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Radiative Corrections to the Neutrino Counting Experiment*

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Abstract

Radiative corrections to radiative neutrino production and to radiative Bhabha scattering are discussed. Concerning the first process, one loop corrections are briefly reviewed. After emphasizing the importance of higher order QED corrections, their implementation using the Structure Functions approach is described. Next, the inclusion of the main weak corrections using the 'star' scheme is explained. Finally some results are presented, showing effects in the range 5-6% with respect to the pure one loop corrected cross-section. Also some comparisons with another calculation featuring multiphoton generation are shown, and good agreement is found between them, both for the total cross-section and the photonic distributions. Concerning the background process, hard photon corrections are computed. After comparing the results with the ones obtained using the EPA, this approximation is used for the virtual and soft part. Some results are presented showing corrections around 3%.

1 Introduction

Radiative neutrino production has been proposed long time ago ([1]) as a clean way of measuring the number of generations within the Standard Model framework. The first calculations were done in the lowest order approximation and using the contact approximation for the W diagrams ([2]). Not until recently complete tree level calculations have been performed ([3], [4]) and also one-loop radiative corrections have been taken into account ([4]-[6]). Since their effect has been found to be large (a common feature of radiative corrections to four fermion neutral current processes around the Z^0 peak), we have undertaken the task of including the main higher order (i.e., more than one-loop) radiative corrections.

Our goal is to achieve a precision better than 1% in the estimation of the total cross-section. This is of the order of the experimental needs, because, although the effect due to an extra generation would be much larger (around 25%), other reactions leading to single photon final states (radiative production of sneutrinos, photinos, etc.) have cross-sections in the per

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cent range. On the other hand, the error in the determination of the absolute luminosity will be the limiting experimental error, and it will not be smaller than 1%. Therefore, this is a sensible choice for the planned accuracy.

Radiative Bhabha scattering was first pointed out to be the main background to the neutrino counting reaction in ref. [7]. The two final state electrons tend to be produced at very small angles, disappearing down the beam pipe undetected. Then, what remains is a single photon just like in the neutrino case. However, since the electrons go at small angle, they have a limited transverse momentum (p_T) which is compensated by the photon. Hence, due to momentum conservation, there is a limitation in the photon p_T ([7]). Therefore, a p_T cut results in an efficient signal over background ratio when running far above the Z peak because it eliminates most of the $e^+e^- \gamma$ events. However, when running close to or exactly at the Z^0 peak, it eliminates the signal events, too. Therefore, in this case, one has to soften the p_T cut and a large fraction of the background events remain. The only way to proceed then is subtracting the expected number of background events to get the signal cross-section. Then it is clear that a good estimation of the theoretical cross-section for the background process is needed at the few per cent level.

In section 2.1 we will review quickly the exact lowest order calculation for $e^+e^- \rightarrow \nu\bar{\nu}\gamma$. Next we explain with no technical details the one-loop corrections in section 2.2. After justifying the need for higher order corrections, we present in section 2.3 the two implementations we have done of the treatment of higher order QED corrections: one based in the 'inductive' approach ([8]) and another one based on the Structure Functions approach ([9]). Section 2.4 deals with the inclusion of the main higher order weak corrections by means of the 'star' scheme ([10]). Also the need for the inclusion of one loop corrections to the imaginary part of the Z^0 self energy is emphasized there. Finally, section 2.5 contains the results of a comparison done between our Monte Carlo program built along the lines explained in the previous sections (NNGG03) and another one based on the Yennie-Frautschi-Suura approach ([11]). These results have been taken from ref. [12].

In chapter 3 we present the calculation of radiative corrections to $e^+e^- \rightarrow e^+e^- \gamma$. Section 3.1 presents the results of the lowest order calculation. Then we will present our calculation of the $O(\alpha)$ QED corrections ([13]) through sections 3.2 and 3.3. Section 3.4 is devoted to the phase space integration and, finally, in section 3.5 we will discuss some results.

2 The process $e^+e^- \rightarrow \nu\bar{\nu}\gamma$

2.1 Lowest order calculation

The diagrams entering the tree level matrix element squared are those depicted in fig. 1. The result for the different helicity amplitudes can be found in ref. [4]. We have taken these results expressing the matrix element squared in terms of α , $\sin^2\theta_W$ and M_Z , i.e. not using the tree level relation between those quantities and G_F to get an 'improved' tree level result which is neither the true tree level result nor the one obtained after the propagator (also called oblique) corrections are included. In Table 1 we can see the results for the integrated cross-section when making different approximations to the exact calculation. The meaning of the different cross-sections is as follows:

- σ_2^0 is the cross-section taking into account only the two Z^0 diagrams.

Table 1: Comparison between different approximations for the tree level cross-section.

$\sqrt{s}(\text{GeV})$	$\sigma_2^0(\text{pb})$	$\sigma_{GGR}^0(\text{pb})$	$\sigma_4^0(\text{pb})$	$\sigma_2^0(\text{pb})$
98	126.5(3)	129.5(3)	128.7(3)	128.7(3)
150	5.80(3)	9.97(3)	7.74(5)	7.79(5)

- σ_{GGR}^0 is the cross-section obtained neglecting the last diagram in fig. 1 and taking the limit $M_W^2 \rightarrow \infty$ in the other W diagrams. It corresponds to the classical result of Gaemers, Gastmans and Renard (GGR) presented in [2].
- σ_4^0 is the result obtained neglecting the last diagram but taking into account without any approximation the one- W diagrams.
- σ_2^0 is, finally, the total exact result without any approximation.

We have taken $M_Z = 93 \text{ GeV}$, $\sin^2 \theta_W = 0.230$, $E_\gamma > 1 \text{ GeV}$ and $|\cos \theta_\gamma| < \cos 15^\circ$.

The main conclusions that we can extract from Table 1 are the following :

- The contribution from the diagram with two W propagators is negligibly small in all the energy range of LEP/SLC.
- The GGR calculation is essentially good enough at LEP1/SLC.
- At LEP1/SLC the bulk of the cross-section is due to the Z diagrams. Hence, we will only need to worry about radiative corrections to the Z diagrams.

In fig. 2 we show the W contribution, to be understood as the difference between the results of σ_2 and σ_4 , as a function of the center of mass energy. We see that for energies between M_Z and 110 GeV it lies between 1% and 4%. This confirms our previous assumption. Therefore, in the following we will only compute radiative corrections to the Z diagrams.

2.2 One-loop corrections

As customarily done, we divide the one-loop QED corrections into virtual ones and real ones. The first are due to the interference between the tree level diagrams and the diagrams shown in fig. 3. The calculation has been performed using the reduction techniques developed by Passarino and Veltman ([14]). The result can be found in [4]-[6].

Concerning the real photon corrections, we have to distinguish between the soft photon corrections and the hard photon ones. The first ones come from the diagrams of fig. 4 where one of the photons has an energy smaller than $x_0 E_s$, with x_0 an arbitrary (small : around $10^{-2} - 10^{-3}$) number. These corrections are computed analytically neglecting the soft photon kinematical effects and combined with the virtual photon ones to give an infrared finite cross-section depending on x_0 .

The hard photon contribution cancels this dependence on x_0 by including events with may look like single photon events because of any of the following reasons:

- One of the photons have fractional energy larger than the minimum detectable photon energy, x_D , and the other one has $x_0 < x < x_D$.

Figure 1: Feynman diagrams of the process $e^+e^- \rightarrow \nu\bar{\nu}\gamma$ in the Born approximation.

Figure 2: Contribution of the W diagrams squared and of their interference with the Z diagrams. $M_Z = 92 \text{ GeV}$

Figure 3: Diagrams contributing to the QED virtual corrections.

Figure 4: Real photon diagrams contributing to the one loop QED radiative corrections.

- For this second photon, $x > x_D$ but it goes too low angle and, then, it cannot be detected.
- The two photons are almost collinear and are seen as a single object.

We have computed this cross-section with the spinor techniques explained, for instance, in ref. [6]. This has proved to be a powerful tool when dealing with calculations involving a lot of diagrams or a lot of particles.

Finally, we have also considered the more important weak corrections, namely (in the 'on-shell' renormalization scheme ([15])) the Z^0 self energy. This correction is taken into account just performing the following substitution wherever appears the Z propagator :

$$\frac{1}{q^2 - M_Z^2 + i\Gamma_Z M_Z} \rightarrow \frac{1}{q^2 - M_Z^2 + \text{Re}\Pi_{ZZ}(q^2) + i\Gamma_Z M_Z} \quad (1)$$

where Π_{ZZ} is the Z^0 self-energy given by the diagram of figure 5. A discussion of the treatment of the propagator corrections is given in section 2.4. At the moment, we just apply (1) to every propagator appearing both in the soft and hard parts. We neglect the non-leading weak corrections ([16]).

The total one-loop corrected cross-section can be expressed as :

$$\begin{aligned} \sigma_{\nu\bar{\nu}\gamma}^{\text{ew}} &= \sigma_{V+S}^{\text{ew}} + \sigma_{\nu\bar{\nu}\gamma}^{\text{ew}} \\ &\equiv \sigma_{\nu\bar{\nu}\gamma}^{\text{ew}}(1 + \delta^V + \delta^S) + \sigma_{\nu\bar{\nu}\gamma}^{\text{ew}} \end{aligned} \quad (2)$$

where the ew superscript means that the Z^0 propagator has been corrected.

For the results given in this section we have used the following set of cuts to define the single photon event :

- Minimum photon energy : $E_D \equiv x_D E_b = 1 \text{ GeV}$.
- Minimum detection angle : $\theta_D = 15^\circ$.
- Veto angle : $\theta_V = 2.4^\circ$.
- Two photon separation angle : $\theta_{\text{RES}} = 1^\circ$.

and the following set of input parameters for the calculation :

- The fine structure function : $\alpha \simeq 1/137$.
- The muon decay constant : $G_\mu = 1.16637 \times 10^{-5} \text{ GeV}^{-2}$.
- The Z^0 mass : $M_Z = 93 \text{ GeV}$.
- The Higgs boson mass : $M_H = 100 \text{ GeV}$.
- The top quark mass : $m_t = 35 \text{ GeV}$.

We use the value of G_μ to fix the value of M_W through the one-loop relation

$$\frac{G_\mu}{\sqrt{2}} = \frac{\pi\alpha}{2s_W^2 M_W^2 (1 - \Delta r)} \quad (3)$$

Table 2: Effect of the one-loop corrections. See in the main text the definitions of the quantities appearing in the table.

$\sqrt{s}(\text{GeV})$	93	100
$\sigma^0(\text{pb})$	77.9 ± 0.7	94.2 ± 0.5
δ^{V+S}	-62.5%	-57.3%
δ_{Z^0}	20.0%	15.3%
$\sigma_{V+S}^{\text{tree}}(\text{pb})$	44.8 ± 0.4	54.6 ± 0.3
$\sigma_{V+S}^{\text{QED}}(\text{pb})$	15.5 ± 0.3	42.8 ± 0.3
$\sigma_{V+S}^{\text{tree}}(\gamma)(\text{pb})$	60.2 ± 0.4	97.5 ± 0.4

We need the top and Higgs masses because they enter the calculations of the Z^0 self energy and of Δr . Finally, one has to choose a mass for the heavy charged leptons which are in the same $SU(2)$ doublet that the extra neutrinos, since they enter the calculation of the Z^0 self energy. We take for all of them $M_{\text{HEAVY}} = 100 \text{ GeV}$. However, the numbers presented here are for $N_\nu = 3$. Furthermore, we have taken the soft-hard separation limit as $k_0 \equiv x_0 E_b = 0.2 \text{ GeV}$.

Table 2 contains the results for the different corrections at two center of mass energies. In this table, we have defined $\delta_{Z^0} \equiv \frac{\sigma_{V+S}^{\text{QED}} - \sigma^0}{\sigma^0}$. We can see there that the QED corrections are extremely large at the Z peak: adding to δ^{V+S} the hard correction $\delta^H \equiv \frac{\sigma_{V+S}^{\text{tree}}}{\sigma^0}$, we find a total QED correction $\delta^{\text{QED}} \equiv \delta^{V+S} + \delta^H = -42.6\%$. Also the Z^0 propagator corrections are sizable. Comparing the first row (tree level calculation) with the last one (final result after the one-loop calculation) we see a huge effect when running at the Z^0 peak ($\sqrt{s} = 93 \text{ GeV}$): around -23% . This is almost the difference in the cross-section when adding a new generation (around 25%). At the higher energy, $\sqrt{s} = 100 \text{ GeV}$, the total effect in the integrated cross-section is rather small ($\sim 3.5\%$). However, it is much more important in the photon distributions. We have shown in fig. 6 b), c) and d) a comparison between the distributions for the photon energy, the cosine of the polar angle and the transverse momentum obtained with the tree level and the one-loop calculations for the two center of mass energies studied before. We can see that the energy distribution at $\sqrt{s} = 100 \text{ GeV}$ is greatly modified by the one-loop corrections, making the spectrum to be softer as a consequence of having the possibility of radiating an extra photon. Then, if an experimental cut is made in the photon energy in order to reduce the background, the actual cross-section will change a lot. So, even at this higher energy, radiative corrections are important. A similar effect can be seen in the transverse momentum distribution.

2.3 Higher order QED corrections

The need for higher order corrections can be summarized in the following points which can be read off the one-loop results:

- The QED initial state corrections are really large in absolute value, around -40% . Therefore, we need higher order initial state QED corrections.

Figure 5: Z^0 self energy diagram.

Figure 6: Comparison between tree level and one-loop results:

- Integrated cross-section.
 - Differential cross-section with respect to the photon energy.
 - Differential cross-section with respect to the photon polar angle.
 - Differential cross-section with respect to the photon transverse momentum.
- Solid line : corrected cross-section.
Dashed line : lowest order.

• The oblique corrections are large, around 15% : we need also higher order oblique corrections.

• From the previous two points we can extract that the effect of considering one-loop QED corrections combined with one-loop oblique corrections will be around -6%. Then we have to consider carefully the QED-oblique interplay.

Let's comment briefly why initial state QED corrections are large around the Z peak. We can write the cross-section for a general $e^+e^- \rightarrow f\bar{f}$ ($f \neq e$) process including one-loop initial state QED corrections as :

$$\sigma_1(s) = \sigma_0(s)(1 + \delta_1 + \beta \ln x_0) + \int_{x_0}^1 \beta \left(\frac{1}{x} - 1 + \frac{x}{2} \right) \sigma_0(s') dx \quad (4)$$

where the first part is the soft cross-section and the rest the hard one; x is the photon energy in units of the beam energy and $s' = s(1-x)$; x_0 is the soft-hard separation cut; δ_1 is the part of the soft radiative corrections which is independent of x_0 , its value being

$$\delta_1 = -\frac{\alpha}{\pi} \left(\frac{3}{2} \ln \frac{s}{m_e^2} + \frac{\pi^2}{3} - 2 \right) = \frac{3}{4} \beta + \frac{\alpha}{\pi} \left(\frac{\pi^2}{3} - \frac{1}{2} \right); \quad (5)$$

β is defined as

$$\beta \equiv \frac{2\alpha}{\pi} \left(\ln \frac{s}{m_e^2} - 1 \right) \quad (6)$$

and it can be regarded as an effective coupling constant for bremsstrahlung. At LEP energies it is large : $\beta \approx 0.11$. This is one of the reasons causing the QED radiative corrections being so large at the Z_0 peak.

The other major reason has to do with the fact that we are close to a (relatively) narrow resonance as the Z^0 one. Since σ_0 is falling rapidly when far from the pole, the upper limit of the integral in (4) is effectively cut to

$$x_M \sim \frac{\Gamma_Z/2}{E_b} \quad (7)$$

where Γ_Z is the Z^0 width and E_b , the beam energy. And then we can approximate

$$\sigma_1(s) \sim \sigma_0(s)(1 + \delta_1 + \beta \ln \Gamma_Z/M_Z) \quad (8)$$

Putting some standard values for the Z width and mass, we find $\beta \ln \Gamma_Z/M_Z \sim -40\%$!

In the case of final state radiation or in absence of any resonance, the restriction in x_M would disappear and we would end up with no large logs.

There is still another, more technical reason for needing higher order QED corrections. We know that the soft-hard limit, x_0 , has no physical content and that any physical result from a Monte-Carlo event generator has to be independent of the value of x_0 in a reasonable range. Since in the soft part we are neglecting the photon energy and the cross-section is a steep function of s , we cannot choose a large value for x_0 if we want a reasonable accuracy. For instance, at the Z peak and for $x_0 = 0.01$ (a standard value) we find

$$\frac{\sigma_0(M_Z^2) - \sigma_0(M_Z^2(1 - 0.01))}{\sigma_0(M_Z^2)} \sim 10\% \quad (9)$$

while for $x_0 = 0.001$ the difference is roughly one per mil. Then, we can conclude that we need to choose $x_0 \sim 0.001$ or lower. Actually, this can be taken into account easily modifying the soft-photon integral. However, choosing a large value for x_0 also affects the accuracy with which some distributions are reproduced. This is specially true for the acollinearity distribution.

On the other hand, if we look at equation (4) we can see that putting x_0 too small would cause the soft part to become negative. Since we have to interpret it as a probability density function, this is unacceptable. We have to put x_0 around 0.001 or higher. So, clearly, we have a problem also here, since there is almost no window left for x_0 . We will see how this problem is also solved when the exponentiation is performed.

After seeing that one-loop corrections are not enough, the first idea has to be to try and compute two loop initial state QED corrections. This has been done in ref. [8] with the following result :

$$\begin{aligned} \sigma_2(s) = & \sigma_0(s)(1 + \delta_1 + \delta_2 + \beta \ln x_0 + \frac{1}{2} \beta^2 \ln^2 x_0) \\ & + \int_{x_0}^1 dx \left[\beta \left(\frac{1}{x} - 1 + \frac{x}{2} \right) (1 + \delta_1 + \beta \ln x) + \delta_2^H \right] \sigma_0(s') \end{aligned} \quad (10)$$

where δ_2 is the $O(\alpha^2)$ correction independent of x_0 and δ_2^H is the $O(\alpha^2)$ non-leading hard correction, the leading one being the one which is proportional to $1/x$. Now x is defined in such a way that $s' = s(1-x)$ is the q^2 in the boson propagator. The explicit expressions for δ_2 and δ_2^H can be found in [8].

If we look carefully at this last equation, it will become clear that it looks like an $O(\alpha^2)$ expansion of the following expression :

$$\begin{aligned} \sigma_{\text{obs}}(s) = & \sigma_0(s)(1 + \delta_1 + \delta_2)x_0^\beta \\ & + \int_{x_0}^1 dx \left[\beta x^{\beta-1} (1 + \delta_1 + \delta_2) - \frac{\beta}{2} (2-x)(1 + \delta_1 + \beta \ln x) + \delta_2^H \right] \sigma_0(s') \end{aligned} \quad (11)$$

which we will take as our exponentiated formula.

With this treatment we have solved the two problems that we had mentioned before :

- The large corrections coming from the x_0 -dependent terms in the initial state corrections have been summed up to all orders. Therefore, the overall precision of the calculation for the initial state QED corrections is around one per mil, the expected size of δ_3 .
- The problem with the value of x_0 has also disappeared. Now we can choose x_0 much smaller than 0.001 without any problem, since the soft part never becomes negative. Actually, we could choose $x_0 = 0$, and this is indeed done in the Structure Functions approach, which will be presented later.

This approach can be implemented very easily in any one-loop Monte Carlo calculation. We have done it following these steps (it has to be noted that, in our problem, σ_0 will be the cross-section for radiative neutrino production, i.e., already with one hard photon) :

- In the virtual + soft photon corrections part : substitute $1 + \delta_1 + \beta \ln x_0$ by $(1 + \delta_1)x_0^\beta$.

- In the hard photon corrections part : substitute the term containing the piece β^2 by $\beta^2 x^{\beta^2}(1+\delta_1)$. Since it is not easy to isolate explicitly from our hard photon cross-section the $\frac{1}{2}$ term, we have just multiplied everything by $x^{\beta^2}(1+\delta_1)$. This makes us to include also the corrections to the hardest piece in an exponentiated way which may not be rigorous. However, we do know that the β^2 piece is taken into account correctly, so the difference with the rigorous treatment will be at most in the β^3 term. But in all the calculation we have neglected those terms, which could amount around 0.1%.

Since we are mainly interested in the total cross-section of the process as a measurement of the number of neutrino generations, this approximation will likely be enough. However, if we want to simulate our process accurately in order to take into account detector effects, for example, we have to worry about how well do we reproduce the differential cross-sections. In this approach we will neglect all the effects of photons carrying a fractional energy smaller than x_0 . This is not a serious problem since now, with the exponentiated formula (11), we can choose x_0 as small as we want, so that the effect will be really negligible. When the energy radiated is above this cut-off, we assume that is carried away by just one photon (besides the visible one). This is clearly an approximation, since what we are doing in (11) is adding up the contribution from the radiation of an infinite number of photons. However, when the radiated energy is large enough, it tends to be taken by just one hard photon or at most two. So our approximation is very good in this case. It has been shown in ref. [12] that the probability of having three photons or more of a noticeable energy is very small around the Z peak. Their effects in the total cross-section are accounted for, but their effects in the distributions are taken into account in an approximate way. This effect cannot produce differences in the observable cross-section at a level larger than some parts in 10^{-3} . Results in section 2.5 will confirm this statement.

The second approach follows the ideas of Kuraev and Fadin and Nicrosini and Trentadue ([9]) of using the Structure Functions formalism. In this approach the initial state electrons are given a structure of electrons and photons. If $D_e(z, s)$ is the probability of finding an electron with fractional momentum z at center of mass energy s , we have

$$\sigma(s) = \int_{x_0}^1 dz_1 \int_{x_0}^1 dz_2 D_e(z_1, s) D_e(z_2, s) \sigma_0(s z_1 z_2) \quad (12)$$

The lower limit is the minimum fractional energy needed to create a pair of final state fermions.

The function $D_e(z, s)$ satisfies the Altarelli-Parisi equation ([17]). It can be solved to all order in the soft photon limit (large z) ([18]). And then the hard photon corrections up to order β^2 can be added by looking at the explicit iterative second order solution. Finally, the second order virtual corrections can be added by comparing with a complete second order calculation like that of [8]. The final result reads ([9]) :

$$D_e(z, s) = \frac{1}{2} \beta (1-z)^{\frac{\beta}{2}-1} \Delta' - \frac{1}{4} (1+z) \beta + \frac{1}{32} \beta^2 \left[(1+z) [3 \ln z - 4 \ln(1-z)] - \frac{4}{1-z} \ln z - 5 - z \right] \quad (13)$$

where Δ' includes the virtual corrections up to $O(\alpha^2)$: $\Delta' = 1 + \frac{\beta}{2} + \dots$. This leads to the following expression for the cross-section

$$\begin{aligned} \sigma(s) &= \int_0^1 dx H(x, s) \sigma_0(s(1-x)) \\ H(x, s) &= \Delta \cdot \beta x^{\beta-1} - \frac{1}{2} \beta (2-x) + \frac{1}{8} \beta^2 \{ (2-x) [3 \ln(1-x) - 4 \ln x] \\ &\quad - 4 \frac{\ln(1-x)}{x} - 6 + x \} + O(\beta^3) \end{aligned} \quad (14)$$

where $\Delta = \Delta'^2 - \frac{\beta^2 x^2}{24}$. Comparing the result obtained for $H(x, s)$ with the integrand in the hard part of (11), we see that they are very similar, so that we would expect the two methods to give very similar results. We will see in the following that this is indeed the case.

We will use the expression with the Structure Functions (12) for the calculation of the total cross-section. However when the total amount of energy radiated is larger than a certain x_0 cut, we will use only for the differential cross-section the exact hard cross-section computed previously corrected with the replacement explained before for the $\frac{1}{2}$ piece. In this way, we are sure that we get the right answer for both the total cross-section and the main distributions. Let's see how we do it in practice :

- First, we generate the fractional energy of the electrons in the collision z_1 and z_2 according to an approximated distribution function $\bar{D}_e(z, s) = \frac{\beta}{2} (1-x)^{\beta/2-1}$, and compute the new center of mass energy for the annihilation $s' = s z_1 z_2$.
- Then we generate the phase-space variables for the two neutrinos and the visible photon according to the differential tree level cross-section at center of mass energy s' and boost them to the lab reference frame.
- Next, we compute the weight of the event as

$$W = \frac{D_e(z_1, s) D_e(z_2, s)}{\bar{D}_e(z_1, s) \bar{D}_e(z_2, s)} \quad (15)$$

Computing the mean of those weights and multiplying by the integral of the approximant we will find the total cross-section and with the help of a rejection algorithm we will have a sample of unweighted events.

- Once we have an unweighted event, we compute the amount of energy taken by the non-visible photons : $x = 1 - z_1 z_2$. If $x < x_0$, we are through and go to next event; if $x > x_0$, we throw away this event and, instead, we generate an unweighted event according to the modified hard differential cross-section. Then, we go to next event.

This second implementation is supposed to give more accurate results than the first one for the distributions. As mentioned before, with the Structure Functions approach we have more kinematical information than with the 'inductive' approach : we know how the radiated energy is shared between the two initial state particles. Furthermore, all the effects of collinear radiation are taken into account in a precise way. With our approach of taking the exact second order matrix element for the hard photon radiation we are sure that we also take into account transverse radiation in a sensible way. For the total cross-sections, they are supposed to give equivalent results. Actually, we will see that this is indeed the case.

Table 3: Comparison between one-loop and exponentiated cross-sections.

$\sqrt{s}(\text{GeV})$	92	94	100
$\sigma^1(\text{pb})$	63.7 ± 0.2	175.4 ± 0.6	96.7 ± 0.4
$\sigma^1(\text{pb})$	67.5 ± 0.3	184.5 ± 0.7	95.6 ± 0.5
$\sigma^{SF}(\text{pb})$	67.6 ± 0.4	185.1 ± 1.3	94.8 ± 0.7

Now we are going to present the results of the contribution of the higher order QED corrections. Throughout this section we are going to use the following set of cuts to define a single photon event :

- Minimum energy : $E_D \equiv x_D E_b = 1 \text{ GeV}$.
- Minimum polar angle : $\theta_D = 15^\circ$.
- veto angle : $\theta_V = \theta_D = 15^\circ$.
- Two photon separation angle : $\theta_{RES} = 1^\circ$.

And the following input parameters for the calculation :

- $M_Z = 92 \text{ GeV}$, $M_H = 100 \text{ GeV}$, $m_t = 60 \text{ GeV}$, $M_{HEAVY} = 100 \text{ GeV}$.

And the standard values for α and G_μ . M_{HEAVY} is the mass of the leptons which are in the doublet with the extra neutrinos. It enters the calculation of the self energy of the Z , for instance. We have taken the soft-hard cut-off as $k_0 \equiv x_0 E_b = 0.2 \text{ GeV}$ for the pure one-loop calculation and 0.002 GeV for the other ones. Of course, the result is independent of the value chosen.

First of all in Table 3 we show a comparison between the pure one-loop result for the integrated cross-section (σ^1), the cross-section obtained after adding the higher order QED corrections via the implementation of the approach of Berends et al. (σ^1), and after adding the higher order QED corrections with the Structure Functions approach (σ^{SF}). In all the cases the only weak correction included is the modification of the Z^0 propagator as indicated by equation (1).

The main conclusions from this table are two : firstly, there is very good agreement between the two ways we have tried for including higher order QED corrections in this region. The results are always compatible and the differences of the central values are between 0.1 and 0.8%. On the other hand, the corrections with respect to the pure $O(\alpha)$ calculation are sizable, especially at the pole. They range from +0.0% at $\sqrt{s} = 92 \text{ GeV}$ to -1.1% at $\sqrt{s} = 100 \text{ GeV}$. Comparing with table 2 where a comparison between tree level and one-loop results is made, we realize that higher order corrections go in the opposite way the order α ones go. They tend to compensate the overestimation of the radiative effects done by the first order corrections.

2.4 Higher order oblique corrections

We have chosen to use the 'star' scheme developed by Kennedy and Lynn ([10]) to include in our calculation the oblique corrections¹. Here we will give just a rough idea of the procedure. The details can be found in ref. [6].

We can write a general four-fermion neutral current process matrix element with the one-loop oblique corrections included through Dyson equations (which means that the leading-log terms are included to all orders) as :

$$M_{NC} = e_0^2 \frac{QQ'}{q^2 + \Pi_{AA}} + \frac{e_0^2}{s_0^2 c_0^2} \left[\frac{I_3 - Q(s_0^2 - s_0 c_0 \frac{\Pi_{ZA}}{q^2 + \Pi_{AA}})}{q^2 - \frac{q^2}{4\sqrt{2}e_0^2 G_\mu \rho_0}} + \text{Re} \Pi_{ZZ} + i \text{Im} \Pi_{ZZ} \right] \left[I_3 - Q'(s_0^2 - s_0 c_0 \frac{\Pi_{ZA}}{q^2 + \Pi_{AA}}) \right] \quad (16)$$

Now defining a set of finite functions (we do not write the imaginary parts)

$$\begin{aligned} c_*^2(q^2) &= \frac{e_0^2}{1 - \frac{\Pi_{AA}(q^2)}{q^2}} \\ s_*^2(q^2) &= \frac{s_0^2 - s_0 c_0 \frac{\Pi_{ZA}(q^2)}{q^2 + \Pi_{AA}(q^2)}}{s_0^2 c_0^2} \\ \frac{1}{4\sqrt{2}G_\mu} (q^2) &= \frac{1}{4\sqrt{2}G_\mu} - \frac{s_0}{c_0} \frac{\Pi_{WW}(q^2) + \frac{s_0 c_0}{e_0} \Pi_{ZA}(q^2) + \frac{s_0^2}{e_0^2} \Pi_{AA}(q^2)}{q^2} \\ \frac{1}{\rho_0} (q^2) &= \frac{1}{\rho_0} - \frac{1}{4\sqrt{2}G_\mu} \left(\Pi_{ZZ}(q^2) - \Pi_{WW}(q^2) \right) \end{aligned} \quad (17)$$

we obtain :

$$M_{NC} = \frac{c_*^2(q^2)QQ'}{q^2} + \frac{e_*^2}{s_*^2 c_*^2} \frac{[I_3 - Q s_*^2(q^2)] [I_3 - Q' s_*^2(q^2)]}{q^2 - \frac{q^2}{4\sqrt{2}e_*^2 G_\mu \rho_*}} \quad (18)$$

We have reabsorbed all the oblique corrections in four finite, universal functions using the Dyson equations. This treatment has some advantages :

- The matrix element looks like the tree level one.
- All the leading-log terms of the oblique corrections are summed up to all orders and the corrections appear automatically in the Z width, for instance. The 'starred' functions being universal, the running coupling constants which appear in the numerator of the neutral current matrix element are the same that appear in the definition of Γ_Z^* and, thus, they cancel at the pole showing that the total cross-section is insensitive to the oblique corrections. Other schemes for computing radiative corrections would have obscured this fact.
- We can now take the new matrix element as an effective Born amplitude and apply to it QED corrections. In this way, we assure that the oblique corrections are not

¹ Recently, there has been some discussion about the gauge-invariance of the 'star' treatment([19]). The main conclusions seem to be that the treatment is gauge-dependent in a formal sense, but that this fact is completely irrelevant numerically, because the pieces missing to achieve gauge-invariance are very small in the 't'Hooft-Feynman gauge in which the authors of ref. [10] work. For a more detailed discussion see, for instance, [6].

overestimated. This would happen if QED corrections were not factorized with respect to the oblique ones. Since, as we have seen, QED corrections are extremely large near the Z pole, this would have implied a large error in the oblique corrections.

One can also absorb the non-abelian vertex corrections (which are also universal) into the 'starred' functions ([6]). Then we will have also these corrections factorized with respect to the QED ones. However, the effect is almost negligible, as we will show later on.

Another important point concerning higher order oblique corrections is that of the corrections to $Im\Pi_{ZZ}$ around the pole, as pointed out for the first time by Wetzel ([20]). If we look at the expression for the matrix element for the Z^0 current, we see that at the Z peak, $q^2 = M_Z^2$, we just have

$$M^{NC} \sim \frac{\alpha A}{iM_Z \Gamma_Z^2(M_Z^2)} \quad (19)$$

where the A in the numerator means something without factors of α in the first approximation. Then the whole numerator is of order α . Since Γ_Z is also an $O(\alpha)$ quantity, the matrix element (19) is of order 1. This means that an order α correction to Γ_Z , which is equivalent to an order α^2 contribution to $Im\Pi_{ZZ}$, is just an order α correction to the matrix element, and, hence, it is probably needed to reach the accuracy we are seeking.

The fact that in front of the 'starred' width we have the 'starred' coupling constants means that we are including the oblique corrections also in the width, as we have pointed out previously. However, these are not the only corrections that have to be taken into account. There are also the direct corrections shown in the diagram of fig. 7. This kind of diagrams have not been computed at the moment. Nevertheless, we do not need to know their contribution at any q^2 , but only when $q^2 \simeq M_Z^2$. We know that exactly at the Z pole the following equations hold

$$\alpha Im\Pi_{ZZ}^{(1)}(M_Z^2) = M_Z \Gamma_Z^{(0)} \quad (20)$$

$$\alpha^2 Im\Pi_{ZZ}^{(2)}(M_Z^2) = M_Z (\Gamma_Z^{(1)} - \Gamma_Z^{(0)}) \quad (21)$$

where $\alpha Im\Pi_{ZZ}^{(1)}$ is the one-loop self-energy, $\alpha^2 Im\Pi_{ZZ}^{(2)}$ is the second order contribution to this self energy, $\Gamma_Z^{(0)}$ is the tree level width and $\Gamma_Z^{(1)}$ is the one-loop corrected Z^0 width, computed in ref. [21]. It includes the complete electroweak corrections together with the QCD corrections to the hadronic branching ratios. This last effect turns out to be the one which is numerically the most important.

We also know that, to good accuracy, the behaviour of the imaginary part of the Z self energy around the peak is

$$Im\Pi_{ZZ}(s) \propto s. \quad (22)$$

Then, a good approximation for the two-loop contributions to $Im\Pi_{ZZ}$ will be

$$Im\Pi_{ZZ}(s) \simeq \alpha Im\Pi_{ZZ}^{(1)}(s) + \frac{s}{M_Z^2} M_Z (\Gamma_Z^{(1)} - \Gamma_Z^{(0)}) \quad (23)$$

This will be a very good approximation near the peak. Far away from it, it will not be so good, but then we do not need such a precision for the imaginary part, since the real part will no longer be small and will dominate the behaviour of the propagator.

Table 4: Comparison between the cross-sections without 'star' scheme (σ^I), with 'star' scheme unmodified (σ^{I*}) and with 'star' scheme modified by us ($\hat{\sigma}^I$).

$\sqrt{s}(\text{GeV})$	92	94	100
$\sigma^I(\text{pb})$	67.5 ± 0.3	184.5 ± 0.7	95.6 ± 0.5
$\sigma^{I*}(\text{pb})$	66.4 ± 0.3	175.9 ± 0.8	93.3 ± 0.5
$\hat{\sigma}^I(\text{pb})$	66.0 ± 0.3	175.6 ± 0.7	92.4 ± 0.5

Table 5: Percentual total corrections due to higher order effects.

$\sqrt{s}(\text{GeV})$	92	94	100
$\delta(\%)$	3.6	0.1	-4.4

Now, we are going to discuss the effect of putting the weak corrections using the 'star' scheme and the effect of the modification we have made to it. We see in Table 4 the cross-section with exponentiation, σ^I from Table 3, and the cross-sections with inductive exponentiation and 'star' scheme without (σ^{I*}) and with our modifications to include the non-abelian vertices ($\hat{\sigma}^I$). Note that since we are including in the Z width the QCD corrections, we need to fix the value of the strong coupling constant, α_s . We have taken $\alpha_s(M_Z^2) = 0.12$.

We see that the largest effect due to the inclusion of the 'star' treatment is at $\sqrt{s} = 94$ GeV, i.e., when we recover the Z peak due to the emission of the visible photon. This effect is important: around -4.6%. Below and above this energy the correction is decreasing up to around -2%. This kind of behaviour is reasonable since most of the effect is due to the inclusion of the one-loop corrections to the width and this is important when the Z^0 is on-shell. The correction due to the inclusion of the non-abelian vertices is almost negligible: it is always less than 1%.

Combining the higher order QED with the higher order weak corrections (with the 'star' scheme modified), we can find the total correction with respect to the one-loop cross-section. It is shown in Table 5 and in figure 8 a).

The QED correction dominates at the peak and hence the total correction is positive. Slightly above the peak, the two corrections are almost of the same size and opposite signs, so that the global correction is almost zero. We cannot see any particular reason for this and, therefore, we think it is completely casual. Finally, above the peak both corrections are negative giving a large total effect.

In fig. 8 b), c) and d) we can see the effect of all the higher order corrections altogether. As we see they do not modify very much the photon distributions, although some effects can be seen in the photon and transverse momentum distributions.

Table 6: Integrated cross-sections (pb) with at least one photon with $E_\gamma > 0.5$ GeV and $15^\circ < \theta_\gamma < 165^\circ$

$\sqrt{s}$ (GeV)	90	92	94	99	110
KORL03	38.9 ± 0.2	143.7 ± 1.0	245.1 ± 1.2	117.0 ± 0.6	46.3 ± 0.3
NNGG03	38.4 ± 0.1	144.9 ± 0.2	245.4 ± 0.5	117.6 ± 0.1	46.0 ± 0.1

2.5 Comparison with other calculations

A thorough comparison has been performed in ref. [12] between our Monte Carlo calculation with the exponentiation à la Berends et al. and the version of KORL03 ([11]) modified by Colas, Mirabito and Wags for the neutrino counting process. All the numbers in this section are taken from ref. [12].

The program in ref. [12] includes multiphoton radiation following the Yennie, Frautschi, Suura approach ([22]) and the oblique radiative corrections have been computed by Stuart ([23]). The neutrino version adds exactly the contributions of the one W diagrams and neglects the two W one.

The results have been obtained with the following set of parameters : $M_Z = 92$ GeV, $m_{top} = 60$ GeV, $M_H = 100$ GeV and $N_c = 3$. Furthermore only the events with at least one photon with $E_\gamma > 0.5$ GeV and $15^\circ < \theta_\gamma < 165^\circ$ are considered.

The agreement in the integrated cross-section has been found to be very good, better than 1% in the region of interest for the neutrino counting experiment, i.e., at and slightly above the Z peak. This is shown in Table 6 and in fig. 9 a) and b).

It has to be noted the fact that KORL03 has larger errors. The reason is that to have a large sample of events a lot of CPU time is needed, since most of the events will not have any hard photon. This is not the case with our program, which has always a photon from the tree level.

Also the main distributions have been compared in ref. [12] and the agreement has been found satisfactory. In fig. 10 a), b) and c) we show the distribution of energy, polar angle (defined in such a way that it is flat for a bremsstrahlung photon) and transverse momentum for the hardest photon at a center of mass energy of 99 GeV. No noticeable differences between the two programs can be seen. The same has been checked at $\sqrt{s} = M_Z = 92$ GeV and at $\sqrt{s} = 110$ GeV.

3 The process $e^+e^- \rightarrow e^+e^-\gamma$

3.1 Lowest order calculation.

In this section we will just review some results found using the calculation of ref. [24]. This is an exact tree level calculation performed with the helicity amplitude method.

We will define a hypothetical experimental set-up with the following conditions :

- veto angle for electrons : $\theta_V = 2.5^\circ$,
- minimum photon energy : $x_D E_b = 1$ GeV, and

- tagging angle for photons : $\theta_D = 15^\circ$.

In fig. 11 a) we can see the integrated cross-section as a function of the center of mass energy compared with that obtained from the tree-level calculation of $e^+e^- \rightarrow \nu\bar{\nu}\gamma$. We have chosen $M_Z = 92$ GeV, $\sin^2\theta_W = 0.23$ and $N_c = 3$. The cross-section for the single photon final state coming from $e^+e^-\gamma$ is totally dominated by the diagrams with t-channel photons, and, hence, we do not see any enhancement of the cross-section near the Z pole. As we can see, in principle, the background is larger than the signal. However, if we look at the photon distributions above the Z peak, at $\sqrt{s} = 100$ GeV (fig. 11 b), 11 c) and 11 d)), we see that they are very different and that a clean selection of events can be made cutting on the transverse momentum (fig. 11 d)).

After obtaining the result for the tree level cross-section, we may worry about higher order corrections. It is clear that since this is a background process which can be mostly eliminated, we do not need as much precision as in the calculation of the signal cross-section. However, there are two reasons which justify the need for computing one-loop corrections to this process :

- The optimum neutrino counting experiment would be performed at an energy some GeV's above the Z peak. However, since most of the luminosity will be taken at the peak, certainly the experiment will also be done there. Then it is very difficult in this case to remove the background by means of kinematical cuts since the signal spectrum is also very soft and a cut on p_T would also cut most of the signal. Thus a background subtraction is needed and a precise calculation of the background cross-section has to be performed.

- The second reason is that a calculation of the one-loop corrections to $e^+e^- \rightarrow e^+e^-\gamma$ was indeed done, and the results (presented in the first version of the SLAC-PUB-4121 preprint, which (after some corrections) led to the article of ref. [25]) showed that the higher order correction was twice as large as the tree level cross-section! Since we were puzzled by this result we decided to start with this calculation.

We will begin the presentation of our one-loop calculation ([13]) by showing the contribution of the hard photon corrections.

3.2 Hard photon corrections.

3.2.1 Exact calculation : Helicity Amplitudes method.

The process

$$e^-(p_1) + e^+(p_2) \longrightarrow e^-(p_3) + e^+(p_4) + \gamma(p_5) + \gamma(p_6) \quad (24)$$

is represented to order g^4 by as many as 80 diagrams. However, for the single photon configuration, the only non negligible contributions are those coming from the 10 diagrams shown in figure 12 plus the 10 additional ones obtained by exchanging the photon labels, i.e., all the diagrams containing a photon in t-channel. From now on, we will restrict ourselves to studying the contributions from those diagrams.

Clearly they form a gauge-invariant subset and we can group them in 4 topologically different groups (A,B,C,D in fig. 12) such that the amplitude for each diagram in one class can be obtained from any other belonging to the same class by an overall permutation of

particle labels and helicities (and, occasionally, conjugation). They cannot be obtained with such a simple transformation from any one of the other classes. Therefore, we just have to calculate the matrix element, M , for one diagram of each class. We will use the helicity amplitudes approach. This method has proved to be very useful when dealing with a great number of diagrams and when large cancellations are expected to occur. When masses are to be kept (and this is indeed our case, due to the collinear peaks of the matrix element squared) the most efficient way to face the problem is the numerical implementation of the helicity amplitudes technique, as described in great detail in ref. [26], [27], [28], [29], among others. The amplitude for diagram A-1 reads:

$$-iM_{A1} = e^4 (b_2 p_2 + b_4 p_4)^{-2} ((b_1 p_1 + b_5 p_5 + b_6 p_6)^2 - m_e^2)^{-1} ((b_1 p_1 + b_5 p_5)^2 - m_e^2)^{-1} \quad (25)$$

$$\frac{[\bar{u}(p_3) \gamma^\mu (b_1 p_1 + m_1) + b_5 p_5 + b_6 p_6] \not{f}_6 (b_1 p_1 + m_1) + b_5 p_5] \not{f}_5 u(p_1)}{[\bar{v}(p_2) \gamma_\mu v(p_4)]}$$

with $b_i = 1$ for incoming particles and -1 for outgoing particles and $m_i = m_e$ for e^- and $-m_e$ for e^+ . In this way, the characteristics of incoming (outgoing) and of particles (antiparticles) have been absorbed in b_i and m_i . Hence, for instance, we can substitute every $v(p_i)$ for an $u(p_i)$. Now the rest of diagrams of class A are obtained via:

$$\begin{aligned} -iM_{A2} &= (-iM_{A1})^* (1 \leftrightarrow 3; 2 \leftrightarrow 4; \lambda_5 \leftrightarrow -\lambda_5; \lambda_6 \leftrightarrow -\lambda_6; 5 \leftrightarrow 6) \\ -iM_{A3} &= (-iM_{A1})^* (1 \leftrightarrow 2; 3 \leftrightarrow 4; \lambda_5 \leftrightarrow -\lambda_5; \lambda_6 \leftrightarrow -\lambda_6) \\ -iM_{A4} &= (-iM_{A1}) (1 \leftrightarrow 4; 2 \leftrightarrow 3; 5 \leftrightarrow 6) \end{aligned} \quad (26)$$

and just changing $5 \leftrightarrow 6$ for the diagrams with the photon label exchanged. This procedure is repeated in a similar way for diagrams B, C and D.

Now, introducing the $Z(p_i, \lambda_i)$ functions as

$$Z(p_i, \lambda_i; p_j, \lambda_j; p_k, \lambda_k; p_l, \lambda_l; C_L, C_R, C'_L, C'_R) = [\bar{u}(p_i, \lambda_i) \gamma^\mu (C_L P_L + C_R P_R) u(p_j, \lambda_j)] [\bar{u}(p_k, \lambda_k) \gamma_\mu (C'_L P_L + C'_R P_R) u(p_l, \lambda_l)], \quad (27)$$

with $P_L \equiv \frac{1}{2}(1 \pm \gamma_5)$, and the polarization vector of the photons as

$$e^\mu(p_i, \lambda_i) = N_i \bar{u}(p_i, \lambda_i) \gamma^\mu u(p_i, \lambda_i), \quad (28)$$

being p any four-momentum occurring in the process different from p_5 and p_6 , and N_i the normalization constant, we can express M_{A1} as

$$-iM_{A1} = e^4 (b_2 p_2 + b_4 p_4)^{-2} ((b_1 p_1 + b_5 p_5 + b_6 p_6)^2 - m_e^2)^{-1} ((b_1 p_1 + b_5 p_5)^2 - m_e^2)^{-1} \cdot T_{A1} \quad (29)$$

and

$$T_{A1} = N_5 N_6 \sum_{\lambda_i \lambda_j} (Z(p_2, \lambda_2; p_4, \lambda_4; p_3, \lambda_3; p_1, \lambda_1; 1, 1, 1, 1) \cdot b_1 \cdot$$

$$\begin{aligned} & \cdot (Z(p_1, \lambda_1; p_5, \lambda_5; p_6, \lambda_6; 1, 1, 1, 1) \cdot b_5 \cdot Z_5 \\ & + Z(p_1, \lambda_1; p_1, \lambda_1; p_6, \lambda_6; 1, 1, 1, 1) \cdot b_1 \cdot Z_1) \\ & + Z(p_2, \lambda_2; p_4, \lambda_4; p_3, \lambda_3; p_5, \lambda_5; 1, 1, 1, 1) \cdot b_5 \cdot \end{aligned}$$

$$\begin{aligned} & \cdot (Z(p_5, \lambda_5; p_5, \lambda_5; p_6, \lambda_6; 1, 1, 1, 1) \cdot b_5 \cdot Z_5 \\ & \cdot Z(p_5, \lambda_5; p_1, \lambda_1; p_6, \lambda_6; 1, 1, 1, 1) \cdot b_1 \cdot Z_1) \\ & + Z(p_2, \lambda_2; p_4, \lambda_4; p_3, \lambda_3; p_6, \lambda_6; 1, 1, 1, 1) \cdot b_6 \cdot \\ & \cdot (Z(p_6, \lambda_6; p_5, \lambda_5; p_6, \lambda_6; 1, 1, 1, 1) \cdot b_5 \cdot Z_5 \\ & \cdot Z(p_6, \lambda_6; p_1, \lambda_1; p_6, \lambda_6; 1, 1, 1, 1) \cdot b_1 \cdot Z_1)) \end{aligned} \quad (30)$$

with

$$\begin{aligned} Z_1 &= Z(p_1, \lambda_1; p_1, \lambda_1; p_5, \lambda_5; p_5, \lambda_5; 1, 1, 1, 1), \\ Z_5 &= Z(p_5, \lambda_5; p_1, \lambda_1; p_5, \lambda_5; p_5, \lambda_5; 1, 1, 1, 1). \end{aligned} \quad (31)$$

Given a particular set of four-momenta and helicities, one can evaluate the Z functions (tabulated in App. D) for diagrams A1, B1, C1 and D1, and, with the appropriate permutations and conjugations, the ones of all the remaining diagrams. Finally

$$|\overline{-iM}|^2 = \frac{1}{8} \sum_{\lambda} |\sum_j (-iM_j)|^2 \quad (32)$$

where j and λ run over all diagrams and helicity configurations respectively and the extra factor $1/2$ accounts for the symmetrization of the final state photons.

Since we are evaluating a complete gauge-invariant set of diagrams and the effect of changing the choice of p is just a gauge transformation, we can check our calculation just by evaluating $|\overline{-iM}|^2$ with different choices of p . We see that the result is not affected at all by this choice. With the definition of $Z(p_i, \lambda_i)$ we have introduced, it is a straightforward exercise to include also the diagrams with an s-channel photon or with a Z^0 , but their contributions are negligible in the single photon configuration.

3.2.2 Approximate calculation : Equivalent Photon Approximation.

In ref. [25] a different way has been chosen to compute the hard cross-section. We describe it here because it is the method we have used for the virtual and soft photon corrections. It is also relevant to try to compare the results obtained for the hard cross-section with both methods to justify the use of the approximation for the virtual and soft photon part.

This method is based on the so called Equivalent Photon Approximation (EPA). In short, it consists in considering the photon exchanged between the electron and the positron as quasi-real. Then the cross-section can be written as

$$d^8 \sigma_{e^+ e^- \rightarrow e^+ e^- \gamma \gamma} = d^3 n_{e^+}(p_+) d^3 n_{e^-}(p_-) d^8 \sigma_{\gamma \gamma}(q_+ q_-) \gamma(p) \gamma(q) \quad (33)$$

where the first piece is the radiation spectrum of the positron and the second one is the differential cross-section for double Compton scattering. This kind of factorization is reasonable when $|t| \ll s$ where $t = (p_+ - q_+)^2$ and $s = (p_- + k)^2$, that is the 'mass' of the virtual photon is much smaller than the center of mass energy of the $\gamma\gamma$ collision.

With this approximation, it is clear that only the diagrams in fig. 13 will be considered. Of course, this could be sufficient when considering that the positron is going at very small angle. Then the charged conjugated diagrams are included just by symmetrization. Therefore, no interference between the set of diagrams in fig. 13 and their charged conjugated ones is included. Of course, no s-channel diagram or Z diagram is taken into account. It is clear

that the approximation will not be valid for general Bhabha scattering at large angles. We will see, however, that it is reasonable at very small angles, which is the region of interest for the evaluation of the background to $e^+e^- \rightarrow \nu\bar{\nu}\gamma$.

Expressions for $d^2 n_{e^+ \rightarrow e^+ \gamma}$ can be found in the literature (for instance, in ref. [30]). It can be written as:

$$d^2 n_{e^+ \rightarrow e^+ \gamma} = \frac{2\alpha}{\pi^2} \frac{1}{s} \left[\frac{s}{1-s} + \frac{1}{2} \left(\frac{s}{s} \right)^2 + \frac{m_e^2}{t} \left(\frac{s}{s} \right)^2 \right] \frac{d^3 \vec{q}_+}{2q_+^0} \quad (34)$$

The expression for the cross-section for the double Compton scattering

$$\gamma(k) e^-(p_-) \rightarrow e^-(q_-) \gamma(k) \gamma(k_s) \quad (35)$$

was first computed by Mandl and Skyrme ([31]). The result can be found in [32]. Combining it with the previous result for the radiation spectrum, we find finally the expression for the double radiative Bhabha cross-section with the EPA:

$$d^2 \sigma_{e^+ e^- \rightarrow e^+ e^- \gamma \gamma} = \frac{\alpha^4}{\pi^4 s} \frac{X_{MS}}{m_e^2} \frac{1}{-t} \left(\frac{s^2 + (s-\hat{s})^2}{s^2} + \frac{2m_e^2}{t} \right) d^3 \vec{q}_+ d^3 \vec{q}_- d^3 \vec{k}_s d^3 \vec{k}_\gamma \quad (36)$$

with

$$\begin{aligned} X_{MS} &= 2(ab-c)((a+b)(x+z) - (ab-c) - 8) - 2x(a^2+b^2) - 8c \\ &+ \frac{4x}{AB} \left[(A+B)(x+1) - (aA+bB)(2+z\frac{1-x}{x}) + x^2(1-z) + 2z \right] \\ &- 2\rho(ab+c(1-x)) \\ a &= \sum_{i=1}^3 \frac{1}{\kappa_i} \frac{1}{\kappa'_i}, \quad b = \sum_{i=1}^3 \frac{1}{\kappa'_i}, \quad c = \sum_{i=1}^3 \frac{1}{\kappa_i \kappa'_i} \\ x &= \sum_{i=1}^3 \kappa_i, \quad y = \sum_{i=1}^3 \kappa'_i, \quad z = \sum_{i=1}^3 \kappa_i \kappa'_i \\ A &= \kappa_1 \kappa_2 \kappa_3, \quad B = \kappa'_1 \kappa'_2 \kappa'_3, \quad \rho = \sum_{i=1}^3 \left(\frac{\kappa_i}{\kappa'_i} + \frac{\kappa'_i}{\kappa_i} \right) \\ m_e^2 \kappa_1 &= p_- k, \quad m_e^2 \kappa_2 = p_- k_s, \quad m_e^2 \kappa_3 = -p_- \vec{k}, \\ m_e^2 \kappa'_1 &= -q_- k, \quad m_e^2 \kappa'_2 = -q_- k_s, \quad m_e^2 \kappa'_3 = q_- \vec{k}, \end{aligned} \quad (37)$$

We have compared the results of this method with the exact tree level matrix element squared found in the previous section. The results of the comparison at some fixed phase space points relevant for the single photon configuration show us that this approximation gives essentially the right number. This reflects the fact that the contribution from diagrams C and D is very small and that the interference between the set formed by A1, A2, B1 and the diagrams with the photon labels exchanged, and their charge conjugated diagrams is also negligible for this configuration, as we have explicitly checked with the complete matrix element squared.

3.3 Virtual photon corrections.

Since no calculations of the virtual and soft photon corrections to the radiative Bhabha scattering exist in the literature at the moment, we have followed [25] in using the EPA for these corrections. Concerning the soft photon part, this approximation implies taking into account only the diagrams in figure 13 when one photon has an energy smaller than k_0 , the soft-hard separation parameter. This seems to be a reasonable approach since, as we have mentioned previously, the EPA works very well for the hard photon part and, as we will show shortly, the total α^4 cross-section calculated in this way does not depend on the soft-hard separation parameter.

Concerning the virtual corrections, we follow the EPA approximation and just include the interference of the diagrams of fig. 14 with the ones in lowest order radiating a photon from the same fermionic line (and the same for the charge conjugated ones). These diagrams are sufficient to cancel out all the infrared divergences coming from the soft diagrams taken into account. We do not consider the interference of the diagrams in fig. 13 with their charge conjugated ones. This interference has an infrared divergence, cancelled by the box diagrams, which are also not considered. The remaining finite contribution, as well as the one coming from the rest of neglected terms, however, seems to be small ([25] and references therein).

The cross-section corresponding to the lowest order contribution and the virtual and soft photon corrections can be expressed as

$$d\sigma^{V+S} = (1+\delta) \cdot d\sigma^0, \quad (38)$$

where δ depends on k_0 . We calculate δ from radiative corrections to Compton scattering. The EPA relates these two processes by

$$d^5 \sigma_{e^+ e^- \rightarrow e^+ e^- \gamma^*} = d^3 n_{e^+ \rightarrow e^+ \gamma^*} \cdot d^2 \sigma_{\gamma^* e^- \rightarrow \gamma e^-}. \quad (39)$$

$d^3 n_{e^+ \rightarrow e^+ \gamma^*}$ is the equivalent photon spectrum, and, hence, δ is the same for the two processes. Then, we can write

$$d\sigma_{e^+ e^- \rightarrow e^+ e^- \gamma^*}^{VS} = (1 + \delta_{VS}^{em}) \cdot d\sigma_{e^+ e^- \rightarrow e^+ e^- \gamma}^0 \quad (40)$$

δ_{VS}^{em} is the virtual and soft photon correction to Compton scattering. It has the following expression ([33], [25])

$$\begin{aligned} \delta &= -\frac{\alpha}{\pi U} \left\{ 2(1-2y)U \ln \left(\frac{2k_0}{m_e} \right) + \frac{\pi^2}{6} \left(4 - 3t - \frac{1}{t} - \frac{2}{E^4 t^3} \right) \right. \\ &+ 4(2-U)y^2 - 4y + \frac{3}{2}U + \frac{2}{E^2 t^2} + 4 \left(1 - \frac{1}{2t} \right) \ln^2 E \\ &+ \left(2t + \frac{1}{t} - 2 + \frac{2}{E^4 t^3} \right) Li_2(1 - E^2 t) \\ &+ \left[2 - 5t - \frac{2}{t} + 4y \left(\frac{2}{t} + t - 2 \right) \right] \ln E \\ &- \frac{1}{2} U \ln^2(1-t) - U Li_2(t) \\ &+ \left[1 - \frac{2}{t} - \frac{2}{E^2 t^2} - \frac{1}{1-E^2 t} + 4y \left(t - 1 + \frac{1}{2t} \right) \right] \ln(E^2 t) \left. \right\} \quad (41) \end{aligned}$$

$$E = \frac{\sqrt{s}}{m_e}, \quad t = \frac{1}{2}(1 + \beta_- \cos \theta_k), \quad y = \ln[E \sin(\theta_k/2)], \quad U = t + 1/t$$

and k_0 has to be specified in the γ^*e^- center of mass frame.

3.4 Monte Carlo implementation and phase-space integration.

In this section we are going to describe the phase space variables we have used for the integration, together with the approximants chosen to handle the extremely sharp peaks showing up in the matrix element squared.

For the three body process including virtual and soft contribution, we have used the same phase space and approximants as those used in ref. [24] for the tree level calculation. This is the normal way to proceed since, in principle, virtual and soft photon corrections do not change the peaking structure of the matrix element squared.

In order to integrate the matrix element squared for the hard part over the allowed phase space of the single photon configuration, we have taken as independent variables:

- the final electron solid angle $d\Omega_3$,
- the final positron energy dE_4 and solid angle $d\Omega_4$,
- the missing photon energy dE_5 and solid angle $d\Omega_5$ with respect to either the initial electron or the final.

The choice of these variables is mainly required by the fact that the solid angle for the particles enumerated above can be very restricted due to the veto condition.

Now, the phase space density reads:

$$W_{ps} = \frac{1}{16} \frac{\beta_3^2 E_3 \beta_4 E_4 E_5}{E_{cm} \beta_3 + E_4 (\beta_4 \cos \theta_{34} - \beta_3) + E_5 (\cos \theta_{35} - \beta_3)} \quad (42)$$

and therefore :

$$dR_4 = W_{ps} d\Omega_3 dE_4 d\Omega_4 dE_5 d\Omega_5, \quad (43)$$

where the directions of the solid angles for particles 3, 4 and 5 can be referred to any defined direction.

After a careful study of the peaking structure of the total integrand (which is proportional to the product $|-iM|^2 W_{ps}$) when variables $\Omega_3, E_4, \Omega_4, E_5$ and Ω_5 are utilized, we apply Importance Sampling to eliminate the main peaks. The final integration variables $\eta_i, i = 1 \dots 8$ are defined in the interval $[0, 1]$ and are related to the previous ones in the following way:

$$\begin{aligned} d\eta_1 &= \kappa_1 \frac{d \cos \theta_3}{1 + \cos \theta_3 + \epsilon'} & d\eta_2 &= \kappa_2 d\phi_3 \\ d\eta_3 &= \kappa_3 \frac{dE_4}{(E_6 - E_4)^2} & d\eta_4 &= \kappa_4 \frac{d \cos \theta_4}{1 - \cos \theta_4 + \delta} \\ d\eta_5 &= \kappa_5 \frac{dE_5}{E_5} & d\eta_6 &= \kappa_6 \frac{d \cos \theta_5}{1 + \cos \theta_5 + \epsilon} \\ d\eta_7 &= \kappa_7 & d\eta_8 &= \kappa_8 d\phi_5, \end{aligned} \quad (44)$$

where κ_i are the transformation Jacobians and

$$\epsilon = \frac{2m_e^2}{s}$$

Table 7: Lowest order + virtual + soft photon corrections (σ^{V+S}), hard photon contribution (σ^H) and total $O(\alpha^4)$ cross-section (σ^4) for different values of $k_0 \equiv x_0 E_b$.

k_0 (GeV)	σ^{V+S} (pb)	σ^H (pb)	σ^4 (pb)
10^{-3}	21.8 ± 0.3	12.75 ± 0.11	34.6 ± 0.3
10^{-2}	26.5 ± 0.4	8.39 ± 0.07	34.9 ± 0.4
10^{-1}	31.0 ± 0.5	4.15 ± 0.09	35.1 ± 0.4

$$\begin{aligned} \delta &= \epsilon \frac{(E_b - \max(E_4))^2}{(\max(E_4))^2} \\ \epsilon' &= \epsilon^2, \end{aligned} \quad (45)$$

being actually the direction of particle 5 generated with a probability of 50 % with respect to the initial and final electron direction. For these integration variables the phase space density reads

$$W'_{ps} = W_{ps} \prod \kappa_i \frac{1}{(1 + \cos \theta_3 + \epsilon') (E_b - E_4)^2 (1 - \cos \theta_4 + \delta) E_5 (1 + \cos \theta_5 + \epsilon)} \quad (46)$$

One proof of the fact that the behaviour of the function W'_{ps} compensates the peaking structure of the matrix element squared resulting in a rather smooth integrand is the fact that the generation efficiency (defined as the ratio between the average value of the total integrand and its absolute maximum) is bigger than 16 % even in extremely demanding cut conditions.

3.5 Results.

Following [25], we have defined a hypothetical experimental set-up such that we consider as single photon events those events fulfilling the following conditions :

- detected photon energy greater than $E_D = 0.5 \text{ GeV}$
- detected photon θ angle between $\theta_D = 30^\circ$ and $180^\circ - \theta_D$
- undetected particles angle between 0° and $\theta_V = 15 \text{ mrad}$ or between $\pi - \theta_V$ and π .

Moreover, we take $\sqrt{s} = 94 \text{ GeV}$ and $\alpha = 1/137.036$.

In Table 7 we see the results we obtain for this configuration with the exact matrix element squared for the hard part (as described in section 3.2.1). We obtain very similar numbers with the EPA. Since, as Table 7 shows, the dependence in the soft-hard cutoff is negligible, we can assume that the approximation we use for the soft and virtual corrections is good enough. Table 8 shows the results of the lowest order (α^3) calculation and of our $O(\alpha^4)$ calculation. The result is that the hard photon corrections to $e^-e^+ \rightarrow e^-e^+\gamma$ happen to cancel almost exactly the soft + virtual corrections, giving an overall correction of about 3% for the single photon configuration. The result at $\sqrt{s} = 98 \text{ GeV}$ with another (maybe more realistic) definition of single photon configuration, namely $E_D = 1 \text{ GeV}$, $\theta_D = 15^\circ$, $\theta_V = 2.4^\circ$, confirms this effect (Table 8). It is clear from Table 8 that we do not observe at all the enhancement that was described in the first version of [25] for the $O(\alpha^4)$ cross-section. Instead of this, we

Table 8: Lowest order (σ^3) vs. one-loop corrected (σ^4) cross-sections. See the caption of the previous table.

	σ^3 (pb)	σ^{V+S} (pb)	σ^H (pb)	σ^4 (pb)	$\delta(\%)$
set-up 1	33.6 ± 0.5	26.5 ± 0.4	8.39 ± 0.07	34.9 ± 0.4	3.9
set-up 2	297.7 ± 0.7	208.4 ± 2.0	95.0 ± 0.9	303.4 ± 2.2	2.0

find a rather modest effect of considering α^4 corrections. Moreover, figures 15 a), 15 b) and 15 c) show that the distributions of the main variables of the detected photon (E_γ , $\cos \theta_\gamma$, $p_{T,\gamma}$) almost do not change from α^3 to α^4 calculation. The most important one in the single photon study, $p_{T,\gamma}$, does not change noticeably (fig. 15 c)) and hence a cut on $p_{T,\gamma}$ reduces this background almost completely.

These results are in agreement with the result found in the definitive version of [25]. The differences between the two calculations are around 2%. This is also the kind of difference that we expect between the EPA result for the virtual and soft photon corrections and the exact one, which has not been computed as of now. Then, in our calculation, using the EPA for the virtual and soft part and our exact calculation for the hard photon part, we expect an accuracy of the order of few per cent. This is sufficient for experimental purposes.

4 Summary and conclusions

We have presented a calculation of $e^+e^- \rightarrow \nu\bar{\nu}\gamma$ including :

- Exact tree level calculation.
- Complete $O(\alpha)$ QED corrections to the Z diagrams.
- Leading QED corrections exponentiated.
- Oblique corrections and non-abelian vertices included via 'star' scheme.
- Corrections to $Im\Pi_{ZZ}$.

The calculation has been implemented as a Monte Carlo event generator of unweighted events. The one-loop corrections give a $\sim -25\%$ effect, while higher order corrections altogether amount something around $\pm 5\%$, depending on the center of mass energy.

Results from a comparison with KORL03 done in ref. [12] have been discussed showing a very good agreement (better than 1%) both in cross-section and main distributions in the region of experimental interest. This is a powerful check of both calculations.

We can conclude that the overall precision of our calculation is of the order of 1%, if not better. Hence, it is precise enough for the experimental needs.

Concerning the background process $e^+e^- \rightarrow e^+e^-\gamma$, we have explained the exact calculation of the hard photon corrections using the helicity amplitudes method. Then a comparison with the Equivalent Photon Approximation (EPA) calculation has been performed showing reasonable agreement between them. We then use the EPA for the virtual and soft photon corrections and after a complicated phase space integration, due to the presence of sharp

peaks in the matrix element squared, we arrive at the final result for the one-loop QED corrections. The total correction to the cross-section lies between 2% and 4% depending on the cut conditions and is almost negligible in the photon distributions. Hence, we conclude that there is no need for higher order corrections.

Due to the uncertainty introduced by the EPA we can only claim a total accuracy of few per cent. However, this is also sufficient for the experiments, since radiative Bhabha scattering is a background and most of it (if not all) can be removed by kinematical cuts.

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Figure 7: Example of one loop corrections to $Im\Pi_{ZZ}$.

Figure 8: Comparison between pure one-loop results and the ones obtained including higher order corrections :

- a) Integrated cross-section.
- b) Differential cross-section with respect to the photon energy.
- c) Differential cross-section with respect to the photon polar angle.
- d) Differential cross-section with respect to the photon transverse momentum.

Solid line : Higher order result with inductive exponentiation.
Dashed line : Higher order result with structure functions exponentiation.
Dotted line : One-loop result.

Figure 9: Comparison between our calculation and the one with KORALZ:

- a) Integrated cross-section.
- b) Ratio of cross-sections.

Figure 10: Comparison between our calculation and the one with KORALZ:

- a) E_γ distribution.
b) $\cos\theta_\gamma$ distribution.
c) $E_{T\gamma}$ distribution.

Figure 11: Results of the tree level calculation of radiative Bhabha scattering.

- a) Integrated cross-section.
b) Energy distribution.
c) Angular distribution.
d) Transverse momentum distribution.

Solid line : $e^+e^- \rightarrow e^+e^-\gamma$.
Dashed line : $e^+e^- \rightarrow \nu\bar{\nu}\gamma$.

Figure 12: Hard photon corrections to the t-channel photon diagrams.

Figure 13: Hard photon diagrams included in the EPA.

Figure 14: Virtual diagrams included in the EPA.

Figure 15: Comparison between tree level and one-loop corrected results for radiative Bhabha scattering:
a) $\cos \theta_\gamma$ distribution.
b) E_γ distribution.
c) $p_{T,\gamma}$ distribution.