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Algebraic determination of covariant anomalies and Schwinger terms

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Abstract

Inclusion of the anti-ghost field in the BRST algebra allows us to derive simultaneously consistent and covariant anomalies and Schwinger terms (as well as Bardeen-Zumino counterterms) by transgression from a Chern-Simons form. Our method which uses the natural variables of the gauged BRST symmetry encompasses all polynomial and descent equations of a previously proposed approach which relies on various homotopy operators. In conclusion, both approaches are generalized to gravitational theories.

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1 Introduction

Anomalies correspond to a breakdown of symmetries of a classical field theory at the quantum level. They appear in different forms known as the consistent and covariant anomalies. The latter emerges incidentally in Fujikawa's path integral approach (see for instance [1]). Schwinger terms occurring in equal-time commutators of gauge group generators or currents represent a different manifestation of these anomalies [2] [3] [4] [5]. They have raised many questions which have been intensively studied these last years and which still remain a subject of active interest (see [6] and references therein).

While it has been known for a long time [7] that candidates for the consistent anomaly may be obtained in an algebraic way by transgression from a Chern-Simons form, it was realized only recently that covariant anomalies can be derived by a similar method [8] [9] [10]. In the first case, the relevant cohomology is that of the BRST operator s modulo the exterior derivative d and in the latter, one relies on an extension which is occasionally referred to as the vertical cohomology [9] [10]. The introduction of a homotopy operator leads to descent equations which contain the covariant anomaly as well as the Bardeen-Zumino (BZ) counterterm [4], which relates the currents whose divergences define the consistent and covariant anomalies, respectively. Introduction of further homotopy derivations allows us to incorporate covariant Schwinger terms in this scheme [10].

In this paper, we present a new algebraic approach to the determination of covariant anomalies, Schwinger terms and BZ-counterterms. Our method is as straightforward as the ordinary transgression and does not require the introduction of homotopy operators. Contrary to the previous approach, it allows us to express the BZ-counterterm and all anomalous terms solely in terms of cocycles of the underlying cohomology. It encompasses all polynomials and descent equations of the first approach and leads to stronger integrability conditions for the (non-integrated) covariant anomalies and Schwinger terms.

The basic idea is to consider the usual cohomology of s modulo d as written in terms of the so-called tilde-fields [11] which are the natural variables of the localized BRST symmetry [12] [11]. The latter amounts to a gauging of the ordinary (global) BRST symmetry. That it is really this algebraic structure which is underlying our construction is best seen in the gravitational case where both the gauge connection and the ghost fields are redefined in the tilde-basis.

This paper is organized as follows. To start with, we consider Yang-Mills theories in flat d -dimensional space-time. We briefly review the definition of consistent anomalies and discuss the algebraic characterization of covariant anomalies (similarly for the Schwinger terms). In section 4, we present our approach which is subsequently compared with the homotopy operator method. Thereafter, the problem of gravitational anomalies is addressed. When generalized to this case, both methods allow the derivation of the covariant Einstein anomaly together with the BZ-counterterm from a Chern-Weil polynomial. All discussions in the body of this paper are performed locally on the space-time manifold. Some remarks on the global formulation are gathered in section 6, which also raises various open problems. An appendix is devoted to the derivation of the tilde-fields which are needed in the text.

2 Algebraic equations for anomalies

In this chapter, we briefly review the different definitions of consistent and covariant anomalies for YM-theories in flat d -dimensional space-time. Our presentation is based on references [7] [4] [13] and discusses, in particular, the algebraic equations satisfied by the covariant anomalies.

2.1 Consistent anomalies

Let $\Gamma[A]$ denote the vacuum functional and $J_\mu^\alpha(x) = \delta\Gamma[A]/\delta A_\mu^\alpha(x)$ the associated gauge field current. In general, in the presence of chiral fermions, the functional Γ is not gauge invariant. If $\delta_\omega A_\mu = \mathcal{D}_\mu \omega \equiv \partial_\mu \omega + [A_\mu, \omega]$ describes an infinitesimal gauge variation with a Lie algebra valued parameter ω and $W(\omega) \equiv \int dx \text{Tr}(\delta_\omega A_\mu) \delta/\delta A_\mu$ the corresponding Ward operator, then the symmetry breaking manifests itself by the occurrence of an (integrated) anomaly $\Delta(\omega, A)$ given by

$$W(\omega)\Gamma[A] = \Delta(\omega, A) \equiv \int dx \text{Tr} [\omega G] \quad , \quad (1)$$

or, equivalently, by

$$\mathcal{D}_\mu J_\mu^\alpha = -G_\alpha(A) \quad . \quad (2)$$

Application of the commutation relations

$$[W(\omega), W(\omega')] = W([\omega, \omega'])$$

to $\Gamma[A]$ immediately leads to the WZ-consistency relation for Δ :

$$W(\omega)\Delta(\omega', A) - W(\omega')\Delta(\omega, A) - \Delta([\omega, \omega'], A) = 0 \quad . \quad (3)$$

Any local functional of A_μ which is linear in ω and satisfies (3) is called a *consistent anomaly*. Equation (3) can be cast into a more compact form by turning ω into a ghost field c . Then, (3) is equivalent to

$$s\Delta(c, A) = 0 \quad , \quad (4)$$

where the action of the BRST operator s on the 1-form $A = A_\mu dx^\mu$ and on c is defined by¹

$$\begin{aligned} sA &= -\mathcal{D}c \\ sc &= -\frac{1}{2}[c, c] \quad . \end{aligned} \quad (5)$$

These equations are equivalent to the so-called Russian formula for the curvature $F = dA + AA = 1/2 F_{\mu\nu} dx^\mu \wedge dx^\nu$:

$$\tilde{F} = F \quad , \quad (6)$$

¹Following Stora [7], we add the ghost number to the form degree and assume all commutators to be graded by this total degree.

where

$$\tilde{F} = \check{d}\check{A} + \check{A}\check{A} \quad , \quad \check{d} = d + s \quad , \quad \check{A} = A + c \quad .$$

In order to determine a solution of the consistency equation (4) in $d = 2(n-1)$ space-time dimensions, one simultaneously applies the Russian formula and the transgression formula

$$\text{Tr } F^n = d T_{2n-1}^0(A, F) \quad (7)$$

for the Chern-Simons polynomial $T_{2n-1}^0(A, F)$:

$$\text{Tr } \tilde{F}^n = \check{d} T_{2n-1}(\check{A}, \check{F}) \quad .$$

By expanding the $(2n-1)$ -form $T_{2n-1}(\check{A})$ in powers of c ,

$$T_{2n-1}(\check{A}) = \sum_{k=0}^{2n-1} T_{2n-1-k}^k(A, c) \quad , \quad (8)$$

where the lower index denotes the form degree and the upper index the ghost number, one obtains the descent equations by matching the corresponding powers of c :

$$s T_{2n-k-1}^k(A, c) + d T_{2n-k-2}^{k+1}(A, c) = 0 \quad \text{for } 0 \leq k \leq 2n-1 \quad . \quad (9)$$

The equation for $k=1$ implies $s \int T_{2n-2}^1(A, c) = 0$ and identifies this descendant as a candidate for the consistent anomaly. It follows from general arguments [14] that this solution of the WZ-integrability condition is non-trivial, i.e. T_{2n-2}^1 is not of the form $s\alpha + d\beta$. The actual occurrence of this anomaly in field theory depends on the representation of the matter fields and on the coefficient of the anomalous term in the Ward identity (1).

2.2 Covariant anomalies

If the consistent anomaly in eq.(2) is non-vanishing, the current $J_a^\mu(x)$ does not transform covariantly under gauge transformations [4]. In some specific cases, a local polynomial $X_a^\mu(A)$ is known which renders the current $J_a^\mu(x)$ covariant [4] :

$$\delta_a \hat{J}^\mu = -[\omega, \hat{J}^\mu] \quad \text{for} \quad \hat{J}^\mu = J^\mu + X^\mu(A) \quad . \quad (10)$$

The polynomial $X^\mu(A)$ is known as the *Bardeen-Zumino counterterm* and the anomaly associated to the covariant current $\hat{J}^\mu$ is usually referred to as the *covariant anomaly* :

$$\mathcal{D}_\mu \hat{J}_a^\mu = -\hat{G}_a \quad . \quad (11)$$

In $d = 2(n-1)$ space-time dimensions, an explicit expression for $\hat{G}$ is well known :

$$\Delta_{\text{cov}}(c, A) \equiv \int dx \text{Tr} [c \hat{G}] = n \int \text{Tr} [c F^{n-1}] \quad , \quad (12)$$

where F is the curvature 2-form. For $d=2$ and $d=4$, the counterterm X^μ has been determined by Bardeen and Zumino :

$$\begin{aligned} d=2 : \quad X_a^\mu &= \epsilon^{\mu\nu} \text{Tr} [\lambda_a A_\nu] \propto \epsilon^{\mu\nu} A_{a\nu} \\ d=4 : \quad X_a^\mu &= \epsilon^{\mu\nu\rho\sigma} \text{Tr} [\lambda_a (A_\nu F_{\rho\sigma} + F_{\rho\sigma} A_\nu - A_\nu A_\rho A_\sigma)] \quad . \end{aligned} \quad (13)$$

Here, ϵ denotes the totally antisymmetric tensor and $\{\lambda_a\}$ is a basis of group generators satisfying $\text{Tr} \lambda_a \lambda_b \propto \delta_{ab}$. (In the previous expressions, we have suppressed the numerical factors which may be evaluated either by perturbative or by topological arguments.)

While the consistent anomaly is characterized by an algebraic condition (eq.(3)), this does not seem to be the case for the covariant anomaly (12). However, in reference [13] it was shown that the covariant vertex anomaly satisfies an algebraic equation, if one considers a gauge field taking its values in $G_L \times G_R$ where G_L and G_R represent two copies of the structure group. More generally, these authors start from a group G with subgroup K . (For $G = G_L \times G_R$, the group K is identified with the diagonal subgroup.) Then, the gauge field A and the gauge parameter ω can be decomposed along the Lie algebra $\text{Lie } K$ and an invariantly defined orthogonal complement $(\text{Lie } K)_\perp$ (i.e. $[\text{Lie } K, (\text{Lie } K)_\perp] \subset (\text{Lie } K)_\perp$):

$$\begin{aligned} A &= A_K + A_\perp & A_K(x), \omega_K(x) &\in \text{Lie } K \\ \omega &= \omega_K + \omega_\perp & A_\perp(x), \omega_\perp(x) &\in (\text{Lie } K)_\perp \quad . \end{aligned}$$

Similarly, the BRST variations (5) can be decomposed in this way. The covariant anomaly then depends on $c_\perp, A_K, A_\perp$ and satisfies the condition

$$s \int \mathcal{A}_{\text{cov}}(c_\perp; A_K, A_\perp) = 0 \quad . \quad (14)$$

For $A_\perp = 0$, the explicit expression for $\int \mathcal{A}_{\text{cov}}(c_\perp; A_K, A_\perp)$ reduces to (12). Yet, the naive projection of the BRST algebra to $A_\perp = 0$ leads to an inconsistent result ($sA_\perp \neq 0$) and therefore the integrability condition (14) does not project to a similar equation for $\int \mathcal{A}_{\text{cov}}(c_\perp; A_K, A_\perp = 0)$. Nevertheless, there exists a consistent truncation of the symmetry transformations in this case:

$$\begin{aligned} sA_K &= -dc_K - [A_K, c_K] \\ sc_K &= -\frac{1}{2} [c_K, c_K] \\ sc_\perp &= -[c_K, c_\perp] \end{aligned} \quad (15)$$

and $sA_\perp = 0$. With this reduction, equation (14) and the explicit result of reference [13] become

$$\begin{aligned} 0 &= s \int \mathcal{A}_{\text{cov}}(c_\perp; A_K) \\ &= s \left\{ n \int \text{Tr} [c_\perp F(A_K)^{n-1}] \right\} \quad . \end{aligned} \quad (16)$$

(Actually, the given explicit expression satisfies the stronger condition $s\mathcal{A}_{\text{cov}}(c_\perp; A_K) = 0$.)

The characterization (15), (16) of the covariant anomaly formally coincides with the one given by Song [8] in terms of the so-called vertical cohomology [8] [9]. The main purpose of the present work is the construction of solutions of eq.(16) with s defined by (15) and the determination of the counterterm X^ν relating the consistent and covariant currents, J^ν and $\tilde{J}^\nu$.

Simplifying the notation in eqs.(15) and (16) by the substitutions $(A_K, c_K, c_L) \rightarrow (A, c, \bar{c})$, the problem can be restated as follows. The (integrated) covariant anomaly is a functional which is linear in $\bar{c}$ and local in A , i.e. it is of the form

$$\Delta_{cov}(\bar{c}, A) = \int dx \text{Tr} [\bar{c} \hat{G}(A)] \quad ,$$

where $\hat{G}(A)$ is a polynomial depending on A_μ and its partial derivatives up to finite order. It satisfies the *covariance condition*

$$s \Delta_{cov}(\bar{c}, A) = 0 \quad (17)$$

with s defined by

$$\begin{aligned} sA &= -\mathcal{D}c \\ sc &= -\frac{1}{2} [c, c] \\ s\bar{c} &= -[c, \bar{c}] \end{aligned} \quad (18)$$

To make contact with field theory and the consistent anomaly occurring in the anomalous Ward identity, one should also require the following property. By a BZ-counterterm, $\Delta_{cov}(\bar{c}, A)$ can be related to $\Delta_{cons}(\bar{c}, A)$ where $\Delta_{cons}(\bar{c}, A)$ represents a solution of the usual WZ-consistency condition, if $\bar{c}$ is substituted by $c : s\Delta_{cons}(c, A) = 0$.

In eqs.(14) and (15), the Lie algebra valued parameter $c_L = \bar{c}$ has a ghost number +1, but eqs.(18) are also compatible with a different ghost number assignment for $\bar{c}$. For the remainder of this paper, we associate a ghost number -1 to $\bar{c}$ and we view it as an anti-ghost field (taking its values in the adjoint representation of the Lie algebra). This assignment turns out to be essential for the approaches considered in this paper.

3 Schwinger terms

As is well known [2] [3] [5], anomalies occurring in Ward identities materialize themselves in other ways in quantum field theory. The so-called Schwinger terms occurring in equal-time commutators of gauge group generators (or time-time commutators of currents) represent a different manifestation of the same phenomenon. In the following, we briefly recall the algebraic characterization of these terms in flat d -dimensional space-time and we discuss the relation with the perturbative results in $d = 2$ and $d = 4$ dimensions [3] [5].

3.1 Consistent Schwinger terms

We suppose the theory has been quantized in a gauge in which the spatial components $\vec{A} = (A_i^a)$ of the vector potential commute at equal times. We denote the gauge group by

$\mathcal{G}$ and its Lie algebra by $Lie \mathcal{G}$. The latter is represented on the vector space of functionals of $\vec{A}$ by a space integral

$$W_t(\omega) = \int d\vec{x} (\delta_\omega A_i^a)(x) \frac{\delta}{\delta A_i^a(x)} = \int d\vec{x} \omega^a(x) X_a(\vec{A}, x) \equiv \omega \cdot X(\vec{A}) \quad ,$$

where $x = (t, \vec{x})$ and where

$$X_a(\vec{A}, x) = -\partial_k \frac{\delta}{\delta A_k^a} - i f_{abc} A_k^b \frac{\delta}{\delta A_k^c}$$

generates gauge transformations of $\vec{A}$. Now, consider the vector space of functionals which depend both on the gauge field and on some matter fields (e.g. chiral fermions). The action of an infinitesimal gauge transformation on such functionals is given by

$$T_a(\vec{A}, x) = X_a(\vec{A}, x) + Y_a(\vec{A}, x) \quad , \quad (19)$$

where Y operates on the matter fields. The generators X_a satisfy normal commutation relations,

$$[\omega_1 \cdot X, \omega_2 \cdot X] = [\omega_1, \omega_2] \cdot X \quad ,$$

but those of T_a generally admit a central extension :

$$[\omega_1 \cdot T(\vec{A}), \omega_2 \cdot T(\vec{A})] = [\omega_1, \omega_2] \cdot T(\vec{A}) + \omega_1 \cdot Z(\vec{A}) \cdot \omega_2 \quad 1 \quad (20)$$

with

$$\omega_1 \cdot Z(\vec{A}) \cdot \omega_2 \equiv \int d\vec{x} \int d\vec{y} \omega_1^a(x) Z_{ab}(x, y, \vec{A}) \omega_2^b(y) \quad (21)$$

As a direct consequence of its definition, the *consistent Schwinger term* Z is antisymmetric:

$$Z_{ab}(x, y, \vec{A}) = -Z_{ba}(y, x, \vec{A}) \quad . \quad (22)$$

Moreover, $\omega_1 \cdot Z(\vec{A}) \cdot \omega_2$ is a local functional of $\vec{A}$, i.e. Z_{ab} is of the form

$$Z_{ab}(x, y, \vec{A}) = \delta^{(3)}(\vec{x} - \vec{y}) P_{ab}(A_i, \partial_j A_i, \dots) \quad . \quad (23)$$

where P_{ab} depends on A_μ and its partial derivatives up to finite order (and transforms like a product of adjoint representations).

The Jacobi identity for $Lie \mathcal{G}$ implies the following consistency condition for the Schwinger term :

$$0 = \sum_{(1,2,3)} ([\omega_1, \omega_2] \cdot Z(\vec{A}) \cdot \omega_3 - \omega_1 \cdot [\omega_2 \cdot T(\vec{A}), Z(\vec{A})] \cdot \omega_3) \quad (24)$$

Here, the sum runs over all cyclic permutations and, in the last term, $\omega_3 \cdot T$ operates on $Z(\vec{A})$ by an infinitesimal gauge transformation on the argument :

$$[\omega \cdot T, A_i^a] = (D_i \omega)^a \quad .$$

To find an explicit expression for the consistent Schwinger term, we turn the gauge parameter ω into a Faddeev-Popov ghost c and consider the BRST operation defined by (5). Then, (24) is equivalent to

$$0 = s \left(c \cdot Z(\vec{A}) \cdot c \right) . \quad (25)$$

Solutions of this condition appear in the descent equations (9) : if the space manifold has no boundary, the $k = 2$ equation implies $s \int T_{2n-3}^2(A, c) = 0$. For $d - 1 = 1$ and $d - 1 = 3$ space dimensions, one obtains the following explicit expressions :

$$\begin{aligned} d = 2 : \quad c \cdot Z(\vec{A}) \cdot c &\propto \int \text{Tr} (c^2 A) \\ d = 4 : \quad c \cdot Z(\vec{A}) \cdot c &\propto \int \text{Tr} (c^2 F A + c[c, A] dA) . \end{aligned}$$

Here and in the remainder of this section, A and F involve only spatial components : $A = A_i dx^i$, $F = 1/2 F_{ij} dx^i \wedge dx^j$.

The solutions of equation (25) are only determined up to space integrals $\int s\alpha + \int d\beta$ and this ambiguity can be used to make the $d = 2$ result independent of the gauge field :

$$d = 2 : \quad c \cdot Z(\vec{A}) \cdot c \propto \int \text{Tr} (c dc) . \quad (26)$$

Choosing the correct normalization factors, this corresponds to the commutation relations

$$[\omega_1 \cdot T(\vec{A}), \omega_2 \cdot T(\vec{A})] = [\omega_1, \omega_2] \cdot T(\vec{A}) + \frac{i}{4\pi} \int \text{Tr} (\omega_1 d\omega_2 - \omega_2 d\omega_1) \mathbf{1} ,$$

or equivalently

$$[T^a(x), T^b(y)] = i f^{abc} T^c(x) \delta(x - y) + \frac{i}{2\pi} \delta^{ab} \partial_x \delta(x - y) \mathbf{1} \quad (27)$$

with $x, y \in \mathbf{R}$. These are the well-known structure relations of an affine Kac-Moody algebra [15] and the Schwinger term coincides with its central extension.

Along the same lines, the $d = 4$ expression can be cast into equivalent forms, although it cannot be rendered independent of A :

$$\begin{aligned} c \cdot Z'(\vec{A}) \cdot c &\propto \int \text{Tr} (\{c, dc\} dA) \\ c \cdot Z''(\vec{A}) \cdot c &\propto \int \text{Tr} \left(\frac{1}{2} [c, c] [dA A + A dA + A^3] + c d[A c A] \right) . \end{aligned} \quad (28)$$

The first of these expressions gives a clue to what a covariant Schwinger term may look like and the second corresponds to the anomalous commutator of the Gauss law operator $T^a(x)$ as evaluated by field theoretic methods [16] :

$$\begin{aligned} [T^a(x), T^b(y)] \delta(x^0 - y^0) &= -i f^{abc} T^c(x) \delta^{(4)}(x - y) + \frac{1}{48\pi^2} \epsilon^{ijk} \text{Tr} ([\lambda^a, \lambda^b] [\partial_i A_j A_k \\ &\quad + A_i \partial_j A_k + A_i A_j A_k] + \lambda^a \partial_i [A_j \lambda^b A_k]) \delta^{(4)}(x - y) . \end{aligned}$$

3.2 Covariant Schwinger terms

Covariant Schwinger terms occur in the time-time commutators of the covariant current $\tilde{J}_\mu^a$, e.g. [16] for $d = 4$:

$$\begin{aligned} [\tilde{J}_0^a(x), \tilde{J}_0^b(y)] \delta(x^0 - y^0) &= -i f^{abc} \tilde{J}_0^c(x) \delta^{(4)}(x - y) \\ &\quad + \frac{1}{16\pi^2} \epsilon^{0ijk} \mathcal{D}_i^c(x) \left[\text{Tr} (\{\lambda^b, \lambda^c\} F_{jk} \delta^{(4)}(x - y)) \right] . \end{aligned}$$

If parametrized in terms of ghost fields, this Schwinger term reads

$$d = 4 : \quad c \cdot Z_{\text{cov}}(\vec{A}) \cdot c \propto \int \text{Tr} (\{c, D_c\} F) ,$$

an expression which represents a covariantization of (28). More generally, in $d = 2(n - 1)$ space-time dimensions, the covariant Schwinger term is given by

$$d = 2(n - 1) : \quad c \cdot Z_{\text{cov}}(\vec{A}) \cdot c \propto \int \text{Tr} (\{c, D_c\} F^{n-2}) . \quad (29)$$

Substituting c by $\bar{c}$, it is easily verified that it satisfies the covariance condition

$$\begin{aligned} 0 &= s \left(\bar{c} \cdot Z_{\text{cov}}(\vec{A}) \cdot \bar{c} \right) \\ &= s \int \text{Tr} (\{\bar{c}, D\bar{c}\} F^{n-2}) , \end{aligned} \quad (30)$$

where the action of s is defined by (18). (As for the covariant anomaly, the integrand in the last expression is already s -invariant.)

Whereas the consistent anomaly and Schwinger term are related by the descent equation $sT_{2n-2}^1 + dT_{2n-3}^2 = 0$, a similar relation does not seem to exist for the covariant anomalous terms.

4 Algebraic determination of anomalous terms

Before proceeding, let us comment on the origin of the variables we are going to use.

It is natural to introduce a new odd differential, the anti-BRST operator $\bar{s}$, which mimics the s -operation (5) :

$$\begin{aligned} \bar{s} A &= -D\bar{c} \\ \bar{s} \bar{c} &= -\frac{1}{2} [\bar{c}, \bar{c}] . \end{aligned} \quad (31)$$

Compatibility with s (i.e. $s\bar{s} + \bar{s}s = 0$) then requires that

$$s\bar{c} + \bar{s}c = -[c, \bar{c}] . \quad (32)$$

Here, c and $\bar{c}$ can be viewed as Faddeev-Popov ghosts and all fields are assumed to take their values in the adjoint representation of the Lie algebra.

Condition (32) leaves $s\bar{c}$ undetermined and for the gauge-fixing procedure in field theory, one usually introduces an auxiliary field b such that $s\bar{c} = -b$, $s\bar{c} = -b - [c, \bar{c}]$ (see [17] and references therein). In a general algebraic formalism, the degeneracy (32) may simply be lifted by

$$s\bar{c} = -[c, \bar{c}] \quad , \quad \bar{s}c = 0 \quad (33)$$

which is in agreement with our s -operation (18). Altogether, the operators s and $\bar{s}$ then satisfy the anti-commutation relations

$$s^2 = s\bar{s} + \bar{s}s = \bar{s}^2 = 0$$

and, by definition, $sd + d\bar{s} = 0 = \bar{s}d + d\bar{s}$. In view of the global mathematical interpretation [9] of the so-defined differentials s and $\bar{s}$, the induced cohomology is sometimes referred to as the vertical cohomology.

Both the s and $\bar{s}$ transformations represent global symmetries which can be rendered local by a gauging procedure [12]. The systematic approach which avoids cumbersome Noetherian constructions was first introduced in reference [11]. In this framework, the gauging of the global (anti-)BRST symmetry requires the determination of a new set of fields, the "tilde"-fields, on which the local (anti-)BRST operator is defined. All fields whose global transformation law involves a derivative of a ghost are redefined in the tilde-basis. In the appendix, we illustrate the general construction for the anti-BRST variations (31) for which the connection form A is the only variable to transform with such a derivative. We find that it is redefined by

$$\tilde{A} = A + \bar{\alpha}\bar{c} \quad , \quad (34)$$

where $\bar{\alpha} = \bar{\alpha}_\mu dx^\mu$ is an Abelian connection 1-form with ghost number +1 corresponding to the gauge field for the localized anti-BRST symmetry.

In the sequel, we may forget about the anti-BRST operator and the origin of the redefinition (34) and simply use this variable in a global BRST context where s is defined by eqs.(18) and

$$s\bar{\alpha} = 0 \quad .$$

Using $A = \tilde{A} - \bar{\alpha}\bar{c}$, we now rewrite the ordinary descent equations (9) in terms of tilde-fields,

$$sT_{2n-k-1}^k(\tilde{A} - \bar{\alpha}\bar{c}, c) + dT_{2n-k-2}^{k+1}(\tilde{A} - \bar{\alpha}\bar{c}, c) = 0 \quad \text{for} \quad 0 \leq k \leq 2n-1 \quad (35)$$

and expand the polynomials T_p^q in $\bar{\alpha}$ and $d\bar{\alpha}$:

$$T_p^q(\tilde{A} - \bar{\alpha}\bar{c}, c) = T_p^q(\tilde{A}, c) - \bar{\alpha} \tilde{T}_{p-1}^{q-1}(\tilde{A}, c, \bar{c}) - d\bar{\alpha} \tilde{T}_{p-2}^{q-1}(\tilde{A}, c, \bar{c}) - \bar{\alpha}^2 \tilde{T}_{p-2}^{q-2}(\tilde{A}, c, \bar{c}) - \bar{\alpha} d\bar{\alpha} \tilde{T}_{p-3}^{q-2}(\tilde{A}, c, \bar{c}) - \dots \quad , \quad (36)$$

Here, the dots stand for terms of higher order in $\bar{\alpha}$ and $d\bar{\alpha}$.

The redefinition (34) of the YM-connection leads to the variation

$$s\tilde{A} = -\tilde{D}c + \bar{\alpha}(s\bar{c} + [c, \bar{c}]) \quad ,$$

where $\tilde{D} = d + [\tilde{A}, \cdot]$. Remarkably enough, the choice (18) for $s\bar{c}$ implies that the original transformation law of A preserves its form under the change of variables $A \rightarrow \tilde{A}$:

$$s\tilde{A} = -\tilde{D}c \quad . \quad (37)$$

Since the other s -operations do not depend on A , they also preserve their form under this change of variables. Taking into account this result and the fact that $\bar{\alpha}$ is s -inert, we conclude that the descent equations (35) have to be satisfied term by term, if T_p^q is expanded as in (36). This leads to the following sets of ladder equations²

$$\begin{aligned} \mathcal{O}(1) : \quad & sT_{2n-k-1}^k(\tilde{A}, c) + dT_{2n-k-2}^{k+1}(\tilde{A}, c) = 0 \quad (0 \leq k \leq 2n-1) \\ \mathcal{O}(\bar{\alpha}) : \quad & s\tilde{T}_{2n-k-2}^{k-1}(\tilde{A}, c, \bar{c}) + d\tilde{T}_{2n-k-3}^k(\tilde{A}, c, \bar{c}) = 0 \quad (0 \leq k \leq 2n-2) \\ \mathcal{O}(d\bar{\alpha}) : \quad & s\tilde{T}_{2n-k-3}^{k-1}(\tilde{A}, c, \bar{c}) + d\tilde{T}_{2n-k-4}^k(\tilde{A}, c, \bar{c}) = \tilde{T}_{2n-k-3}^k(\tilde{A}, c, \bar{c}) \quad (0 \leq k \leq 2n-3) \\ \mathcal{O}(\bar{\alpha}^2) : \quad & s\tilde{T}_{2n-k-4}^{k-2}(\tilde{A}, c, \bar{c}) + d\tilde{T}_{2n-k-5}^{k-1}(\tilde{A}, c, \bar{c}) = 0 \quad (0 \leq k \leq 2n-4) \\ \mathcal{O}(\bar{\alpha}d\bar{\alpha}) : \quad & s\tilde{T}_{2n-k-4}^{k-2}(\tilde{A}, c, \bar{c}) + d\tilde{T}_{2n-k-5}^{k-1}(\tilde{A}, c, \bar{c}) = 2\tilde{T}_{2n-k-4}^{k-1}(\tilde{A}, c, \bar{c}) \quad (0 \leq k \leq 2n-4) \end{aligned}$$

The first set is nothing other than the ordinary descent equations modulo the substitution $A \rightarrow \tilde{A}$. For $k=1$ ($k=2$), they contain the *consistent anomaly* (Schwinger term), both regarded as functions of c and $\tilde{A}$. The other sets are new descent equations. For $k=0$, the second (fourth) set contains a candidate for the covariant anomaly (Schwinger term), since these equations imply the covariance conditions

$$s \int_{\text{space-time}} \tilde{T}_{2n-2}^{-1}(\tilde{A}, \bar{c}) = 0 = s \int_{\text{space}} \tilde{T}_{2n-3}^{-2}(\tilde{A}, \bar{c}) \quad . \quad (38)$$

The third (fifth) set can be substituted into the second (fourth) set to give

$$\begin{aligned} s \left(\tilde{T}_{2n-k-2}^{k-1} - d\tilde{T}_{2n-k-3}^{k-1} \right) &= 0 \quad \text{for} \quad 0 \leq k \leq 2n-2 \\ s \left(\tilde{T}_{2n-k-3}^{k-2} - \frac{1}{2} d\tilde{T}_{2n-k-4}^{k-2} \right) &= 0 \quad \text{for} \quad 0 \leq k \leq 2n-3 \end{aligned}$$

For $k=0$, these equations read

$$\begin{aligned} s \left(\tilde{T}_{2n-2}^{-1} - d\tilde{T}_{2n-3}^{-1} \right) &= 0 \\ s \left(\tilde{T}_{2n-3}^{-2} - \frac{1}{2} d\tilde{T}_{2n-4}^{-2} \right) &= 0 \end{aligned} \quad (39)$$

In view of eqs.(38)(39), we identify the non-integrated covariant anomaly and Schwinger term as

$$\begin{aligned} A_{\text{cov}}(\tilde{A}, \bar{c}) &\equiv \tilde{T}_{2n-2}^{-1} - d\tilde{T}_{2n-3}^{-1} \\ \left(\bar{c} \cdot Z_{\text{cov}}(\tilde{A}) \cdot \bar{c} \right)_{\text{integrated}} &\equiv \tilde{T}_{2n-3}^{-2} - \frac{1}{2} d\tilde{T}_{2n-4}^{-2} \quad (n \geq 3) \end{aligned} \quad (40)$$

²We remark that the higher order terms in the expansion (36) yield further equations which are not of interest here.

By construction, they satisfy the strong covariance condition

$$s \mathcal{A}_{\text{cov}}(\tilde{A}, \tilde{c}) = 0 = s \left(\tilde{c} \cdot Z_{\text{cov}}(\tilde{A}) \cdot \tilde{c} \right)_{\text{integrated}}. \quad (41)$$

As we will see shortly, the last term in eq.(40), i.e. $\tilde{T}_{2n-3}^{-1}$, corresponds to the *BZ-counterterm*.

Let us now illustrate the general approach for $d = 2$ and $d = 4$ (corresponding to $n = 2$ and $n = 3$, respectively). When expressed in terms of tilde-fields, the $n = 2$ Chern-Simons form

$$T_3^0(A) = \text{Tr} \left[A dA + \frac{2}{3} A^3 \right] \quad (42)$$

becomes

$$T_3^0(A) = T_3^0(\tilde{A}) - \bar{\alpha} \text{Tr} \left[\tilde{c} \left(d\tilde{A} + 2\tilde{A}^2 \right) + d\tilde{c}\tilde{A} \right] - d\tilde{\alpha} \text{Tr} \left[\tilde{c}\tilde{A} \right] - \dots \quad (43)$$

The same procedure for the $n = 3$ Chern-Simons form

$$T_5^0(A) = \text{Tr} \left[A dA dA + \frac{3}{2} A^3 dA + \frac{3}{5} A^5 \right] \quad (44)$$

gives

$$\begin{aligned} T_5^0(A) = & T_5^0(\tilde{A}) - \bar{\alpha} \text{Tr} \left[\tilde{c} \left(d\tilde{A} d\tilde{A} + \frac{3}{2} d\tilde{A} \tilde{A}^2 + \frac{3}{2} \tilde{A} d\tilde{A} \tilde{A} + \frac{3}{2} \tilde{A}^2 d\tilde{A} + 3\tilde{A}^4 \right) \right. \\ & + d\tilde{c} \left(d\tilde{A} \tilde{A} + \tilde{A} d\tilde{A} + \frac{3}{2} \tilde{A}^3 \right) \left. \right] - d\tilde{\alpha} \text{Tr} \left[\tilde{c} \left(d\tilde{A} \tilde{A} + \tilde{A} d\tilde{A} + \frac{3}{2} \tilde{A}^3 \right) \right] \\ & + \bar{\alpha}^2 \text{Tr} \left[\tilde{c} d\tilde{c} \left(d\tilde{A} + \frac{3}{2} \tilde{A}^2 \right) + d\tilde{c}\tilde{c} \left(d\tilde{A} + \frac{3}{2} \tilde{A}^2 \right) + d\tilde{c}d\tilde{c} \tilde{A} \right. \\ & + \frac{3}{2} \tilde{c} \tilde{A} d\tilde{c} \tilde{A} + \tilde{c}^2 \left(\frac{3}{2} d\tilde{A} \tilde{A} + \frac{3}{2} \tilde{A} d\tilde{A} + 3\tilde{A}^3 \right) + \tilde{c} \tilde{A} \tilde{c} \left(\frac{3}{2} d\tilde{A} + 3\tilde{A}^2 \right) \left. \right] \\ & + \bar{\alpha} d\tilde{\alpha} \text{Tr} \left[\tilde{c} d\tilde{c} \tilde{A} + d\tilde{c}\tilde{c} \tilde{A} + \frac{3}{2} \tilde{c} \tilde{A} \tilde{c} \tilde{A} \right] - \dots \end{aligned} \quad (45)$$

The non-integrated covariant anomaly, regarded as a function of $\tilde{c}$ and $\tilde{A}$, is then given by

$$\begin{aligned} d = 2 : & \quad \mathcal{A}_{\text{cov}}(\tilde{A}, \tilde{c}) = \tilde{T}_2^{-1} - d\tilde{T}_1^{-1} = 2 \text{Tr} \left[\tilde{c} \tilde{F} \right] \\ d = 4 : & \quad \mathcal{A}_{\text{cov}}(\tilde{A}, \tilde{c}) = \tilde{T}_4^{-1} - d\tilde{T}_3^{-1} = 3 \text{Tr} \left[\tilde{c} \tilde{F}^2 \right], \end{aligned} \quad (46)$$

where $\tilde{F} = d\tilde{A} + \tilde{A} \tilde{A}$. Comparison with the expressions (13) shows that the $d\tilde{\alpha}$ -term in the expansions (43) and (45) (i.e. $\tilde{T}_1^{-1}$ and $\tilde{T}_3^{-1}$, respectively) corresponds to the BZ-counterterm X_a^μ . Quite generally, in $d = 2(n-1)$ dimensions, we have

$$\tilde{c}^a \cdot X_a^\mu(\tilde{A}) = \epsilon^{\mu\nu_1 \dots \nu_{n-3}} \left(\tilde{T}_{2n-3}^{-1} \right)_{\nu_1 \dots \nu_{n-3}}, \quad (47)$$

where ϵ denotes the totally antisymmetric tensor and $(\)_{\nu_1 \dots \nu_{n-3}}$ the components of the corresponding $(2n-3)$ -form.

For the covariant Schwinger term, one finds the following result

$$d = 4 : \quad \left(\tilde{c} \cdot Z_{\text{cov}}(\tilde{A}) \cdot \tilde{c} \right)_{\text{integrated}} = \tilde{T}_3^{-2} - \frac{1}{2} d\tilde{T}_2^{-2} = -\frac{3}{2} \text{Tr} \{ \tilde{c}, \tilde{D}\tilde{c} \} \tilde{F}.$$

In summary, use of the natural variables of the gauged anti-BRST symmetry allows for an algebraic determination of all anomalous terms and of the BZ-counterterm. It leads to strong integrability conditions for the non-integrated covariant anomaly and Schwinger term.

5 Homotopy operator approach

In this chapter, we compare our algebraic approach to the one of references [8] and [9] which we first reformulate in terms of our compact notations.

The Russian formula, $\tilde{F} = F$, can be postulated to include the anti-BRST transformations [17] :

$$\tilde{\tilde{F}} = \tilde{\tilde{A}} + \tilde{\tilde{A}} \tilde{\tilde{A}}, \quad \tilde{\tilde{d}} = d + s + \tilde{s}, \quad \tilde{\tilde{A}} = A + c + \tilde{c}.$$

In this case, $\tilde{\tilde{F}} = F$ is equivalent to the BRST/anti-BRST variations (5) (31) and the compatibility condition (32). Combining this form of the Russian formula with the transgression formula (7) leads to descent equations which can be separated into a set corresponding to the s -cohomology,

$$s T_{2n-k-1}^{k-2j} (A, c) + d T_{2n-k-2}^{k-2j+1} (A, c) = 0 \quad \text{for} \quad 0 \leq k \leq 2n-1, \quad (48)$$

a mirror set corresponding to the $\tilde{s}$ cohomology,

$$\tilde{s} T_{2n-k-1}^{k-2j} (A, c) + d T_{2n-k-2}^{k-2j+1} (A, c) = 0 \quad \text{for} \quad 0 \leq k \leq 2n-1 \quad (49)$$

and a mixed set of equations :

$$\begin{aligned} s T_{2n-k-1}^{k-2j} (A, c, \tilde{c}) + \tilde{s} T_{2n-k-1}^{k-2j+2} (A, c, \tilde{c}) + d T_{2n-k-2}^{k-2j+1} (A, c, \tilde{c}) &= 0 \\ \text{for} \quad 1 \leq j \leq k \quad \text{and} \quad 1 \leq k \leq 2n-1. \end{aligned} \quad (50)$$

As in eq.(8), the polynomials T_p^q have a form degree p and a ghost number q .

The candidate for the consistent anomaly in the s -cohomology (resp. $\tilde{s}$ -cohomology) has form degree d and ghost number $+1$ (resp. -1). It appears for $k=1$ in eq.(48) (resp. (49)), from which relations it follows that $s \int T_{2n-2}^1 (A, c) = 0$, resp.

$$\tilde{s} \int T_{2n-2}^{-1} (A, \tilde{c}) = 0, \quad (51)$$

the integration being over space-time. Similarly, the consistent Schwinger term arises at ghost number $+2$ (-2).

The basic idea of the present approach to covariant anomalies consists of the introduction of a homotopy operator l satisfying certain commutation relations with d, s and $\tilde{s}$ in

such a way that it is possible to translate the consistency condition (51) into the covariance equation

$$s \int Q_{2n-2}^{-1}(A, \bar{c}) = 0, \quad (52)$$

Q_{2n-2}^{-1} being constructed from the polynomials T_p^g by use of the l -operation. To be more concrete [9], an even operator l is defined on the algebra generated by the basic fields and their derivatives by

$$\begin{aligned} lA &= 0, & lc &= 0, & l\bar{c} &= 0 \\ l(dA) &= \bar{s}A, & l(dc) &= 0, & l(d\bar{c}) &= \bar{s}\bar{c}. \end{aligned} \quad (53)$$

This operator lowers the form degree and the ghost number by one unit and satisfies

$$ld - dl = \bar{s}, \quad sl - ls = 0. \quad (54)$$

Now, consider eq.(50) for $j = 1$. By virtue of $\bar{s} = ld - dl$, it is equivalent to

$$sT_{2n-k-1}^{k-2} + ldT_{2n-k-1}^k + d(T_{2n-k-2}^{k-1} - lT_{2n-k-1}^k) = 0.$$

Substituting dT_{2n-k-1}^k by eq.(48) and using $sl = ls$ leads to

$$sQ_{2n-k-1}^{k-2} + dQ_{2n-k-2}^{k-1} = 0 \quad \text{for} \quad 1 \leq k \leq 2n-1, \quad (55)$$

where we defined

$$Q_p^g = T_p^g - lT_{p+1}^{g+1}. \quad (56)$$

For $k = 1$, the new set of descent equations (55) provides explicit expressions for the covariant anomaly, since it implies eq.(52).

Modulo the substitution $A \rightarrow \bar{A}$, the set of equations (55) formally coincides with the descent equations which appeared at order $\bar{\alpha}$ in our approach. In fact, explicit calculations to be indicated below show that

$$Q_{2n-k-2}^{k-1}(A, c, \bar{c}) = \bar{T}_{2n-k-2}^{k-1}(A, c, \bar{c}) \quad \text{for} \quad 0 \leq k \leq 2n-2, \quad (57)$$

where $\bar{T}_p^g$ are the polynomials introduced in eq.(36).

For definiteness, we consider the cases of $d = 2$ and $d = 4$ for which the known expressions for T_p^g lead to

$$\begin{aligned} d = 2 : \quad Q_2^{-1}(A, \bar{c}) &= T_2^{-1} - lT_3^0 \\ &= \text{Tr} \left[\bar{c} (dA + 2A^2) + d\bar{c}A \right] \\ &= 2 \text{Tr} [\bar{c}F] + d \text{Tr} [\bar{c}A] \\ d = 4 : \quad Q_4^{-1}(A, \bar{c}) &= T_4^{-1} - lT_5^0 \\ &= \text{Tr} \left[\bar{c} (dA dA + \frac{3}{2} dA A^2 + \frac{3}{2} A dA A + \frac{3}{2} A^2 dA + 3A^4) \right. \\ &\quad \left. + d\bar{c} (dA A + A dA + \frac{3}{2} A^3) \right] \\ &= 3 \text{Tr} [\bar{c}F^2] + d \text{Tr} \left[\bar{c} (dA A + A dA + \frac{3}{2} A^3) \right]. \end{aligned} \quad (58) \quad (59)$$

Thus, $Q_{2n-2}^{-1} = \mathcal{A}_{\text{cov}}(\bar{c}, A) + d(\dots)$ where $\mathcal{A}_{\text{cov}}(\bar{c}, A)$ represents the canonical, i.e. s -invariant expression for the non-integrated covariant anomaly. The derivative term corresponds to the BZ-counterterm, e.g. for $d = 2$:

$$d \text{Tr} [\bar{c}A] = \partial_\mu \text{Tr} [\bar{c} (\epsilon^{\mu\nu} A_\nu)] dx^0 \wedge dx^1 = \partial_\mu \text{Tr} [\bar{c} X^\mu] dx^0 \wedge dx^1.$$

The descent equations (55) do not contain a polynomial of the form Q_{2n-3}^{-2} which could be identified as a candidate for the covariant Schwinger term. These anomalous terms are incorporated into the present framework by the introduction of another homotopy derivation [10] which is defined by

$$\begin{aligned} mA &= \bar{c}, & mc &= 0, & m\bar{c} &= 0 \\ mF &= 0, & m(dc) &= 0, & m(d\bar{c}) &= \bar{s}\bar{c}. \end{aligned} \quad (60)$$

The covariant Schwinger term in $d = 2(n-1)$ space-time dimensions then reads

$$S_{2n-3}^{-2}(A, \bar{c}) \equiv T_{2n-3}^{-2} - lT_{2n-2}^{-1} + \frac{1}{2}(l^2 + ml - ml)T_{2n-1}^0, \quad (61)$$

where the T_p^g are the polynomials occurring in eqs.(49). As a consequence of the commutation relations

$$md - dm = \bar{s}, \quad ms = sm, \quad m\bar{s} = \bar{s}m,$$

it satisfies the covariance condition $s \int S_{2n-3}^{-2}(A, \bar{c}) = 0$, an explicit expression in four dimensions being given by

$$d = 4 : \quad S_3^{-2} = \frac{3}{2} \text{Tr} \{ \bar{c}, D\bar{c} \} F + d \text{Tr} \left[\bar{c} (d\bar{c}A - \frac{1}{2} \bar{c} dA + \frac{3}{4} A \bar{c}A) \right]. \quad (62)$$

Unlike our method, the approach presented in this section does not allow us to express the BZ-counterterm and the canonical expressions for the anomalous terms in terms of cocycles of the cohomology (i.e. solely in terms of the Q_p^g). Also, it does not lead to the strong consistency conditions (41) for the canonical expressions.

6 Gravitational theories

In the following, we extend the analysis of the last two sections to gravitational theories. After presenting the general framework for d -dimensional Riemannian manifolds, we derive explicit expressions in the two-dimensional case.

The rôle of the YM-field A is now played by the 1-form Γ_μ^λ associated to the Levi-Civita connection symbols $\Gamma_{\mu\nu}^\lambda$ according to $\Gamma_\mu^\lambda = \Gamma_{\mu\nu}^\lambda dx^\nu$. The action of s on this field is defined by [4]

$$\begin{aligned} s\Gamma_\mu^\lambda &= L_\ell \Gamma_\mu^\lambda - \mathcal{D}(\partial_\mu \xi^\lambda) \\ &= L_\ell \Gamma_\mu^\lambda - d(\partial_\mu \xi^\lambda) + \Gamma_\mu^\rho (\partial_\rho \xi^\lambda) - \Gamma_\rho^\lambda (\partial_\mu \xi^\rho), \end{aligned} \quad (63)$$

where $L_\ell = i_\ell d - d i_\ell$ is the Lie derivative along ξ , the ghost vector field of diffeomorphisms. The operator i_ℓ denotes the contraction of forms by the vector field ξ and the latter transforms with the graded Lie bracket :

$$s\xi = \frac{1}{2} L_\ell \xi = \frac{1}{2} [\xi, \xi] = \xi^\mu (\partial_\mu \xi^\nu) \partial_\nu . \quad (64)$$

The algebra (63)(64) can be extended by an anti-BRST operation $\bar{s}$ defined in a similar way:

$$\begin{aligned} \bar{s} \Gamma_\mu^\lambda &= L_\ell \Gamma_\mu^\lambda - \mathcal{D}(\partial_\mu \tilde{\xi}^\lambda) \\ &= L_\ell \Gamma_\mu^\lambda - d(\partial_\mu \tilde{\xi}^\lambda) + \Gamma_\mu^\sigma (\partial_\sigma \tilde{\xi}^\lambda) - \Gamma_\sigma^\lambda (\partial_\mu \tilde{\xi}^\sigma) \\ \bar{s} \tilde{\xi} &= \frac{1}{2} L_\ell \tilde{\xi} = \frac{1}{2} [\tilde{\xi}, \tilde{\xi}] = \tilde{\xi}^\mu (\partial_\mu \tilde{\xi}^\nu) \partial_\nu . \end{aligned} \quad (65)$$

Both actions are related by

$$\bar{s}\xi + s\tilde{\xi} = [\xi, \tilde{\xi}]$$

and, in analogy to YM theories, we lift this degeneracy by the choice

$$s\tilde{\xi} = L_\ell \tilde{\xi} = [\xi, \tilde{\xi}] , \quad \bar{s}\xi = 0 . \quad (66)$$

For definiteness, we now specialize to two dimensions where the (non-integrated) consistent anomaly follows by transgression from

$$T_4^0 = d \left(\Gamma_\mu^\lambda d\Gamma_\lambda^\mu + \frac{2}{3} \Gamma_\mu^\lambda \Gamma_\nu^\mu \Gamma_\lambda^\nu \right) \equiv dT_3^0 . \quad (67)$$

The resulting expression reads

$$T_2^1(\xi, \Gamma) = (\partial_\mu \xi^\lambda) d\Gamma_\lambda^\mu \quad (68)$$

and satisfies the WZ-consistency condition

$$sT_2^1 + dT_1^2 = 0$$

with $T_1^2(\xi, \Gamma) = (\partial_\mu \xi^\lambda) i_\ell d\Gamma_\lambda^\mu - (\partial_\rho \xi^\lambda) (\partial_\mu \xi^\rho) \Gamma_\lambda^\mu$.

For our approach to covariant anomalies, we need to know the tilde-variables of the gauged anti-BRST symmetry. Since the global anti-BRST variations of both the connection and the ghost fields involve derivative terms, all these variables have to be redefined in the tilde-basis. According to the results of the appendix,

$$\begin{aligned} \tilde{\tilde{\xi}}^\mu &= \tilde{\xi}^\mu + \tilde{\xi}^\nu \tilde{\alpha}_\nu \tilde{\xi}^\mu = (1 - i_{\tilde{\alpha}}) \tilde{\xi}^\mu , \\ \tilde{\tilde{\xi}}^\mu &= (1 + i_{\tilde{\alpha}}) \tilde{\tilde{\xi}}^\mu \end{aligned} \quad (69)$$

and

$$\begin{aligned} \tilde{\Gamma}_\mu^\lambda &= \Gamma_\mu^\lambda + \tilde{\alpha} \Lambda_\mu^\lambda(\tilde{\xi}) - \mathcal{D}(\tilde{\alpha}_\mu \tilde{\xi}^\lambda) + \mathcal{O}(\tilde{\alpha}^2) \\ \Gamma_\mu^\lambda &= \tilde{\Gamma}_\mu^\lambda - \tilde{\alpha} \tilde{\Lambda}_\mu^\lambda(\tilde{\xi}) + \tilde{\mathcal{D}}(\tilde{\alpha}_\mu \tilde{\tilde{\xi}}^\lambda) + \mathcal{O}(\tilde{\alpha}^2) \end{aligned} \quad (70)$$

where

$$\Lambda_\mu^\lambda(\tilde{\xi}) \equiv \partial_\mu \tilde{\xi}^\lambda + i_\ell \Gamma_\mu^\lambda$$

and $\tilde{\Lambda}_\mu^\lambda(\tilde{\xi}) = \partial_\mu \tilde{\tilde{\xi}}^\lambda + i_{\tilde{\ell}} \tilde{\Gamma}_\mu^\lambda$. In eq.(70), we did not specify the terms of higher order in $\tilde{\alpha}, d\tilde{\alpha}$, since these are irrelevant for the determination of the anomaly.

From the Chern-Simons form $T_3^0(\Gamma)$ of eq.(67), we now deduce

$$T_3^0(\Gamma) = T_3^0(\tilde{\Gamma}) - 2\tilde{\alpha} \tilde{\Lambda}_\mu^\lambda(\tilde{\xi}) \tilde{R}_\lambda^\mu - d \left(\tilde{\alpha} \tilde{\Lambda}_\mu^\lambda(\tilde{\xi}) \tilde{\Gamma}_\lambda^\mu \right) + \dots \quad (71)$$

where $\tilde{R}_\lambda^\mu = d\tilde{\Gamma}_\lambda^\mu + \tilde{\Gamma}_\rho^\mu \tilde{\Gamma}_\lambda^\rho$ is the curvature 2-form associated to $\tilde{\Gamma}$. The second term on the r.h.s. of eq.(71) represents the covariant gravitational anomaly $\mathcal{A}_{\text{cov}}(\tilde{\xi}, \Gamma)$ and the last one the BZ-counterterm [4] [18].

Contrary to the YM-case, the s -operator acting in the tilde-basis does not have the same form as in the original global algebra (eqs.(63)(64)(66)). A further dependence on $\tilde{\alpha}$ occurs and therefore the descent equations are no longer satisfied term by term in the $\tilde{\alpha}$ -expansion. As a result, the conclusion (41) for the non-integrated anomaly cannot be drawn in the present case. In fact, it does not hold. An explicit calculation shows that

$$\begin{aligned} s\Lambda_\mu^\lambda(\tilde{\xi}) &= L_\ell \Lambda_\mu^\lambda + \Lambda_\mu^\nu (\partial_\nu \xi^\lambda) + (\partial_\mu \xi^\nu) \Lambda_\nu^\lambda \\ s\mathcal{A}_{\text{cov}}(\tilde{\xi}, \Gamma) &= L_\ell \mathcal{A}_{\text{cov}}(\tilde{\xi}, \Gamma) \\ &= -d i_\ell \mathcal{A}_{\text{cov}}(\tilde{\xi}, \Gamma) , \end{aligned} \quad (72)$$

the last line following from the fact that a 3-form identically vanishes in two dimensions. In conclusion, only the integrated covariant Einstein anomaly is s -invariant :

$$s \int \mathcal{A}_{\text{cov}}(\tilde{\xi}, \Gamma) = 0 . \quad (73)$$

We will now show that these results can also be derived by generalizing the homotopy operator approach to d -dimensional Riemannian manifolds. For this purpose, we introduce a derivation l on the algebra generated by $\Gamma, \xi^\mu, \tilde{\xi}^\mu$ and their derivatives:

$$\begin{aligned} l\Gamma &= 0 , \quad l\xi^\mu = 0 = l(\partial_\nu \xi^\mu) , \quad l\tilde{\xi}^\mu = 0 , \quad l(\partial_\nu \tilde{\xi}^\mu) = 0 \\ l(d\Gamma) &= s\Gamma , \quad l(d\xi^\mu) = 0 = l(d\partial_\nu \xi^\mu) , \quad l(d\tilde{\xi}^\mu) = s\tilde{\xi}^\mu , \quad l(d\partial_\nu \tilde{\xi}^\mu) = s(\partial_\nu \tilde{\xi}^\mu) . \end{aligned}$$

This action is compatible with the assumption that $[l, i_\ell] = 0 = [l, L_\ell]$. To derive the covariant form of the Einstein anomaly in two dimensions, we consider the polynomial T_2^{-1} which is given by eq.(68) with ξ replaced by $\tilde{\xi}$. Using (67) and the definition (56) as well as the 2-dimensional identity $(i_\ell \Gamma_\mu^\lambda)(d\Gamma_\lambda^\mu) + \Gamma_\mu^\lambda (i_\ell d\Gamma_\lambda^\mu) = 0$, we obtain

$$Q_2^{-1}(\tilde{\xi}, \Gamma) \equiv T_2^{-1} - lT_3^0 = 2\Lambda_\mu^\lambda(\tilde{\xi}) R_\lambda^\mu + d \left(\Lambda_\mu^\lambda(\tilde{\xi}) \Gamma_\lambda^\mu \right) \quad (74)$$

with $R_\lambda^\mu = d\Gamma_\lambda^\mu + \Gamma_\rho^\mu \Gamma_\lambda^\rho$. Thus, one recovers the previous results modulo the substitutions $\tilde{\Gamma} \rightarrow \Gamma, \tilde{\xi} \rightarrow \xi$.

7 Concluding remarks

In conclusion, we would like to point out some open problems and make some comments on the uniqueness of the results as well as on the global formulation of the theory.

By concrete calculations, we have found that our method contains, amongst others, the polynomials Q_p^g and the corresponding descent equations of the homotopy operator approach. It would be nice to obtain this construction from first principles.

Up to s -exact terms, the quantity $\int \text{Tr } \bar{c} F^{n-1}$ appears to be the general solution of

$$s \int \text{Tr } [\bar{c} \hat{G}(A)] = 0, \quad (75)$$

where $\hat{G}$ depends only on the forms A and dA . For consistent anomalies, the general solution of the WZ-condition $s \int \text{Tr } cG(A) = 0$, with G depending on the forms A and dA , has been determined some time ago [19]. Quite recently, it was shown that the more general problem where G depends on the coefficients $A_\mu, \partial_\nu A_\mu, \dots$ does not lead to new solutions [20]; we expect the same to be true for covariant anomalies viewed as solutions of eq.(75) with $\hat{G}$ depending on A_μ and its partial derivatives up to finite order.

As pointed out in section 4, the pairs $(c, \bar{c})$ and $(s, \bar{s})$ of the vertical cohomology correspond to the usual generators and coboundary operators of the BRST/anti-BRST algebra without auxiliary fields. Therefore, the global geometric description given by Zhang [9] for these transformations can be viewed as a geometric interpretation of the BRST/anti-BRST symmetry. We should mention that similar descriptions on the fibre bundle of the gauge theory have already been presented by Bonora and Cotta-Ramusino [21]. It should be interesting to understand the relation between these different formalisms and, more specifically, to find a vertical cohomology interpretation of the result [21] that the non-integrated covariant anomalies are in one-to-one correspondence with the elements of $H_{loc}^{d,1}(P)$ (i.e. the local, d -th de Rham cohomology group with cocycles in the principal fibre bundle P over the d -manifold M , linear in the ghost parameter).

The descent equations for consistent anomalous terms are familiar in mathematics, where they appear in a more general setting known as the de Rham bar construction or as Milnor's construction of the universal bundle [22]. Our approach to covariant anomalies yields further descent equations and it should be worthwhile analyzing if these equations also fit into such a scheme and to get a better understanding of the general mathematical formalism underlying our construction.

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A Gauged BRST symmetry

As pointed out in reference [11], the gauged BRST algebra results from a combination of the global BRST symmetry and the ghost number symmetry, the latter being turned into a local one. The systematic approach developed in reference [11] requires the introduction of a new set of fields, the "tilde"-fields on which the local BRST operator is defined. In the following, we illustrate the general construction for the anti-BRST symmetry in YM-theories and in gravitational theories.

We note that it is possible to gauge simultaneously the BRST and anti-BRST symmetries, but such a generalization is not relevant for our purposes.

A.1 Yang-Mills theories

To start with, we rescale the ghost fields by the exponential of a real, commuting function $\omega(x)$,

$$\bar{c} \longrightarrow e^\omega \bar{c}, \quad c \longrightarrow e^{-\omega} c,$$

the sign of the exponent being dictated by the ghost number of the field. Then, we postulate the anti-BRST variations to preserve their form under this rescaling :

$$\begin{aligned} \bar{s}_{loc} A &= -\mathcal{D}(e^\omega \bar{c}) \\ \bar{s}_{loc}(e^\omega \bar{c}) &= -\frac{1}{2}[e^\omega \bar{c}, e^\omega \bar{c}] \\ \bar{s}_{loc}(e^{-\omega} c) &= 0. \end{aligned} \quad (76)$$

From $\bar{s}_{loc} \omega \equiv 0$, it immediately follows that

$$\begin{aligned} \bar{s}_{loc} \bar{c} &= -\frac{1}{2} e^\omega [\bar{c}, \bar{c}] \\ \bar{s}_{loc} c &= 0. \end{aligned}$$

To eliminate the dependence on derivatives of e^ω in the localized transformation law of A , one introduces an Abelian connection 1-form $\bar{\alpha}$ (with ghost number +1) associated to the local anti-BRST symmetry (76). This gauge field is assumed to transform according to

$$\begin{aligned} \bar{s}_{loc} \bar{\alpha} &= d e^\omega + e^\omega \bar{t} \\ \bar{s}_{loc} \bar{t} &= 0. \end{aligned} \quad (77)$$

Here, the external source field $\bar{t}$ denotes a 1-form with ghost number zero; it is to be identified with the connection of the ghost number symmetry. We now replace the derivative of e^ω occurring in the local transformation law of A by $\bar{s}_{loc} \bar{\alpha}$ and collect the resulting terms in a $\bar{s}_{loc}$ -variation :

$$\begin{aligned} \bar{s}_{loc} A &= -(d e^\omega) \bar{c} - e^\omega \mathcal{D} \bar{c} \\ &= -(\bar{s}_{loc} \bar{\alpha} - e^\omega \bar{t}) \bar{c} - e^\omega \mathcal{D} \bar{c} \\ &= -\bar{s}_{loc}(\bar{\alpha} \bar{c}) + \bar{\alpha}(\bar{s}_{loc} \bar{c}) - e^\omega(-\bar{t} \bar{c} + \mathcal{D} \bar{c}). \end{aligned}$$

Bringing the $\tilde{s}_{loc}$ -variation on the r.h.s. to the l.h.s., we end up with

$$\begin{aligned}\tilde{s}_{loc} \tilde{A} &= -e^\omega \left(\tilde{\mathcal{D}} \tilde{c} - \frac{1}{2} \tilde{\alpha} [\tilde{c}, \tilde{c}] - \tilde{t} \tilde{c} \right) \\ \tilde{s}_{loc} \tilde{c} &= -\frac{1}{2} e^\omega [\tilde{c}, \tilde{c}] \\ \tilde{s}_{loc} \tilde{c} &= 0\end{aligned}\quad (78)$$

where

$$\tilde{A} = A + \tilde{\alpha} \tilde{c}, \quad \tilde{c} = c, \quad \tilde{c} = \tilde{c} \quad (79)$$

and $\tilde{\mathcal{D}} = d + [\tilde{A}, \cdot]$. Eqs.(78) represent the gauged version of the anti-BRST symmetry and expressions (79) are the tilde-variables we are looking for.

A.2 Gravitational theories

As for YM-theories, we first rescale the ghost fields according to

$$\tilde{\xi} \longrightarrow \tilde{\xi}_\omega \equiv e^\omega \tilde{\xi}, \quad \xi \longrightarrow \xi_\omega \equiv e^{-\omega} \xi$$

and postulate the global anti-BRST variations (65)(66) to hold for $\tilde{\xi}_\omega, \xi_\omega$:

$$\begin{aligned}\tilde{s}_{loc} \Gamma_\mu^\lambda &= L_\mu \Gamma_\mu^\lambda - d(\partial_\mu \tilde{\xi}_\omega^\lambda) + \Gamma_\mu^\rho (\partial_\rho \tilde{\xi}_\omega^\lambda) - \Gamma_\rho^\lambda (\partial_\mu \tilde{\xi}_\omega^\rho) \\ \tilde{s}_{loc} \tilde{\xi}_\omega &= \frac{1}{2} [\tilde{\xi}_\omega, \tilde{\xi}_\omega] \\ \tilde{s}_{loc} \xi_\omega &= 0.\end{aligned}\quad (80)$$

As before, all derivatives of e^ω occurring in these equations can be replaced by $\tilde{s}_{loc} \tilde{\alpha}$ using eq.(77). Multiplying $\tilde{s}_{loc} \tilde{\xi}_\omega$ by $e^{-\omega}$, we immediately get

$$\begin{aligned}\tilde{s}_{loc} \tilde{\xi}^\mu &= \tilde{\xi}^\nu \partial_\nu (e^\omega \tilde{\xi}^\mu) \\ &= -\tilde{s}_{loc} (\tilde{\xi}^\nu \tilde{\alpha}_\nu \tilde{\xi}^\mu) + e^\omega \tilde{\xi}^\nu \partial_\nu \tilde{\xi}^\mu + \mathcal{O}(\tilde{\alpha}, \tilde{t}),\end{aligned}$$

henceforth

$$\tilde{\xi}^\mu = \tilde{\xi}^\mu + \tilde{\xi}^\nu \tilde{\alpha}_\nu \tilde{\xi}^\mu = (1 - i_{\tilde{t}} \tilde{\alpha}) \tilde{\xi}^\mu. \quad (81)$$

It should be noted [11] that this relation is easy to invert in two dimensions: $\tilde{\xi}^\mu = (1 + i_{\tilde{t}} \tilde{\alpha}) \tilde{\xi}^\mu$. Similarly, the replacement of derivatives of e^ω in $\tilde{s}_{loc} \Gamma_\mu^\lambda$ leads to

$$\begin{aligned}\tilde{s}_{loc} \Gamma_\mu^\lambda &= \tilde{s}_{loc} (-\tilde{\alpha} \partial_\mu \tilde{\xi}^\lambda - \tilde{\alpha} i_{\tilde{t}} \Gamma_\mu^\lambda + (d \tilde{\alpha}_\mu) \tilde{\xi}^\lambda - \tilde{\alpha}_\mu d \tilde{\xi}^\lambda - \Gamma_\mu^\rho \tilde{\alpha}_\rho \tilde{\xi}^\lambda + \Gamma_\rho^\lambda \tilde{\alpha}_\mu \tilde{\xi}^\rho) \\ &\quad + e^\omega (L_\mu \Gamma_\mu^\lambda - d(\partial_\mu \tilde{\xi}^\lambda) + \Gamma_\mu^\rho (\partial_\rho \tilde{\xi}^\lambda) - \Gamma_\rho^\lambda (\partial_\mu \tilde{\xi}^\rho)) + \dots\end{aligned}\quad (82)$$

In the course of this process, new terms again containing derivatives of e^ω emerge, e.g. from the $d e^\omega$ -terms in the first of eqs.(80). In principle, the whole procedure has to be reiterated until all derivatives in e^ω have dropped out (at the expense of introducing quadratic terms in $\tilde{\alpha}$). However, for the determination of the anomaly, it is enough to know the terms linear in $\tilde{\alpha}$, $d \tilde{\alpha}$ and (82) is sufficient to express Γ in the tilde-basis to this order, see eq.(70).

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