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Helicity Amplitudes Calculation *

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Abstract

A method for computing Feynman diagrams at the tree level using an helicity amplitudes formalism is reviewed. This method allows the calculation of as many diagrams as needed and keeping all masses in a very simple and elegant way which fully exploits the symmetries of the problem. The method has been successfully applied in many processes and, in practice, the only actual limitation on the complexity of the problem treated is the actual computation power available for the user.

1 Introduction

The HELICITY AMPLITUDES method as we will describe it here, can be considered just as a practical way of computing Feynman diagrams at the amplitude level. This makes sense only if we specify the momenta and *spin state* of all the particles involved in the process since the amplitudes can be only added when the initial and final states are completely specified, that is, the matrix element squared is computed in the following way:

$$|iM|^2 = \sum_{\lambda} \left| \sum_j (iM_j) \right|^2 \quad (1)$$

where

j runs over all possible diagrams
 λ runs over all the spin states into consideration

The particular choice we use to define the spin states is in the case of massless particles the helicity (equivalent in this case to the chirality) because of its easy representation and interesting properties.

The "standard method" used to compute Feynman diagrams is based on the spin sum properties:

$$\sum_{\lambda} \epsilon_{\lambda}^{\mu} \epsilon_{\lambda}^{\nu} = -g^{\mu\nu} \quad (2)$$

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for massless spin 1 gauge bosons and

$$\sum_{\lambda} u_{\lambda}(p, m) \bar{u}_{\lambda}(p, m) = \not{p} + m \quad (3)$$

for spin 1/2 particles, which are used when combining amplitudes to compute terms of the matrix element squared to close traces. This method in a natural way gets rid off the need of spin state information. The final result of the calculation usually can be expressed in terms of SCALAR products of the type

$$(p_i \cdot p_j) \quad (4)$$

being p_i and p_j particle four-momenta.

In opposition, the basic idea of the Helicity Amplitudes method is that if chiral spinors are used then

$$u_{\lambda}(p) \bar{u}_{\lambda}(p) = \left(\frac{1 + \lambda \gamma_5}{2} \right) \not{p} \quad (5)$$

that is, no sum over λ has to be implemented and then each amplitude can be computed separately for a given helicity configuration. The final result can be expressed in terms of SPINOR products of the type

$$[\bar{u}_{\lambda_i}(p_i) u_{\lambda_j}(p_j)] \quad (6)$$

which are complex functions of p_i, p_j, λ_i and λ_j .

At least a couple of reasons can be pointed out justifying the use of this second method, namely

i.- the fact that the amplitude of each diagram can be computed independently allows to fully exploit the symmetry of the diagrams. Usually all the Feynman diagrams contributing to a given process show enormous symmetries: several amplitudes can be obtained from another just by exchanging particle labels or "rotating" external legs. In practice, only few topologically different diagrams can be found for any process, and therefore is enough computing just the amplitude for the GENERIC DIAGRAMS. Doing so:

a.- we are enabled to attack processes with a large number of diagrams (very impractical with the standard trace-closing technique) being in the method we are going to explain here our computational power the only practical limitation.

b.- it is trivial to obtain a simple and clear codification and also very compact expressions.

ii.- the fact that diagrams are added at the amplitude level provides an optimal control of the cancellations among them. In processes in which there are strong cancellations among the contributions coming from different diagrams this method allows the cancellation to occur directly between the relevant pieces instead of between terms of the matrix element squared (as is the case in the standard method), and therefore:

a.- the needed accuracy is just half the number of digits required when working directly with matrix element squared.

b.- there is no need for mixing different spin configurations (which actually correspond to different physical processes).

Additionally in our approach all particle masses are taken into account without any additional effort and if there are massless gauge bosons as external particles (which is most of the time the practical case) the check of the whole calculation is automatic.

Few different approaches have been suggested and successfully used in the last years [1-2]. We could classify them in two groups:

- (a) the approaches foreseen to be used to produce analytical results (at the prize of neglecting particle masses) and
- (b) the ones designed to be used in computer programs to lead to numerical results (keeping everything on).

As we are mainly interested in calculational techniques for Monte Carlo event generators, that is for numerical calculations, we are going to focus in the second approach which is, in fact, more general but we will show also how the first one can be easily deduced from it when neglecting the particle masses. We are going to follow the work of the CALKUL coll., with the important modifications introduced by Berends, Daverveldt and Kleiss [2] and further developments done by ourselves [3].

2 Feynman diagrams

The starting point for building helicity amplitudes is analyzing the structure of the amplitudes that we obtain when applying Feynman rules to write down the Feynman diagrams of a given process.

The external spin 1/2 particles are described by spinors $u(p, m)$ verifying

$$\sum_{\lambda} u_{\lambda}(p, m) \bar{u}_{\lambda}(p, m) = \not{p} + m \quad (7)$$

and antiparticles show up as particles of negative mass and reversed "fermionic number" flow. These spinors show up paired delimiting the ends of fermionic lines with a well defined "fermionic number" flow direction. This actually represents bilinears of the form

$$[\bar{u}_{\lambda}(p_i, m_i) \Gamma u_{\lambda}(p_j, m_j)] \quad (8)$$

being Γ a set of γ matrices taking into account:

- i.- the couplings with vector bosons
- ii.- the propagator of virtual fermions

The external spin 1 massless gauge bosons¹ are described by polarization vectors $\epsilon_{\lambda}^{\mu}(k)$ verifying

$$\begin{aligned} \epsilon_{\lambda} \cdot k &= 0 \\ \epsilon_{-\lambda} &= \epsilon_{\lambda}^* \end{aligned} \quad \begin{aligned} \epsilon_{\lambda} \cdot \epsilon_{\lambda} &= 0 \\ \epsilon_{\lambda} \cdot \epsilon_{-\lambda} &= 1 \end{aligned} \quad (9)$$

¹ Actually the only stable ones

in such a way that any objects verifying the above relations make an acceptable choice for $\epsilon_{\lambda}^{\mu}(k)$. For instance

$$\epsilon_{\lambda}^{\mu*} = N [\bar{u}_{\lambda}(k) \gamma^{\mu} u_{\lambda}(p)] \quad (10)$$

being

- N a normalization factor (which we will compute later)
- p^{μ} any 4-vector non-proportional to k^{μ}

Therefore an amplitude can be seen just as the product of a set of bilinears interconnected among them through Lorenz indices, namely

$$iM_i = \alpha [\bar{u}_1 \Gamma_A \gamma^{\mu} \Gamma_B u_2] [\bar{u}_3 \Gamma_C \gamma_{\mu} \Gamma_D \gamma_{\nu} \Gamma_E u_4] [\bar{u}_5 \Gamma_F \gamma^{\nu} \Gamma_G u_6] \quad (11)$$

where now the matrices Γ_i are basically products of

- i.- chiral projectors $P_R = \frac{1+\gamma_5}{2}$ and $P_L = \frac{1-\gamma_5}{2}$
- ii.- matrices $\not{p}$

Shifting the chiral projectors and substituting the matrices as

$$\not{p}_i = \left(\sum_{\lambda} [u_{\lambda}(p) \bar{u}_{\lambda}(p)] \right) - m_i \quad (12)$$

we end up essentially with bilinears of the following kind:

$$\begin{aligned} [\bar{u}_{\lambda_1}(p_1) (C_L P_L + C_R P_R) u_{\lambda_2}(p_2)] &\longrightarrow \text{Spinor products ("Y" functions)} \\ [\bar{u}_{\lambda_1}(p_1) \Gamma^{\mu} u_{\lambda_2}(p_2)] [\bar{u}_{\lambda_3}(p_3) \Gamma^{\nu} u_{\lambda_4}(p_4)] &\longrightarrow \text{"Z" functions} \end{aligned} \quad (13)$$

being

$$\Gamma^{\mu} \equiv \gamma^{\mu} (C_L P_L + C_R P_R) \quad (14)$$

3 Basic spinors

The standard way of building up spinors explicitly is solving Dirac's equation in the rest frame of the particle to obtain BASIC SPINORS.

$$\begin{aligned} m \gamma^0 u_i(0, m) &= m u_i(0, m) \\ m \gamma^0 v_i(0, m) &= -m v_i(0, m) \end{aligned} \quad (15)$$

taking an explicit representation of the γ matrices and afterwards boosting this solution to a frame with velocity $\vec{\beta} = \vec{p}/E$, namely

$$\begin{aligned} u_i(p, m) &= \frac{\not{p} + m}{\sqrt{2m(m+E)}} u_i(0, m) \\ v_i(p, m) &= \frac{-\not{p} + m}{\sqrt{2m(m+E)}} v_i(0, m) \end{aligned} \quad (16)$$

for $i = 1, 2$.

This spinors verify Dirac equation and obviously

$$\begin{aligned}\sum_i u_i(p) \bar{u}_i(p) &= \not{p} + m \\ \sum_i v_i(p) \bar{v}_i(p) &= \not{p} - m\end{aligned}\quad (17)$$

The problem is that this kind of procedure does not work for building spinors for massless particles. To do so we can instead start from the last equation and then for massless particles

$$\sum_i w_i(k_0) \bar{w}_i(k_0) = \not{k}_0 \quad \text{being } k_0 \cdot k_0 = 0 \text{ (massless 4-momentum)} \quad (18)$$

in such a way that defining the two spin states as the chiral ones (the most natural choice for massless particles) we can write

$$w_\lambda(k_0) \bar{w}_\lambda(k_0) = \frac{1 + \lambda \gamma_5}{2} \not{k}_0 \quad (19)$$

being λ the helicity $\lambda = \pm 1$. The main point is that such spinors can act as BASIC SPINORS too, that is, can be transformed to produce the spinor of a particle of momentum p in the following way:

$$u_\lambda(p, 0) = \frac{\not{p}}{\eta} w_{-\lambda}(k_0) \quad \text{being } \eta = \sqrt{2(p \cdot k_0)} \quad (20)$$

where u_λ has all the properties of a chiral spinor of helicity λ and the transformation applied can be regarded as a sort of boost. The result that it becomes singular for $p \cdot k_0 = 0$ reflects the fact that no Lorentz transformation can "flip" the spin of a massless particle.

Moreover, we are enabled to produce also the spinor of a particle of momentum p and mass m in the following way:

$$u_\lambda(p, m) = \frac{\not{p} + m}{\eta} w_{-\lambda}(k_0) \quad \text{being } \eta = \sqrt{2(p \cdot k_0)} \text{ as before} \quad (21)$$

where u_λ is a spinor for which λ denotes the two possible spin projections along

$$s^\mu = \frac{p^\mu}{m} - \frac{m}{(p \cdot k_0)} k_0^\mu \quad (22)$$

that is

$$u_\lambda(p, m) \bar{u}_\lambda(p, m) = \frac{1 + \lambda \gamma_5 \not{s}}{2} (\not{p} + m) \quad (23)$$

Obviously now the transformation is not a boost but has a similar structure and produces the desired results. Thus it has to be considered as the natural extension of the massless boost to the case of massive particles.

Therefore $w_\lambda(k_0)$ can be regarded as the BASIC SPINORS we need to fix up the phase among all the spinors occurring in a process because we have a well defined method to build any spinor as a function of the same $w_\lambda(k_0)$. The defining properties of $w_\lambda(k_0)$ are

$$w_\lambda(k_0) \bar{w}_\lambda(k_0) = \frac{1 + \lambda \gamma_5}{2} \not{k}_0 \quad (24)$$

which defines λ as a chiral index and

$$w_\lambda(k_0) = \lambda \not{k}_1 w_{-\lambda}(k_0) \quad (25)$$

which fixes up the phase between both chiral states. k_0 and k_1 are any 4-vector satisfying:

$$\begin{aligned}k_0 \cdot k_0 &= 0 && \text{massless vector} \\ k_1 \cdot k_1 &= -1 && \text{space-like (rotation)} \\ k_0 \cdot k_1 &= 0 && \text{orthogonality}\end{aligned}\quad (26)$$

4 Spinor product

4.1 Massless case

For massless fermions with momenta p_1 and p_2 there are just two non-zero products

$$\begin{aligned}[u_+(p_1) u_-(p_2)] &= S_+(p_1, p_2) \\ [u_-(p_1) u_+(p_2)] &= S_-(p_1, p_2) = -[S_+(p_1, p_2)]^*\end{aligned}\quad (27)$$

being

$$\begin{aligned}S_+(p_1, p_2) &= \frac{[\bar{w}_-(k_0) \not{p}_1 \not{p}_2 w_+(k_0)]}{\eta_1 \eta_2} = \frac{[\bar{w}_-(k_0) \not{p}_1 \not{p}_2 \not{k}_1 w_-(k_0)]}{\eta_1 \eta_2} \\ &= \frac{T_+ \left\{ \frac{1+\gamma_5}{2} \not{k}_0 \not{p}_1 \not{p}_2 \not{k}_1 \right\}}{\eta_1 \eta_2} \\ &= 2 \frac{(p_1 \cdot k_0)(p_2 \cdot k_1) - (p_1 \cdot k_1)(p_2 \cdot k_0) - i\epsilon(k_0, k_1, p_1, p_2)}{\eta_1 \eta_2}\end{aligned}\quad (28)$$

As the values of k_0 and k_1 are arbitrary we can evaluate the previous expression taking a simple and suitable choice. For instance

$$\begin{aligned}k_0^\mu &= (1, 1, 0, 0) \\ k_1^\mu &= (0, 0, 1, 0)\end{aligned}\quad (29)$$

turns out to be a good choice if the majority of the particles tend to go in the z direction. If we take $p_i^\mu = (E_i, p_i^x, p_i^y, p_i^z)$ then we can write

$$\begin{aligned}S_\lambda(p_1, p_2) &= \lambda \left((p_1^y + \lambda i p_1^z) \frac{\eta_2}{\eta_1} - (p_2^y + \lambda i p_2^z) \frac{\eta_1}{\eta_2} \right) \\ &= \lambda \left((p_1^y + \lambda i p_1^z) \sqrt{\frac{E_2 - p_2^z}{E_1 - p_1^z}} - (p_2^y + \lambda i p_2^z) \sqrt{\frac{E_1 - p_1^z}{E_2 - p_2^z}} \right)\end{aligned}\quad (30)$$

where

$$\eta_i = \sqrt{2(p_i \cdot k_0)} = \sqrt{2(E_i - p_i^z)} \quad (31)$$

This expression is the basic building block of helicity amplitudes in the same way that scalar products build $|M|^2$. In fact

$$2(p_1 p_2) = |S(p_1, p_2)|^2 \quad (32)$$

4.2 Massive case

For massive spinors the previous calculation can be carried out in the same way and leads to the same result, namely

$$\begin{aligned} [\bar{u}_+(p_1)u_+(p_2)] &= S_+(p_1, p_2) \\ [\bar{u}_-(p_1)u_+(p_2)] &= S_-(p_1, p_2) \end{aligned} \quad (33)$$

being $S_{\pm}$ the same function obtained before, but now there are two additional non-vanishing spinor products:

$$\begin{aligned} [\bar{u}_+(p_1)u_+(p_2)] &= Y_+(p_1, p_2) \\ [\bar{u}_-(p_1)u_-(p_2)] &= Y_-(p_1, p_2) = Y_+(p_1, p_2) \end{aligned} \quad (34)$$

being Y a function which can be computed as before using the definition properties of u_{λ} , namely

$$\begin{aligned} Y(p_1, p_2) &= \frac{[\bar{u}_+(k_0)(p_1 + m)(p_2 + m)w_+(k_0)]}{\eta_1 \eta_2} \\ &= m_1 \frac{Tr\{\frac{1+\gamma_5}{2} k_0 p_2\}}{\eta_1 \eta_2} + m_2 \frac{Tr\{\frac{1+\gamma_5}{2} k_0 p_1\}}{\eta_1 \eta_2} \\ &= m_1 \left(\frac{\eta_2}{\eta_1} + m_2 \left(\frac{\eta_1}{\eta_2} \right) \right) \end{aligned} \quad (35)$$

so that

$$[\bar{u}_{\lambda}(p_1)u_{\lambda}(p_2)] = Y(p_1, p_2) = 2 m_1 \quad (36)$$

as expected because of the spinor normalization used. In this case, the scalar product of the particle 4-momenta satisfies:

$$2(p_1, p_2) = |S(p_1, p_2)|^2 + m_1^2 \left(\frac{\eta_2}{\eta_1} \right)^2 + m_2^2 \left(\frac{\eta_1}{\eta_2} \right)^2 \quad (37)$$

5 Z functions

As said at the beginning, the links between fermionic lines shows up as functions of the form

$$Z(p_1, \lambda_1; p_2, \lambda_2; p_3, \lambda_3; p_4, \lambda_4) \equiv [\bar{u}_{\lambda_1}(p_1)\Gamma^{\mu}u_{\lambda_2}(p_2)][\bar{u}_{\lambda_3}(p_3)\Gamma^{\nu}u_{\lambda_4}(p_4)] \quad (38)$$

being in the general case

$$\Gamma^{\mu} \equiv \gamma^{\mu}(C_L P_L + C_R P_R) \quad P_L = \left(\frac{1 - \gamma_5}{2} \right) \quad P_R = \left(\frac{1 + \gamma_5}{2} \right) \quad (39)$$

We will see now how these functions can be computed.

5.1 Massless case

Lets realize that for chiral spinors the effect of chiral projectors P_L and P_R can be trivially taken into account. It is also trivial to demonstrate that in this case

$$[\bar{u}_{\lambda}(p_1)\gamma^{\mu}u_{\lambda}(p_2)] \quad \gamma_{\mu} = 2 u_{-\lambda}(p_2)\bar{u}_{\lambda}(p_1) + 2 u_{-\lambda}(p_1)\bar{u}_{-\lambda}(p_2) \quad (40)$$

which is the so-called Chisholm identity. Therefore, as the massless spinors we have are chiral the helicities should necessarily verify $\lambda_1 = \lambda_2 = \lambda$ and $\lambda_3 = \lambda_4 = \lambda'$ and therefore, we can trivially write

$$\begin{aligned} Z(\lambda, p_1, p_2; \lambda', p_3, p_4) &\equiv [\bar{u}_{\lambda}(p_1)\gamma^{\mu}u_{\lambda}(p_2)][\bar{u}_{\lambda'}(p_3)\gamma_{\mu}u_{\lambda'}(p_4)] \\ &= 2 [\bar{u}_{\lambda'}(p_3)u_{\lambda}(p_2)][\bar{u}_{\lambda}(p_1)u_{\lambda'}(p_4)] \\ &\quad + 2 [\bar{u}_{\lambda'}(p_3)u_{-\lambda}(p_1)][\bar{u}_{-\lambda}(p_2)u_{\lambda'}(p_4)] \end{aligned} \quad (41)$$

so that we finally have

$$Z(\lambda, p_1, p_2; \lambda', p_3, p_4) = -2 \delta_{\lambda\lambda'} S_{\lambda}(p_1, p_2) S_{-\lambda}(p_3, p_4) - 2 \delta_{-\lambda\lambda'} S_{\lambda}(p_1, p_4) S_{-\lambda}(p_2, p_3) \quad (42)$$

5.2 Massive case

In the massive case, spinors are not chiral and therefore we do not have any trivial simplification of the P_L and P_R projectors. Moreover Chisholm identity does not hold directly and therefore we need to proceed in a different manner.

One possible way of attacking the calculation of Z is decomposing the u spinors as follows:

$$\begin{aligned} u_{\lambda}(p, m) &= \frac{\not{p} + m}{\eta} w_{-\lambda}(k_0) = \frac{\not{p}}{\eta} w_{-\lambda}(k_0) + \frac{m}{\eta} w_{-\lambda}(k_0) \\ &= w_{\lambda}(p) + \mu w_{-\lambda}(k_0) \end{aligned} \quad (43)$$

being

$$w_{\lambda}(p) \equiv \frac{\not{p}}{\eta} w_{-\lambda}(k_0) \quad \mu \equiv \frac{m}{\eta} \quad (44)$$

$w_{\lambda}(p)$ does not verify Dirac equation but is certainly a chiral object by definition. Additionally it is easy to demonstrate that

$$[\bar{w}_{\lambda}(p_1)w_{\lambda}(p_2)] = S_{\lambda}(p_1, p_2) \quad (45)$$

Moreover, if we define

$$\tilde{Z}(\lambda, q_1, q_2; \lambda', q_3, q_4) \equiv F^{\mu}(\lambda, q_1, q_2) F_{\mu}(\lambda', q_3, q_4) \quad (46)$$

where

$$F^{\mu}(\lambda, q_1, q_2) \equiv [\bar{w}_{\lambda}(q_1)\gamma_{\mu}w_{\lambda}(q_2)] \quad (47)$$

being $q_i \equiv p_i$ or k_0 where p_i denotes a particle 4-momentum, it is a lengthy but straightforward task to prove that $\tilde{Z}$ can be computed easily if at the level of F^{μ} we make explicit the difference between $w_{\lambda}(p_i)$ and $w_{\lambda}(k_0)$ by using in all the occurrences the substitution rule:

$$\begin{aligned} F^{\mu}(\lambda, k_0, q_i) &= -f^{\mu}(-\lambda, q_i, k_0) \\ F^{\mu}(\lambda, p_i, q_j) &= f^{\mu}(\lambda, p_i, q_j) \end{aligned} \quad (48)$$

and applying afterwards the index elimination property (Chisholm identity like)

$$f^\mu(\lambda, q_1, q_2) f_\mu(\lambda', q_3, q_4) = -2 \delta_{\lambda\lambda'} SM_\lambda(q_1, q_2) SM_{-\lambda}(q_3, q_4) - 2 \delta_{-\lambda\lambda'} SM_\lambda(q_1, q_4) SM_{-\lambda}(q_2, q_3) \quad (49)$$

being the quantities $SM_\lambda(q_i, q_j)$ a natural extension of the spinor products $S_\lambda(p_i, p_j)$ defined as:

$$\begin{aligned} SM_\lambda(p_i, p_j) &= -SM_\lambda(p_j, p_i) = [\bar{u}_\lambda(p_i) u_{-\lambda}(p_j)] = S_\lambda(p_i, p_j) \\ SM_\lambda(p_i, k_0) &= -SM_\lambda(k_0, p_i) = \eta \end{aligned} \quad (50)$$

that is, $SM_\lambda(q_i, q_j)$ is an antisymmetric function of q_i and q_j and therefore:

$$SM_\lambda(k_0, k_0) = SM_\lambda(p_i, p_i) = 0 \quad (51)$$

After applying these rules the calculation of all the possible values of the Z function is straightforward, and defining

$$Z(\lambda_i, \lambda_j, \lambda_k, \lambda_l) \equiv Z(p_i, \lambda_i; p_j, \lambda_j; p_k, \lambda_k; p_l, \lambda_l) \quad (52)$$

we find the following expressions for half of the possible helicity configurations:

$$\begin{aligned} Z(+, +, +, +) &= -2\{SM_+(p_3, p_1)SM_-(p_4, p_2)C'_R C_R - \mu_1 \mu_2 \eta_3 \eta_4 C'_R C_L \\ &\quad - \eta_1 \eta_2 \mu_3 \mu_4 C'_L C_R\} \\ Z(+, +, +, -) &= -2\eta_2 C_R \{SM_+(p_4, p_1) \mu_3 C'_L - SM_+(p_3, p_1) \mu_4 C'_R\} \\ Z(+, +, -, +) &= -2\eta_1 C_R \{SM_-(p_2, p_3) \mu_4 C'_L - SM_-(p_2, p_4) \mu_3 C'_R\} \\ Z(+, +, -, -) &= -2\{SM_+(p_1, p_4)SM_-(p_2, p_3)C'_L C_R - \mu_1 \mu_2 \eta_3 \eta_4 C'_L C_L \\ &\quad - \eta_1 \eta_2 \mu_3 \mu_4 C'_R C_R\} \\ Z(+, -, +, +) &= -2\eta_4 C'_R \{SM_+(p_3, p_1) \mu_2 C'_R - SM_+(p_3, p_2) \mu_1 C'_L\} \\ Z(+, -, +, -) &= 0 \\ Z(+, -, -, +) &= -2\{\mu_1 \mu_4 \eta_2 \eta_3 C'_L C_L + \mu_2 \mu_3 \eta_1 \eta_4 C'_R C_R - \mu_2 \mu_1 \eta_1 \eta_3 C'_L C_R \\ &\quad - \mu_1 \mu_3 \eta_2 \eta_4 C'_R C_L\} \\ Z(+, -, -, -) &= -2\eta_3 C'_L \{SM_+(p_2, p_1) \mu_1 C'_L - SM_+(p_1, p_4) \mu_2 C'_R\} \end{aligned} \quad (53)$$

where C_L and C_R are the left and right handed couplings of the first bilinear and C'_L and C'_R are the ones of the second bilinear. The rest can be obtained by exchanging $+$ $\leftrightarrow$ $-$ and $L \leftrightarrow R$.

6 Calculational procedure

The actual steps needed to perform a calculation using the helicity amplitudes method are:

- (1) draw all the Feynman diagrams you want to compute to figure out how many topologically different you have. Obviously gauge invariant subsets have to be complete.

- (2) write down the amplitude for one representative of each topology. In order to allow an easy transformation of such amplitude to obtain the one from any diagram topologically equivalent is important to use standard Feynman rules plus the following prescriptions:

$$i.- \text{ take } m_i = \begin{cases} m & \text{for particles} \\ -m & \text{for antiparticles} \end{cases} \quad (54)$$

and then just substitute every

$$\begin{aligned} v_i &\rightarrow u_i \\ \bar{v}_i &\rightarrow \bar{u}_i \end{aligned} \quad (55)$$

- ii.- write each 4-momentum showing up explicitly in the amplitude absorbing its sign in the way

$$b_i p_i \quad (56)$$

where

$$b_i = \begin{cases} 1 & \text{if } p_i \text{ is incoming} \\ -1 & \text{if } p_i \text{ is outgoing} \end{cases} \quad (57)$$

these two rules allow absorbing the characteristics of *particle* $\leftrightarrow$ *antiparticle* and *incoming* $\leftrightarrow$ *outgoing* of the specific amplitude studied, in the masses m_i and the direction parameters b_i in such a way that the obtained amplitude corresponds to a diagram in which the only information relevant in each external leg is just one label. Therefore, we have build up the **GENERIC AMPLITUDE** for the diagram topology studied. Any diagram topologically equivalent to the one studied can be obtained from the generic amplitude just by an overall permutation of particle labels and helicities (and occasionally conjugation). This would be more clearly seen in the examples discussed later.

- (3) express the generic amplitude in terms of Z and Y functions (massive case) or directly in terms of *spinor products* (massless case) by using the two following substitutions:

$$i.- \text{ if you have external massless gauge bosons: } \epsilon_\lambda^{\mu\nu} = N [\bar{u}_\lambda(k) \gamma^\mu u_\lambda(p)] \quad (58)$$

being

$$\begin{aligned} N &= (2[SM_\lambda(k, p)]^2)^{-1/2} \text{ the normalization} \\ p &= \text{any 4-momentum but } k \end{aligned}$$

This substitution allows you to transform the vector boson polarization vector in a bilinear. The existing freedom in choosing p corresponds to fixing the gauge and therefore different p correspond to different gauges. As the final result of a calculation must be gauge-independent, we must obtain the same result regardless on the election of p . The fact that p is actually just a simple label, allows to easily check for the consistency of the calculation.

ii.- if you have propagators in a fermionic line:

$$P_i = \left\{ \sum_{\lambda} u_{\lambda}(p_i) \bar{u}_{\lambda}(p_i) \right\} - m_i \quad (59)$$

regardless on whether P_i is a fermion or a boson 4-momentum

- (4) In the case of massless particles you can obtain rather compact expressions by just adding analytically all the amplitudes for each helicity configuration λ_i , whereas in the case of massive particles the complexity of the Z and Y functions strongly suggests using directly a computer program in which the generic amplitudes are written down as callable functions of the particle labels and helicities to proceed to a numerical calculation.

7 Monte Carlo generators

When using this method to implement a Monte Carlo event generator the general way of proceeding is the following:

- (1) Generate a phase space point with a suitable distribution density mapping as much as possible the relevant $|M|^2$ peaks
- (2) Fill the arrays $SM(i,j)$ (and $Y(i,j)$ if needed) where the indices i and j run over all particle labels and also k_0
- (3) Take a helicity configuration λ_i and add the contribution of all the diagrams by using the generic amplitude functions which in turn are written in terms of functions Z (this is just a numerical complex number calculation)
- (4) For this helicity configuration $M(\lambda_i)$ is just a complex number and $|M(\lambda_i)|^2$ is proportional to the probability of this configuration in the generated phase space point.
- (5) Add (or average) over the desired helicity configurations and then you will have your matrix element squared

8 Example: $e^-(p_1) + e^+(p_2) \rightarrow e^-(p_3) + e^+(p_4) + \gamma(k)$

For this process, at the tree level in the Electroweak theory there are 16 possible diagrams (fig. 1) and in some detection configurations it is very important keeping the electron mass, namely collinear photon or outgoing electron (positron) parallel to the incoming one. Using the standard method the calculation of this process shows the following problems:

- (1) there are $16 \times 17/2 = 136$ terms in $|M|^2$ which have to be computed
- (2) keeping masses produces very long expressions which in practice can be handled only by using an algebraic computer language such as REDUCE for instance
- (3) during the calculation of $|M|^2$ in phase space points giving the bulk of the contribution in collinear configurations you have numerical instability even using FORTRAN double precision variables due to strong cancellations among different diagram contributions

Figure 1: Feynman diagrams of the process $e^-(p_1) + e^+(p_2) \rightarrow e^-(p_3) + e^+(p_4) + \gamma(k)$ at the tree level

These problems vanish using the helicity amplitudes method as we will see now. Looking at the diagrams of figure 1 it is clear that all of them have exactly the same topology. Therefore it is enough to build one generic amplitude for the calculation of all the diagrams. Lets take for instance diagram $g-1$ (or $z-1$) and write down the amplitude using standard Feynman rules (fig.2):

$$iM_1 = \frac{ie^3 T_1(p_1, \lambda_1)}{((p_1 - k)^2 - m_e^2) F((p_2 - p_4)^2)} \quad (60)$$

being

$$T_1(p_1, \lambda_1) = [\bar{u}(p_3, \lambda_3) \Gamma^\mu (\not{p}_1 - \not{k} + m_e) \not{\epsilon}(k, \lambda_k) u(p_1, \lambda_1)] [\bar{v}(p_2, \lambda_2) \Gamma^\nu v(p_4, \lambda_4)] \quad (61)$$

where

Figure 2: g-1 diagram

Figure 3: Generic diagram

$$\Gamma^\mu = \begin{cases} \gamma^\mu & \text{photon} \\ \gamma^\mu(C_L P_L + C_R P_R) & Z^0 \end{cases} \quad (62)$$

and

$$F(y) = \begin{cases} y & \text{photon} \\ (y - M_Z)^2 + iM_Z \Gamma_Z & Z^0 \end{cases} \quad (63)$$

As we are going to use this diagram as the generic one, we have to re-write its amplitude by using the rules recommended:

$$iM_1 = \frac{ie^3 T_1(p_i, \lambda_i)}{\{(b_1 p_1 + b_k k)^2 - m_e^2\} F((b_2 p_2 + b_4 p_4)^2)} \quad (64)$$

being

$$T_1(p_i, \lambda_i) = [\bar{u}(p_3, \lambda_3) \Gamma^\mu (b_1(p_1 + m_e) + b_k \not{k}) \not{f}(k, \lambda_k) u(p_1, \lambda_1)] [\bar{u}(p_2, \lambda_2) \Gamma_\mu u(p_4, \lambda_4)] \quad (65)$$

taking

$$\begin{aligned} m_1 &= m_3 = m_e \\ m_2 &= m_4 = -m_e \end{aligned} \quad (66)$$

and

$$\begin{aligned} b_1 &= b_3 = 1 \\ b_2 &= b_4 = b_k = -1 \end{aligned} \quad (67)$$

What we get is an amplitude which in fact corresponds to the diagram of figure 3 where in b_i, p_i and m_i is just a label which can be used for any particle in the process. Then it is trivial to see that for instance, for diagrams $g-5$ ($z-5$) we have (fig.4):

$$iM_5 = -iM_1(2 \leftrightarrow 3) \quad (68)$$

Figure 4: g-5 diagram

Figure 5: g-2 diagram

where the $(-)$ sign comes from the t channel to s channel relative sign. In the same way, for diagrams $g-2$ ($z-2$) we have (fig.5):

$$iM_2 = -(iM_1)^*(1 \leftrightarrow 2, 3 \leftrightarrow 4; \lambda_k \leftrightarrow -\lambda_k) \quad (69)$$

where the conjugation reverses the fermionic arrows and changes the photon helicity (besides producing an overall $(-)$ sign due to an odd number of couplings).

Therefore, as all the amplitudes can be computed using M_1 (and permutations of labels and conjugations) we just have to compute M_1 . To do so, following the rules explained above, let's rewrite the photon polarization vector as

$$e^{\mu*}(k, \lambda_k) = N \bar{u}(k, \lambda_k) \gamma^\mu u(p, \lambda_k) \quad (70)$$

being

$$N = \frac{1}{\sqrt{2|SM_\lambda(k, p)|^2}} \quad (71)$$

p any 4-momentum different from k

and let's substitute

$$\begin{aligned} p_1 + m_1 &= \sum_\lambda u(p_1, \lambda) \bar{u}(p_1, \lambda) \\ \not{k} &= \sum_\lambda u(k, \lambda) \bar{u}(k, \lambda) \end{aligned} \quad (72)$$

and then we can write

$$\begin{aligned} T_1(p_i, \lambda_i) &= N \sum_\lambda (b_1 [\bar{u}_{\lambda_3}(p_3) \Gamma^\mu u_{\lambda_1}(p_1)] [\bar{u}_{\lambda_2}(p_2) \Gamma_\mu u_{\lambda_4}(p_4)] \\ &\quad - b_k [\bar{u}_{\lambda_1}(p_1) \Gamma^\mu u_{\lambda_3}(p_3)] [\bar{u}_{\lambda_4}(k) \gamma_\mu u_{\lambda_2}(k)] \\ &\quad - b_k [\bar{u}_{\lambda_3}(p_3) \Gamma^\mu u_{\lambda_1}(k)] [\bar{u}_{\lambda_2}(p_2) \Gamma_\mu u_{\lambda_4}(p_4)] \\ &\quad - b_k [\bar{u}_{\lambda_1}(k) \gamma^\mu u_{\lambda_3}(p_1)] [\bar{u}_{\lambda_4}(k) \gamma_\mu u_{\lambda_2}(p)]]) \end{aligned} \quad (73)$$

so that in terms of Z functions we simply have

$$\begin{aligned} T_1(p_i, \lambda_i) &= N \sum_\lambda (b_1 Z(p_3, \lambda_3; p_1, \lambda_1; p_2, \lambda_2; p_4, \lambda_4; C_L, C_R, C_L, C_R) \\ &\quad - b_k Z(p_1, \lambda_1; p_1, \lambda_1; k, \lambda_k; p_1, \lambda_1; 1, 1, 1, 1) \\ &\quad - b_k Z(p_3, \lambda_3; k, \lambda_k; p_2, \lambda_2; p_4, \lambda_4; C_L, C_R, C_L, C_R) \\ &\quad - b_k Z(p_3, \lambda_1; p_1, \lambda_1; k, \lambda_k; p_1, \lambda_1; 1, 1, 1, 1)) \end{aligned} \quad (74)$$

being $C_L = C_R = 1$ for photonic diagrams

Given a particular set of 4-momenta and helicities p_i, λ_i one can evaluate the Z functions for the generic diagram and, with the appropriate permutations and conjugations the actual amplitudes for all the diagrams contributing to the process. Finally

$$|\bar{M}|^2 = \frac{1}{4} \sum_\lambda |\sum_j (iM_j)|^2 \quad (75)$$

where j runs over diagrams and λ runs over helicities.

Choosing different possibilities for p (that is, choosing labels 1, ..., 4) the consistency of the calculation is trivially checked.

9 Conclusions

We have sketched a powerful method for computing tree level Feynman diagrams which we have already used in many processes [3-4]. For instance

$$\begin{aligned} e^+ e^- &\rightarrow e^+ e^- \gamma \\ e^+ e^- &\rightarrow e^+ e^- \gamma \gamma \\ e^+ e^- &\rightarrow \nu \bar{\nu} \gamma \end{aligned} \quad (76)$$

The basic tool are just few Fortran functions computing spinor products and Z functions which are available from the authors on request. The only actual limitation on the complexity of the problem attacked are your available computer resources.

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