}(f) + \text{Re Tr}H_{21}(f)] - 2 \text{Re} \xi, \quad (4.6)$$

where $f: \Omega \rightarrow \xi \Omega$.

For example, for symmetry A the fixed-point submanifold is

$$\begin{pmatrix} 3 & || & 2 & 2 & 0 & 0 & 1 \\ 3 & || & 0 & 0 & 2 & 2 & 1 \end{pmatrix} \quad (4.7)$$

and from Tables I and II (4.6) reads

$$-32 = 2[1 + (-4) - 12] - 2 \text{Re} \xi, \quad (4.8)$$

which implies $\text{Re} \xi = 1$ or $\xi = 1$. In other words Ω is invariant under A and so A is not an R symmetry. Similarly, neither B nor S are R symmetries.

This completes the study of predominantly mathematical properties of our manifold. We now have all the quark and lepton fields which have survived the compactification to $d=4$ and the associated topological symmetry breaking via Wilson flux lines. We also know all the symmetries of K and the transformation properties of the fields under them. There is still a large intermediate gauge symmetry based on $SU(4)_C \times SU(2)_L \times SU(2)_R \times U(1)$ group. The next step is to discuss the phenomenological consequences for the low-energy weak-interaction phenomena.

V. PHENOMENOLOGICAL CONSIDERATIONS

We address here a number of phenomenological aspects of this manifold.

A. Proton decay

As always in supersymmetric theories, one has to worry about the dimension-4 baryon-violating operators, through the existence of $\bar{d}^c$, the down squark, whose mass is supposed to be in the TeV region. Clearly, we

must find a symmetry, which forbids this, i.e., the so-called *matter parity*. It turns out that both A and B symmetries can play this role, if augmented by an element from E_6 :

$$U = \text{diag}(-1, -1)_L \otimes \text{diag}(-1, -1)_R \otimes \text{diag}(1, 1, 1, 1)_C. \quad (5.1)$$

To check this, one can simply recall the criterium of Bento, Hall, and Ross,²⁶ which says that the discrete symmetry of the manifold can be used as a matter parity, if the number of singlet fields under it from 27 's and $\overline{27}$'s is precisely the number of light generations, i.e., 4 in our case. This is certainly true of A , B , or AB ; let us, to be specific, choose A and so let matter parity P_M be

$$P_M = AU. \quad (5.2)$$

Of course, one must then guarantee that P_M is not broken in the subsequent process of symmetry breaking. Imagine for the moment that P_M remains unbroken; then this implies that the light fields of the theory will be among

$$(q, l)_{1;2;4;5}, \quad H_{1;2;4;5}, \quad (q^c, l^2)_{1;2;4;5} \quad (5.3)$$

and so

$$\begin{aligned} P_M(q, l) &= -(q, l), \quad P_M(q^c, l^c) = -(q^c, l^c), \\ P_M H &= H. \end{aligned} \quad (5.4)$$

Clearly P_M acts right on the matter fields, quarks and leptons and H can play the role of Higgs fields at low energies.

Before we discuss this in more detail, we must first address the issue of symmetry breaking.

B. Intermediate scale symmetry breaking

Another nontrivial test that the superstring theory faces is the requirement of symmetry breaking down to $SU(3)_C \times SU(2)_L \times U(1)$ without breaking supersymmetry. The large gauge symmetry $SU(4)_C \times SU(2)_L \times SU(2)_R \times U(1)$ is easily broken through the vacuum expectation values of n and ν^c fields; the question is, how large does the scale of symmetry breaking (called M_I here) have to be? The limit is provided by the dangerous dimension-5 proton decay operators through g and g^c quarks (not forbidden by P_M), which can get the mass through $\langle n \rangle$ and $\langle \nu^c \rangle$ of order M_I ; and so $M_I \geq 10^{15}$ GeV. It is obvious that a potential has to be rather flat in order for supersymmetry to remain all the way down to the TeV region.

Now, for the matter parity in (5.2) to remain unbroken only the following fields are allowed to have nonvanishing vacuum expectation values:

$$\begin{aligned} n_{1;2;3;4;5;6;7;8}, \quad \bar{n}_{1;2;4;5}, \\ \nu_{3;6}^c, \quad \bar{\nu}_{1;2}^c. \end{aligned} \quad (5.5)$$

The next step would be to construct the quartic superpotential (recall that 27^3 or $\overline{27}^3$ triplic superpotential is automatically flat, since it contains only terms linear in $n, \bar{n}$

and $\nu^c, \bar{\nu}^c$) and check for the flatness of (5.5). This is not as simple as one would imagine for the following reasons.

The first obstacle is the fact that the geometric interpretation of $(27 \times \overline{27})^2$ couplings is still missing and one does not know how to compute these couplings. One possibility is to write down all the couplings allowed by the symmetries of the manifold (A , B , and S); but this could be misleading, for it is possible that the geometric miracle could occur, with some of such couplings vanishing. Still, we can analyze what would happen in a general case; it is easy to see then that there is no natural way to ensure F flatness. Namely, besides other couplings, W_4 contains at least terms

$$W_4 = \frac{\lambda_{ij}}{M_C} 27_i^2 \overline{27}_j^2 + \dots, \quad (5.6)$$

where M_C is the compactification scale ($M_C \simeq \text{Planck scale}$); and $\lambda_{ij} \neq 0$ for all i, j . It is obvious that without unnatural adjustments W_4 will not be flat. But, of course, geometry could imply vanishing of some λ_{ij} and before one knows how to compute W_4 this issue cannot and should not be settled.

Another important point is that the light spectrum of the theory contains also E_6 singlets S , whose precise number or their interactions with 27 's and $\overline{27}$'s are still not completely understood (at least not for all such singlets). We will not go into details here, again because of our inability to compute all the relevant couplings, but the superpotential is then

$$W = W_4 + f_{ijk} 27_i \overline{27}_j S_k + g_{ijk} S_i S_j S_k + \dots, \quad (5.7)$$

where we drop the $27^3, \overline{27}^3$ terms due to their automatic flatness. It is the flatness of (5.7) that remains to be checked; we will come back to it in a future publication, when armed with a better understanding of singlet couplings. It does appear that the same geometric miracle may be needed in order for (5.7) to be flat, but one simply cannot know at this point. Recall also that at least some of the couplings in (5.7) could be suppressed by nonperturbative (instanton) effects, which could make life easier.

C. Light quark and lepton mass matrices and the issue of the light Higgs field

Even if (5.7) does pass the test and allows for F -flatness at M_I , there remain, of course, still some important issues to be settled before one can hope for the theory to be realistic: (i) we need a naturally light Higgs field(s) to complete the symmetry breaking; (ii) simultaneously we wish to ensure that all g and g^c fields pick up a mass of order M_I through $\langle n_i \rangle$; (iii) maybe less dramatic requirement, but it would be nice to get rid of the extra families and their mirrors at low energies.

(i) Light Higgs field? Let us for simplicity ignore $\langle \nu^c \rangle$'s and $\langle \bar{\nu}^c \rangle$'s in order to illustrate the problem. Imagine the scenario with

$$\langle n_1 \rangle, \langle \bar{n}_1 \rangle, \langle \bar{n}_2 \rangle \neq 0. \quad (5.8)$$

This is sufficient to eliminate the "junk" from low energies, so that the light spectrum is as in (5.3). The Higgs

field then lies in $H_{1;2;4;5}$; with $H=(H_u, H_d)$. From the $c_{ijk} n_i H_j^u H_k^d$ couplings, we get the mass matrix for H fields:

$$\begin{array}{c} H_u^1 \\ H_u^2 \\ H_u^4 \\ H_u^5 \end{array} \begin{array}{c} \left[\begin{array}{cccc} c_{111} \langle n_1 \rangle & 0 & 0 & 0 \\ 0 & c_{122} \langle n_1 \rangle & 0 & 0 \\ 0 & 0 & c_{144} \langle n_1 \rangle & 0 \\ 0 & 0 & 0 & c_{155} \langle n_1 \rangle \end{array} \right] \end{array} \quad (5.9)$$

Again, unless some of the couplings $c_{111}, c_{122}, c_{144}, c_{155}$ vanish for geometric reasons, there will be no light Higgs fields. This is a potential blow to the theory and should encourage the calculation of at least 27^3 -type couplings. It is not an easy task; we plan, however, to come back to it. Recently, Mohapatra²⁷ has noticed this problem too and tried to argue against the theory; in our opinion, rather prematurely. We simply know nothing about the c_{ijk} couplings.

(ii) The exotic quarks pick up their masses through $d_{ijk} n_i g_j g_k^c$ -type couplings. We would like, of course, the d_{1jk} couplings general enough to ensure the massiveness of g, g^c quarks; this is the required geometric “miracle” to make the theory work: it should produce the conspiracy to ensure the light Higgs fields, but not light (g, g^c) quarks. Since d and c couplings are unrelated, this issue can be only settled after the computation of Yukawa couplings. It makes no sense to us whatsoever, to argue one way or another before this is done. This is where we find absolutely no justification for Mohapatra’s claim that the theory cannot be realistic; at this stage the above are just interesting and important issues that should stimulate the profound exercise of computing the superpotential.

VI. SUMMARY AND OUTLOOK

We would be very happy if you have actually read this paper. However, it is very likely that you have just glanced through the Introduction and so the summary of our results may be called for. We have managed to construct a new four-generation Calabi-Yau manifold with two main characteristics:

(i) $h_{21}=10$, $h_{11}=6$ and so this manifold does not suffer from the usual problem associated with $h_{11}=1$ as previous examples, such as $(4||5)$ and $(7||2222)$. This was our main motivation.

(ii) The nontrivial and suggestive embedding of our freely acting $Z_2 \times Z_2$ symmetry in E_6 leads to (among many other choices) Pati-Salam intermediate symmetry. Even if the fourth generation may not be found soon, this still provides a nontrivial departure from the three-generation Tian-Yau manifold. Of course, one will eventually have to address the issue of Yukawa couplings, which requires in general large symmetries. Can we increase the honest symmetry G_F discussed in Sec. IV by a specific choice of the a_i , b_i , and c_i parameters? It seems not. One may even wonder if there is a simple choice of these parameters consistent with the conditions for transversality discussed in the Appendix. Here fortunately the answer is yes. A particularly elegant and

consistent complex structure of (2.10) is given by

$$a_k = e^{i(2\pi/5)k}, \quad c_k = a_k^*, \quad k=0,1,2,3,4. \quad (6.1)$$

This implies a Z_5 cyclic symmetry on K_0 :

$$P: x_i \rightarrow x_{i+1}, \quad y_i \rightarrow y_{i+1} \quad (6.2)$$

with

$$\begin{aligned} P_1 &\rightarrow P_1, \quad P_2 \rightarrow e^{-i(2\pi/5)} P_2, \quad P_3 \rightarrow e^{i(2\pi/5)} P_3, \\ P_4 &\rightarrow e^{-i(2\pi/5)} P_4, \quad P_5 \rightarrow P_5 \end{aligned} \quad (6.3)$$

so that new polynomials still vanish. For this to be an honest symmetry on K , we need

$$P^{-1} g P \in G_F \quad (6.4)$$

for every $g \in G_F$. It is easily verified that

$$P^{-1} g_1 P = g_1 g_2 \in G_F$$

but

$$P^{-1} g_1 g_2 P = \text{diag}(1, -1, 1, 1, -1) \notin G_F. \quad (6.5)$$

Of course, (5.1) is still interesting, since P symmetry will be useful in evaluating Yukawa couplings. Also, it is easily shown, that P is not an R symmetry. It is actually the largest possible symmetry on K_0 consistent with the transversality conditions derived in the Appendix.

It is maybe worth commenting that we cannot enlarge G_F by including the element in (6.5), since one can easily see that it does not act freely on K_0 when multiplied by (say) $g_1 g_2$. In other words, in this way K cannot be made into a two-generation manifold. Motivated by the fact that K_0 satisfies the criteria of Aspinwall, Greene, Kirklín, and Miron to actually accommodate two light generations and by the suggestion²⁸ that the extra generation (and its mirrors) may come from all the stuff in (3.11), we tried to construct a two-generation manifold based on K_0 . However, this appears to be impossible.

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APPENDIX

Here we show that the polynomials (2.10) are transverse for a generic choice of the coefficients a_i, c_i and obtain the constraints on these coefficients. This proves that K_0 is a smooth manifold.

We need to check that $\prod_{i=1}^5 dP_i = dP_1 \wedge dP_2 \wedge dP_3 \wedge dP_4 \wedge dP_5$ does not vanish on K_0 . From (2.10),

$$0 = \prod_{i=1}^5 dP_i = A(x, y) \wedge B(y) + B(x) \wedge A(y, x), \quad (\text{A1})$$

where

$$A(x, y) = \sum_{i,j,k=0}^4 a_j c_k x_i x_j y_k dx_i \wedge dx_j \wedge dx_k, \quad (\text{A2})$$

$$B(x) = \sum_{i,j=0}^4 a_j x_i x_j dx_j \wedge dx_j. \quad (\text{A2})$$

Since they are linearly independent, each term in (A1) vanishes separately. Furthermore, as long as

$$a_m - a_n \neq 0 \text{ for } m \neq n, \quad (\text{A3})$$

B 's cannot vanish. Hence

$$A(x, y) = 0 = A(y, x). \quad (\text{A4})$$

The first condition is a set of ten coupled equations:

$$x_0 x_1 y_2 (a_1 - a_0) c_2 - x_0 x_2 y_1 (a_2 - a_0) c_1 + x_1 x_2 y_0 (a_2 - a_1) c_0 = 0, \quad (\text{A5a})$$

$$x_0 x_1 y_3 (a_1 - a_0) c_3 - x_0 x_3 y_1 (a_3 - a_0) c_1 + x_1 x_3 y_0 (a_3 - a_1) c_0 = 0, \quad (\text{A5b})$$

$$x_0 x_1 y_4 (a_1 - a_0) c_4 - x_0 x_4 y_1 (a_4 - a_0) c_1 + x_1 x_4 y_0 (a_4 - a_1) c_0 = 0, \quad (\text{A5c})$$

$$x_0 x_2 y_3 (a_2 - a_0) c_3 - x_0 x_3 y_2 (a_3 - a_0) c_2 + x_2 x_3 y_0 (a_3 - a_2) c_0 = 0, \quad (\text{A5d})$$

$$x_0 x_2 y_4 (a_2 - a_0) c_4 - x_0 x_4 y_2 (a_4 - a_0) c_2 + x_2 x_4 y_0 (a_4 - a_2) c_0 = 0, \quad (\text{A5e})$$

$$x_0 x_3 y_4 (a_3 - a_0) c_4 - x_0 x_4 y_3 (a_4 - a_0) c_3 + x_3 x_4 y_0 (a_4 - a_3) c_0 = 0, \quad (\text{A5f})$$

$$x_1 x_2 y_3 (a_2 - a_1) c_3 - x_1 x_3 y_2 (a_3 - a_1) c_2 + x_2 x_3 y_1 (a_3 - a_2) c_1 = 0, \quad (\text{A5g})$$

$$x_1 x_2 y_4 (a_2 - a_1) c_4 - x_1 x_4 y_2 (a_4 - a_1) c_2 + x_2 x_4 y_1 (a_4 - a_2) c_1 = 0, \quad (\text{A5h})$$

$$x_1 x_3 y_4 (a_3 - a_1) c_4 - x_1 x_4 y_3 (a_4 - a_1) c_3 + x_3 x_4 y_1 (a_4 - a_3) c_1 = 0, \quad (\text{A5i})$$

$$x_2 x_3 y_4 (a_3 - a_2) c_4 - x_2 x_4 y_3 (a_4 - a_2) c_3 + x_3 x_4 y_2 (a_4 - a_3) c_2 = 0. \quad (\text{A5j})$$

The second condition is obtained from the above equations by the substitution $x \leftrightarrow y$. We now show that (A5) has no solution.

(i) Assume first all $x_i, y_i \neq 0$. Take the ratio of Eqs. (A5a) and (A5b), and the same ratio of their symmetric sisters:

$$\frac{c_2}{c_3} \frac{y_2}{y_3} = \frac{x_2}{x_3} \frac{(a_2 - a_0) c_1 x_0 y_1 - (a_2 - a_1) c_0 x_1 y_0}{(a_3 - a_0) c_1 x_0 y_1 - (a_3 - a_1) c_0 x_1 y_0}, \quad (\text{A6})$$

$$\frac{c_2}{c_3} \frac{x_2}{x_3} = \frac{y_2}{y_3} \frac{(a_2 - a_0) c_1 x_1 y_0 - (a_2 - a_1) c_0 x_0 y_1}{(a_3 - a_0) c_1 x_1 y_0 - (a_3 - a_1) c_0 x_0 y_1}.$$

Define $u \equiv x_0 y_1 / x_1 y_0$; (A6) gives a quadratic equation for u :

$$u^2 - Bu + 1 = 0, \quad (\text{A7})$$

where

$$B = \frac{c_3^2 [(a_2 - a_0)^2 c_1^2 + (a_2 - a_1)^2 c_0^2] - c_2^2 [(a_3 - a_0)^2 c_1^2 + (a_3 - a_1)^2 c_0^2]}{c_0 c_1 [c_3^2 (a_2 - a_0)(a_2 - a_1) - c_2^2 (a_3 - a_0)(a_3 - a_1)]}. \quad (\text{A8})$$

Similarly, for (A5a)/(A5c) we get another equation for u , with

$$B' = B(a_3 \rightarrow a_4, c_3 \rightarrow c_4). \quad (\text{A9})$$

In order for (A7)–(A9) to have no solution an additional condition emerges

$$B \neq B'. \quad (\text{A10})$$

Strictly speaking, it is sufficient that any of the corresponding conditions for ratios such as $a_1 y_3 / x_3 y_1$, $x_1 y_4 / x_4 y_1$, $\dots$, is satisfied.

(ii) Next, let us assume that only one x coordinate vanishes, say $x_0 = 0$. From (A5a), $y_0 = 0$, too, which means that we are searching for the possibility $x_0 = y_0 = 0$ and the rest $x_i, y_i \neq 0$. Conditions (A5a), $\dots$, (A5f) then vanish identically. As in case (i) we can still determine (but not over determine) ratios $x_1 y_2 / x_2 y_1$, $x_1 y_3 /$

$x_3 y_1$, $x_1 y_4 / x_4 y_1$. We write them as

$$\frac{y_2}{y_1} = \alpha \frac{x_2}{x_i}, \quad \frac{y_3}{y_1} = \beta \frac{x_3}{x_1}, \quad \frac{y_4}{y_1} = \gamma \frac{x_4}{x_1}, \quad (\text{A11})$$

$$\alpha, \beta, \gamma = \frac{B_{1,2,3} \pm (B_{1,2,3}^2 - 4)^{1/2}}{2},$$

where

$$B_1 = B(a_i \rightarrow a_{i+1}, c_i \rightarrow c_{i+1}),$$

$$B_2 = B(a_0 \rightarrow a_1 \rightarrow a_3 \rightarrow a_4, c_0 \rightarrow c_1 \rightarrow c_3 \rightarrow c_4), \quad (\text{A12})$$

$$B_3 = B(a_0 \rightarrow a_1 \rightarrow a_4, c_0 \rightarrow c_1 \rightarrow c_4).$$

From $P_i = 0$ ($i = 1, \dots, 5$), we get then

$$\begin{aligned}
x_1^2 + x_2^2 + x_3^2 + x_4^2 &= 0, \\
a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + a_4 x_4^2 &= 0, \\
c_1 x_1^2 + c_2 \alpha x_2^2 + c_3 \beta x_3^2 + c_4 \gamma x_4^2 &= 0, \\
a_1 x_1^2 + a_2 \alpha^2 x_2^2 + a_3 \beta^2 x_3^2 + a_4 \gamma^2 x_4^2 &= 0, \\
x_1^2 + \alpha^2 x_2^2 + \beta^2 x_3^2 + \gamma^2 x_4^2 &= 0.
\end{aligned} \tag{A13}$$

But $x_1 \neq 0$ by assumption and on CP_4 we can always scale it to be 1. This implies five equations with three unknowns (x_2, x_3, x_4), which in general is expected to have no solution. To be more specific, by subtracting the first equation from the rest in (A13), we derive four homogeneous equations:

$$\begin{aligned}
(a_2 - a_1)x_2^2 + (a_3 - a_1)x_3^2 + (a_4 - a_1)x_4^2 &= 0, \\
(c_2 \alpha - c_1)x_2^2 + (c_3 \beta - c_1)x_3^2 + (c_4 \gamma - c_1)x_4^2 &= 0, \\
(a_2 \alpha^2 - a_1)x_2^2 + (a_3 \beta^2 - a_1)x_3^2 + (a_4 \gamma^2 - a_1)x_4^2 &= 0, \\
(\alpha^2 - 1)x_2^2 + (\beta^2 - 1)x_3^2 + (\gamma^2 - 1)x_4^2 &= 0.
\end{aligned} \tag{A14}$$

Since in this case we assumed $x_2, x_3, x_4 \neq 0$, as long as any one of the 3×3 subdeterminants of the above system is

nonvanishing, there will be no solution to (A14).

Although it is not illuminating to write all conditions, the reader should be aware of the analogous demands in order not to have any of the other $x_i = y_i = 0$ possibilities. Needless to say, all of them must be verified when a specific set of complex structures is chosen.

(iii) Finally, we consider the case $x_0 = y_0 = 0$ and say $x_1 = 0$. From (A5) then either $x_2 = 0$ or $y_1 = 0$. Now, $P_1 = P_2 = 0$ and the condition (A3) imply immediately no solution with $x_2 = 0$. In other words, *case $x_0 = y_0 = x_1 = y_1$ is the last one we have to analyze.*

$P_i = 0$ then gives a set of equations completely analogous to (2.13). Recall the conditions (2.14) [and use (A3)]

$$Q_{012} = c_0(a_1 - a_2) \pm c_1(a_2 - a_0) \pm c_2(a_0 - a_1) \neq 0. \tag{A15}$$

In complete analogy, we need (in obvious notation)

$$Q_{234} \neq 0. \tag{A16}$$

Again, to eliminate all the solutions with two x 's and y 's vanishing we require altogether ten conditions $Q_{rst} \neq 0$, for any $r \neq s \neq t$. Notice that these conditions guarantee the absence of fixed points for the dividing $Z_2 \times Z_2$ symmetry.

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