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**EXCLUSIVE EXPONENTIATION IN THE MONTE CARLO
YENNIE-FRAUTSCHI-SUURA APPROACH**

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ABSTRACT

A closer look is taken into various types of the so-called "exponentiation" procedures in QED. In particular the question of the difference between the common (ad hoc) inclusive exponentiation and the exclusive Yennie-Frautschi-Suura exponentiation is examined. The discussion is limited to the initial state bremsstrahlung in e^+e^- annihilation. Numerical results from the YFS2 Monte Carlo are used to illustrate the discussion.

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1. Introduction

The procedure of summing up to infinite order the infrared divergent contributions in QED, so called "exponentiation", is known to be an efficient way of including the higher order QED effects in the perturbative calculations. Typically, the first order perturbative calculation combined with the exponentiation procedure may give a result quite close to the complete unexponentiated second order result [1].

The common exponentiation procedure, say for the first order calculation, consists in combining in an ad hoc way two types of the calculations. The conventional calculation provides the energy spectrum of the single hard photon. On the other hand, from the classical papers on the exponentiation [2,3] one knows the distribution of the *total* energy of many soft photons. The two distributions are interpolated at some intermediate energy point and the resulting formula is used for predicting (successfully) the fermion pair effective mass spectrum and the total cross section.

The modern detectors can control, however, the topology of the experimental events (photon multiplicities and four-momenta) to a large extent and the above exponentiation procedure is not sufficient because it only provides a rather limited information on the inclusive spectra and the total event rate.

A closer look at the classical work on the exponentiation [2] reveals that the exclusive multi-photon distributions are well defined but they were not exploited for practical calculations due to a necessity of integration over the multibody Lorentz-invariant phase space.

In the recent works we have developed technical means for integrating the relevant phase space integrals using the Monte Carlo method [4,5,6]. The related computer programs simulate directly the multiple real photon emission - something what is very valuable for the present experiments in $e^\pm$ annihilation.

In the talk we shall shortly comment on the relation of the common inclusive exponentiation to the exclusive one. Some examples of the numerical results will be given and the applicability limits of both methods will be indicated.

The discussion will be mainly related to the YFS2 Monte Carlo [6] for the initial state bremsstrahlung in the electron-positron annihilation which is also a part of the KORALZ program [7]. This talk together with that of ref. [8] may be treated as an introduction to more technical papers on the exclusive exponentiation like [5,6].

The outline of the talk is the following: In the first part we shall recall what is the common (ad hoc) exponentiation procedure and we shall point out what are the conceptual and numerical uncertainties related to this procedure. Then, in the second part, we describe briefly the Yennie-Frautschi-Suura exponentiation scheme, and its implementation in the Monte Carlo, and we show how the above uncertainties can be resolved or better understood. Section 4 contains some summary remarks.

2. Common exponentiation – unanswerable questions

In the following we shall recall what is usually meant as an exponentiation procedure (we call it inclusive or ad hoc exponentiation) on the textbook example of exponentiating the Bonneau-Martin, $O(\alpha)$, expression for the total cross section in the annihilation process

$$e^+(p_1) + e^-(p_2) \rightarrow \mu^+(q_1) + \mu^-(q_2) (+\gamma(k)).$$

The Bonneau-Martin formula reads as follows

$$\sigma = \int_0^1 dv \rho_{\text{BM}}(v) \sigma^{\text{Born}}(s(1-v)), \quad (2.1)$$

where

$$\rho_{\text{BM}}(v) = \delta(k)(1 + \gamma \ln \varepsilon + \delta_{VS}^{(1)}) + \gamma \Theta(v - \varepsilon) \frac{1 + (1 - v)^2}{2v} \quad (2.2)$$

$$\gamma = 2 \left(\frac{\alpha}{\pi} \right) (L - 1), \quad L = \ln \frac{s}{m_e^2}, \quad \delta_{VS}^{(1)} = \left(\frac{\alpha}{\pi} \right) \left(\frac{3}{2} L - 2 + \frac{1}{3} \pi^2 \right).$$

and $v = 1 - (q_1 + q_2)^2 / (p_1 + p_2)^2 = 1 - s' / s$. In the single photon case v is simply a fraction of the beam energy carried by the photon, $v = 2k^0 / \sqrt{s}$.^{*} What is the usual ad hoc "exponentiation" procedure in the case of the above formula? It amounts to the following, essentially by hand, modification of the photon spectrum

$$\rho_{\text{KF}} = \gamma v^{\gamma-1} (1 + \delta_{VS}^{(1)} + \delta_H^{(1)}(v)), \quad \delta_H^{(1)}(v) = \gamma(-1 + \frac{1}{2}v). \quad (2.3)$$

There is definitely some freedom in the way it is done and we have followed here the prescription of ref. [10].

Let us examine more closely the question: *What is the meaning of the above procedure?* The exponentiated photon distribution $\rho(v)$ has the following two properties:

1. The original Bonneau-Martin $\rho(v)$ distribution (2.2) and the exponentiated one (2.3) coincide, up to terms of $O(\gamma^2)$, for hard photon, $v \sim 1$.
2. The main modification is made in soft photon region, $v \sim 0$, and it is done in such a way that, the cross section $\sigma(v < \varepsilon)$, when expanded up to $O(\gamma)$, remains the same.

There are convincing reasons for such a modification for $v \ll 0$. It is well known that in the region $v < \varepsilon \ll 1$ one has to sum up the contributions from the infinite number of soft virtual and real photons and, furthermore, it is known quite precisely [2] what are the resulting differential and integrated cross sections $d\sigma/dv$ and $\sigma(v < \varepsilon)$.[†]

Summarizing, the exponentiated formula (2.3) is well founded (A) in the region $v \sim 1$ where it becomes

$$\rho_{\text{hard}} = \gamma \frac{1 + (1 - v)^2}{2v} \quad (2.4)$$

and represents the *exclusive* single real photon bremsstrahlung contribution (as obtained directly from the Feynman rules) and (B) in the soft region $v \ll 1$

$$\rho_{\text{soft}} = \gamma v^{\gamma-1} (1 + \delta_{VS}^{(1)}), \quad (2.5)$$

where it represents the *inclusive* cross section summed/integrated over the infinite number of real and virtual soft photons with the total energy constrained to some small value: $\frac{2}{\sqrt{s}} \sum_{i=1}^n k_i^0 = v$, $v \ll 1$.

Our main observation is the following: *the usual exponentiation procedure is essentially the interpolation between the two expressions: exclusive hard of eq. (2.4) and inclusive soft of eq. (2.5).* Strictly speaking interpolation is made by ad hoc recipe following some sort of "continuity principle".

* In the multiple photon case $v \sim \sum_i 2k_i^0 / \sqrt{s}$ as long as there are no two anti-collinear hard photons.

† The necessity of such a modification is also signaled by the fact that the Bonneau-Martin cross section $\sigma(v < \varepsilon)$, in the limit $\varepsilon \rightarrow 0$ is ill defined - it becomes negative.

* Of course, such an ansatz can be quite successful. Its validity may be checked experimentally.

Fig. 1 The various types of the distribution $v\rho(v)$. The two cases (a) and (b) represent $O(\alpha)$ and $O(\alpha^2)$ results correspondingly. The solid curve rising towards $v = 1$ represents $v\rho_{\text{soft}}(v)$, see for example eq. (2.5), and the other solid curve represents $v\rho_{\text{hard}}(v)$, see eq. (2.4). The dashed curve shows the distribution $v\rho(v)$ exponentiated/interpolated according to ref. [10] (see also eq. (2.3)) and the dotted curve shows exponentiation/interpolation according to ref. [6]. Note that the area under the "soft" curve $v\rho_{\text{soft}}(v)$ is roughly equal to unity.

In the Fig. 1 (a) it is shown how the above interpolation works in practice. We plot the two source distributions $v\rho_{soft}(v)$ and $v\rho_{hard}(v)$ and the interpolation $v\rho_{KF}(v)$ as a function of $\ln v$ at $s = M_Z^2$. As we see, the exponentiated distribution coincides with $v\rho_{hard}(v)$ in the region (A), $0.3 < v < 1$, and with $v\rho_{soft}(v)$ in the region (B), $0 < v < 0.002$. The interpolation takes place within the interval (C) $0.02 < v < 0.3$.

The exponentiated distribution may be quite uncertain in the transition region (C). Furthermore, it is not clear what is the precise meaning of the result of the interpolation. This is because we interpolate among the quantities which represent rather different things: in the soft region (B) the contribution from many photons and in the hard region (A) the "Born-like" contribution from one photon only. The other (soft) photons disappear somehow in between, in the intermediate region (C). The related questions are: How are the many soft photons distributed over the multiphoton phase space? Are their momenta limited somehow? Do the resulting cross sections and distributions depend on the limits imposed on the (additional) soft photons? These questions are quite relevant experimentally since modern detectors can see directly and indirectly photons down to 100 MeV and because the precision level in the LEP experiments will reach the level of $\delta\sigma/\sigma \simeq 10^{-3}$.

We postpone answering these questions to the next Section and we shall elaborate further on some interesting aspects of the "common exponentiation". More specifically, the usual way of getting some hint on the uncertainty resulting from the interpolation among soft and hard formulas is to take another "ad hoc" interpolation or to go to a higher perturbative order. Figs. 1 (a) and (b) illustrate both cases.

In Fig. 1 (a) we show what is the result of switching to another $O(\alpha)$ exponentiation/interpolation recipe. The dotted curve is obtained following the exponentiation prescription of ref. [6]^{*}

$$\begin{aligned} \rho_{YFS}(v) &= \frac{e^{-C\gamma}}{\Gamma(1+\gamma)} e^{\delta_{YFS}} \gamma v^{\gamma-1} (1 + \Delta_S^{(1)} + \Delta_H^{(1)}(v)), \quad \delta_{YFS} = \left(\frac{\alpha}{\pi}\right) \left(\frac{1}{2}L - 1 + \frac{\pi^2}{3}\right), \\ \Delta_S^{(1)} &= \left(\frac{\alpha}{\pi}\right)(L-1), \quad \Delta_H^{(1)}(v) = v \left(-1 + \frac{1}{2}v\right), \quad C = 0.57721566... \end{aligned} \quad (2.6)$$

As we see in Fig. 1 (a) the above change of the type of exponentiation affects the distribution $d\sigma/dv$ around $v \sim 0.2$ by 2-3%.

The advantage of this new (also ad hoc) exponentiation is that the soft photon limit ($v \ll 1$) of the new formula (2.6) coincides precisely with the exact expression resulting from the soft photon resummation, as given in ref. [2][†],

$$\begin{aligned} d\sigma/dv &= \sigma_{\text{Born}}(s) e^{\delta_{YFS}} \int \frac{dx}{2\pi} \exp \left[ixvW + \int \frac{d^3k}{k^0} \tilde{S}(p_1, p_2, k) (e^{-ixk} - 1) \right] \\ &= \sigma_{\text{Born}}(s) e^{\gamma \ln s + \delta_{YFS}} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=1}^n \int_{k_i^0 \geq \epsilon W} \frac{d^3k_i}{k_i^0} \tilde{S}(p_1, p_2, k_i) \delta^4(vW - \sum_{i=1}^n k_i^0) \quad (2.7) \\ &= \sigma_{\text{Born}}(s) \frac{e^{-C\gamma}}{\Gamma(1+\gamma)} e^{\delta_{YFS}} \gamma v^{\gamma-1}, \quad W = \frac{1}{2}\sqrt{s}, \end{aligned}$$

where

^{*} The main difference is that $\gamma v^{\gamma-1}$ multiplies now the hard photon correction $\Delta_H(v)$.

[†] We set here $\Delta_S = 0$ for simplicity.

$$\begin{aligned}\delta_{\text{YFS}} &= 2\alpha B + \bar{B} \\ &= \text{Re} \frac{i\alpha}{4\pi^2} \int \frac{d^4 k}{k^2} \left(\frac{2p_1 - k}{k^2 - 2p_1 \cdot k} + \frac{2p_2 + k}{k^2 + 2p_2 \cdot k} \right)^2 + \int_{k^0 < W} \frac{d^3 k}{k^0} \tilde{S}(p_1, p_2, k),\end{aligned}\quad (2.8)$$

$$\tilde{S}(p_1, p_2, k) = -\frac{\alpha}{4\pi^2} \left(\frac{p_1}{p_1 \cdot k} - \frac{p_2}{p_2 \cdot k} \right)^2.$$

We would like to stress that the eqs. (2.7) are exact identities and we refer the reader to refs. [6] and [8] for formal proofs – in particular on the transition from the Mellin type inclusive representation to the exclusive representation with the explicit integration/summation over soft photons.

In Fig. 1 (b) we show what happens if we go to $O(\alpha^2)$ exponentiation, i.e. interpolation. The starting “building blocks” for exponentiation are the “soft distribution”*

$$\rho_{\text{soft}}(v) = \frac{e^{-C\gamma}}{\Gamma(1+\gamma)} e^{\delta_{\text{YFS}}} \gamma v^{\gamma-1} (1 + \Delta_S^{(1)} + \Delta_S^{(2)}), \quad \Delta_S^{(2)} = \frac{1}{2} \left(\frac{\alpha}{\pi} \right)^2 L^2 \quad (2.9)$$

which includes summation over infinite number of soft photons and the “hard distribution”

$$\begin{aligned}\rho_{\text{hard}} &= \gamma \frac{1 + (1-v)^2}{2v} + \left(\frac{\alpha}{\pi} \right)^2 L^2 \left(-\frac{1 + (1-v)^2}{v} \ln(1-v) \right. \\ &\quad \left. + (2+v) \left(\frac{1}{2} \ln(1-v) - 2 \ln v \right) - \frac{5}{2} - \frac{1}{2}(1-v) + \frac{3}{v} + \frac{4}{v} \ln v \right),\end{aligned}\quad (2.10)$$

representing the second order (double bremsstrahlung) result without any exponentiation.

Before we interpolate the two distributions, let us note the remarkable difference with the $O(\alpha)$ case. The “hard photon” curve does not cross the “soft” one any more. In fact the “hard” one goes to $-\infty$ at $v \rightarrow 0$. This is related to the fact that, in the $O(\alpha^2)$, at fixed v , there are two types of contributions to $\rho(v)$: one from two real hard photons and another from one real and one virtual photon. The latter one (as noted in ref. [6]) becomes negative for $v \rightarrow 0$.

Now we “exponentiate” the above $O(\alpha^2)$ raw distributions (2.9) and (2.10), that is, we simply *interpolate* them. We do it either with the help of the formula (2.3) where we append $\delta_{VS}^{(1)}$ and $\delta_H^{(1)}$, see ref. [10,12], with the following $O(\alpha^2)$ contributions

$$\begin{aligned}\delta_{VS}^{(2)} &= \left(\frac{\alpha}{\pi} \right)^2 L^2 \left(\frac{9}{8} - \frac{1}{3} \pi^2 \right), \quad \delta_H^{(2)}(v) = \left(\frac{\alpha}{\pi} \right)^2 L^2 \left(-\frac{1 + (1-v)^2}{v} \ln(1-v) \right. \\ &\quad \left. + (2+v) \left(\frac{1}{2} \ln(1-v) - 2 \ln v \right) - \frac{5}{2} - \frac{1}{2}(1-v) \right),\end{aligned}\quad (2.11)$$

or with the help of eq. (2.6) where we add $\Delta_S^{(2)}$ of eq. (2.9) and

$$\Delta_H^{(2)}(v) = \left(\frac{\alpha}{\pi} \right) L \left(-\frac{1}{4} (4 - 6v + 3v^2) \ln(1-v) - v \right), \quad (2.12)$$

see ref. [6]. The difference among the two types of the exponentiation/interpolation is now rather small, of order 0.2% at most.

* In these expressions we omitted some subleading, numerically unimportant, terms as in refs. [11,12].

Let us conclude by recalling main deficiencies of the common exponentiation:

1. There is no general and systematical framework of the type "calculations order by order" for this type of exponentiation – it is a matter of inventing recipes.
2. The phase space for real (soft) photons remains unspecified.
3. Finally, one should keep in mind that the common exponentiation is defined for some specific choice of the "principal" kinematical variable ($v = 1 - s'/s$ in the case presented above). It has to be repeated from the scratch if one decides to switch to another variable.[†]

3. Yennie-Frautschi-Suura exclusive exponentiation

The aim of the following part of the talk is to explain what is the Yennie-Frautschi-Suura exclusive exponentiation. We shall show that the questions which could not be answered in the case of the common (ad hoc) exponentiation can be here either answered or shown to be not relevant. Let us indicate immediately what are the answers to the most disturbing questions from the former Section: The question "How many of the soft photon are there and where are they?" is answered in the Yennie-Frautschi-Suura exclusive exponentiation as follows: there are no events without soft photons – soft real photons are always there; there is no sharp distinction in a form of cut-off between soft and hard real photons – the important and relevant distinction (in a given perturbative order) is between the part of the phase space where the actual differential cross section is or is not sufficiently precise with respect to the (experimental) precision requirements in mind. Furthermore, this type of exponentiation offers a definite prescription how, by going to higher orders, to improve in a systematical way on the precision of the calculation and, since Monte Carlo method is used, there is no need to rely on some specific choice of a "principal variable" for exponentiation nor on some specific kinematical cut-off's.

In the following we shall present the main aspects of the Yennie-Frautschi-Suura exponentiation. Only some most important aspects of the Monte Carlo phase space integration algorithm will be mentioned.

3.1 GENERAL PROPERTIES

Let us take as a starting point of discussion the Yennie-Frautschi-Suura formula for the entire perturbative series in a form suitable for the Monte-Carlo integration[‡]

$$\begin{aligned} \sigma = & \int \frac{d^3 q_1}{q_1^0} \frac{d^3 q_2}{q_2^0} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=1}^n \int_{k_i^0 < \epsilon W} \frac{d^3 k_i}{k_i^0} \tilde{S}(p_1, p_2, k_i) \delta^4(p_1 + p_2 - q_1 - q_2 - \sum_{i=1}^n k_i) \\ & e^{\gamma \ln \epsilon + \delta_{YFS}} \left\{ \tilde{\beta}_0(p_1, p_2, q_1, q_2) + \sum_{l=1}^n \frac{\tilde{\beta}_1(p_1, p_2, q_1, q_2, k_l)}{\tilde{S}(k_l)} \right. \\ & \left. + \frac{1}{2} \sum_{\substack{l,j=1 \\ l \neq j}}^n \frac{\tilde{\beta}_2(p_1, p_2, q_1, q_2, k_l, k_j)}{\tilde{S}(k_l) \tilde{S}(k_j)} + \dots \right\}. \end{aligned} \quad (3.1)$$

The other, Mellin-type representation, similar to that in eq. (2.7) used in the original YFS paper [2] and in many other papers on exponentiation [13] is algebraically and numerically equivalent to the above representation and we refer the reader to ref. [6] and [8] for further comments on the transition from one to another formulation.

[†] For example to sum of photon energies, upper cut on energy of any photon or a combination of the angular and energy cut-off.

[‡] Here and in the rest of the paper we limit ourself to the initial state bremsstrahlung only.

Let us summarize on the most important properties of the above multi-distribution:

1. Eq. (3.1) represents the entire perturbative series (no approximations) reorganized in such a way that all infrared divergences cancel in the form-factor $\exp(\gamma \ln \epsilon + \delta_{YFS})$.
2. The total cross section and any measurable quantity do not depend on the infrared cut-off (dummy) parameter ϵ which may be set arbitrarily low.
3. In the Monte Carlo real photons are explicitly generated above the energy threshold ϵW all over the entire phase space, i.e., many hard photons are generated as well. The average multiplicity of generated photons is approximately $\gamma \ln \frac{1}{\epsilon}$.
4. The functions $\tilde{\beta}_i$ are infrared free and are calculable from the Feynman diagrams order by order. There is certain amount of freedom in defining $\tilde{\beta}'s$ related to the way of extracting the factors $\tilde{S}(p_1, p_2, k)$ from the differential distributions provided by the Feynman rules.[◊]
5. In the N -th perturbative order the sum of $\tilde{\beta}'s$ gets truncated and we deal in practice with $\tilde{\beta}_i, i = 1, 2, \dots, N$. This we call the N -th order exclusive YFS exponentiation. In particular, in the lowest order exclusive exponentiation^{*} we deal only with $\tilde{\beta}_0$.

The (truncated) formula (3.1) answers immediately the principal question of the previous Section: *Where are soft/hard real photons and how they are distributed in the phase space.* As we have noted already, the real photons are distributed over the complete phase space and, as we see, the eq. (3.1) defines the differential cross section everywhere. (The Monte Carlo is merely an exact, up to a statistical error, method of numerical integration over the phase space.) The limitation is hidden somewhere else. As usual in the perturbative expansion, due to the truncation in series of $\tilde{\beta}'s$, the differential cross section may be not precise enough. If it is true then one has to go to higher order and include more $\tilde{\beta}'s$ in the expansion. In the following we shall discuss this point in more detail.

We would like to stress that the truncation of the series in $\tilde{\beta}'s$ and the truncation of the perturbative expansion for each $\tilde{\beta}$ is the normal perturbative procedure. In fact we adhere to the following (conservative) algorithm of the perturbative calculation:

- {1} Start with lowest order calculation $n = 0$,
- {2} Calculate the quantity of interest $X^{(n)}$ in order n and $n + 1$,
- {3} If $|X^{(n+1)} - X^{(n)}| < \delta$ where δ is required (experimental) precision then finish calculation else go to point {2}.

In practice such an algorithm would require to perform $O(\alpha^3)$ calculation of the QED calculation around Z and, in our opinion, such a calculation should be done at least in some approximation, to estimate the "theoretical precision" of the $O(\alpha^2)$ calculation [14].

◊ As long as the complete sum over $\tilde{\beta}'s$ is kept this freedom does not influence the integrated and/or differential cross sections at all. This is analogous to the first step in the renormalization group technique – the scheme dependence comes when the series gets truncated.

• The results may now slightly depend on the way of defining $\tilde{\beta}$ functions.

3.2 ZERO ORDER EXPONENTIATION

The zero-order exponentiation, with $\tilde{\beta}_0$ alone, is the case when only the most infrared-singular part is isolated and summed up to infinite order. As was proven in ref. [2] the virtual photon divergences sum up to $e^{\alpha B}$ and the real photon leading singularity is proportional to $\prod_{i=1}^n \tilde{S}(k_i)$. The function $\tilde{\beta}_0$ is what remains after factorization of these singularities

$$\rho_n(p_i, q_i, k_1, k_2, \dots, k_n) = \prod_{i=1}^n \tilde{S}(p_1, p_2, k_i) e^{2\alpha \text{Re} B} \tilde{\beta}_0(p_1, p_2, q_1, q_2). \quad (3.2)$$

It should be stressed that for any number n of real photons we get the same residue $\tilde{\beta}_0$! Of course, it is calculable most easily for $n = 0$. In general, $\tilde{\beta}_0$ is an expansion in a series of the coupling constant and in the zero-order case it coincides simply with the Born angular distribution

$$\tilde{\beta}_0^{(0)}(p_1, p_2, q_1, q_2) = \frac{d\sigma^{\text{Born}}}{d\Omega}(\mathcal{R}p_1, \mathcal{R}p_2, \mathcal{R}q_1, \mathcal{R}q_2) = \frac{d\sigma^{\text{Born}}}{d\Omega}(\theta_{\mathcal{R}}, s(1-v)). \quad (3.3)$$

The only complication is that, strictly speaking, $d\sigma^{\text{Born}}/d\Omega$ is defined within the 2-body phase space $p_1 + p_2 - q_1 - q_2 = 0$ while $\tilde{\beta}_0$, see eq. (3.2) has to be also defined for $p_1 + p_2 - q_1 - q_2 \neq 0$, see also the discussion in ref. [2]. In eq. (3.3) we do it by means of mapping $p_i \rightarrow \mathcal{R}p_i$, $q_i \rightarrow \mathcal{R}q_i$ such that $\mathcal{R}p_1 + \mathcal{R}p_2 - \mathcal{R}q_1 - \mathcal{R}q_2 = 0$. (The angle $\theta_{\mathcal{R}}$ is calculated using the reduced momenta.) This so called "reduction procedure" has to obey two basic rules: (i) it has to become an identity transformation when $p_1 + p_2 - q_1 - q_2 \rightarrow 0$ and (ii) it should be the same for arbitrary configuration of the photons which sum up to the same total four-momentum $K = \sum_{i=1}^n k_i = p_1 + p_2 - q_1 - q_2$.[∇] In eq. (3.3) we have implicitly imposed the additional condition $(\mathcal{R}p_1 + \mathcal{R}p_2)^2 = (q_1 + q_2)^2 = s(1-v)$ anticipating the resonance formation in the s -channel. The reduction is done in practice by means of boosts, rotations and rescaling of the momenta, see ref. [2] for more details. Summarizing, the zero-order version of the eq. (3.1) reads[†]

$$\sigma = e^{\gamma \ln \epsilon + \delta_{\text{YFS}}} \int dv \frac{d\Omega_q}{4\pi} \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=1}^n \int_{k^0 < \epsilon W} \frac{d^3 k_i}{k_i^0} \tilde{S}(p_1, p_2, k_i) \delta^4\left(v - \frac{(p_1 + p_2 - \sum k_i)^2}{s}\right) \frac{d\sigma^{\text{Born}}}{d\Omega}(\theta_{\mathcal{R}}, s(1-v)). \quad (3.4)$$

The above zero-order exclusive exponentiation is of little practical use and we presented it in order to discuss some aspects of the inclusive exponentiation on the maximally simple example. The main deficiency of the zero-order exponentiation is twofold: (i) The distribution of the real photons is a simple product of infrared factors $\prod \tilde{S}(k_i)$ – something which could be wrong if there is at least one hard photon, (ii) The virtual correction of the type $1 + \delta_S$ is missing. The total cross section may be imprecise by about 5% due to these defects. This will be corrected in the $O(\alpha)$ case.

[∇] In particular the reduction knows nothing about the number of photons.

[†] Note that a distribution very close to this one is generated in the early stage in the YFS2 Monte Carlo – the contributions due to $\tilde{\beta}_1$ and $\tilde{\beta}_2$ are added by rejection.

3.3 FIRST AND SECOND ORDER EXPONENTIATION

In the $O(\alpha)$ case the $\tilde{\beta}_0$ is calculated (up to a normalization constant) as follows

$$\begin{aligned}\tilde{\beta}_0^{(1)} &= |e^{-2\alpha B} \mathcal{M}(\mathcal{O}(\alpha))|^2 = (1 - 2\alpha \text{Re} B)(1 + 2\text{Re} F_1) |\mathcal{M}_{\text{Born}}|^2 \\ &= (1 - 2\alpha \text{Re} B + 2\text{Re} F_1) |\mathcal{M}_{\text{Born}}|^2 = (1 + \Delta_S^{(1)}) |\mathcal{M}_{\text{Born}}|^2\end{aligned}\quad (3.5)$$

where $\mathcal{M}(\mathcal{O}(\alpha)) = (1 + F_1) \mathcal{M}_{\text{Born}}$ is the one loop result obtained directly from the Feynman rules (Born + vertex correction). The reduction procedure is understood to be applied here similarly as in the previous zero order case. We expect, now, that the differential cross section will be improved in the limit $v \rightarrow 0$ due to the $1 + \Delta_S^{(1)}$ factor.

What about the cases with hard photons? In the case of $O(\alpha)$ exponentiation we include in (3.1) the $\tilde{\beta}_1$ term. Its definition results, see ref. [2], from the careful analysis of the distributions obtained from the Feynman rules. It is quite easy, however, to deduce its definition indirectly as follows: If in the sum

$$d\sigma_n \sim \tilde{\beta}_0 \prod_{i=1}^n \tilde{S}(k_i) + \sum_{l=1}^n \tilde{\beta}_1(k_l) \prod_{i \neq l} \tilde{S}(k_i), \quad (3.6)$$

all photons are soft, $k_i^0 \ll \sqrt{s}/2$, then the second sum is negligible because $\tilde{\beta}_1(k)$ is not singular in the infrared limit $k^0 \rightarrow 0$. If one photon, say $l = L$, is hard, $k_L^0 \sim \sqrt{s}/2$, then in the second sum only this term, $l = L$, is non-negligible and we can rewrite (3.6) as follows

$$\tilde{S}(k_1) \dots \tilde{S}(k_{L-1}) (\tilde{\beta}_0(k_L) \tilde{S}(k_L) + \tilde{\beta}_1(k_L)) \dots \tilde{S}(k_n) = \tilde{S}(k_1) \dots \tilde{S}(k_{L-1}) \rho_1(k_L) \dots \tilde{S}(k_n),$$

where the $\rho_1(k)$ should obviously coincide with the conventional single bremsstrahlung matrix element squared. The above implies the definition of the $\tilde{\beta}_1(k)$, see also refs. [2,6,15,5],

$$\tilde{\beta}_1^{(1)}(k) = \rho_1^{(1)}(k) - \tilde{S}(k) \tilde{\beta}_0^{(0)}. \quad (3.7)$$

Again, as is the case of $\tilde{\beta}_0$, the definition of $\tilde{\beta}_1$ must necessarily include the reduction procedure which eliminates from the four-momentum balance all photons but one.[†]

It should be stressed that in the above definition one uses the lowest order* $\tilde{\beta}_0^{(0)}$ while in the first term of the sum (3.6) one uses rather the $O(\alpha)$ $\tilde{\beta}_1^{(1)}$. To summarize: *the $\tilde{\beta}_1$ contribution is negligible if all photons are soft (compared with beam momentum) and its role is to correct the distribution of the events with one hard and (possibly) many soft photons.*

To proceed from the $O(\alpha)$ to $O(\alpha^2)$ exponentiation is quite straightforward. The $\tilde{\beta}_0$ is calculated now as follows

$$\begin{aligned}\tilde{\beta}_0^{(2)} &= |e^{-2\alpha B} \mathcal{M}(\mathcal{O}(\alpha^2))|^2 = (1 - 2\alpha \text{Re} B + \frac{1}{2}(2\alpha \text{Re} B)^2)(1 + 2\text{Re} F_1) |\mathcal{M}_{\text{Born}}|^2 \\ &= (1 + \Delta_S^{(1)} + \Delta_S^{(2)}) |\mathcal{M}_{\text{Born}}|^2\end{aligned}\quad (3.8)$$

where

[†] The reduction procedure for $\tilde{\beta}_1(k_l)$ which preserves the momentum k_l and eliminates the other ones has to fulfil another important constraint [6]: it has to preserve the value of $\tilde{S}(p_1, p_2, k_l) \simeq 1/k_{l,T}^2$, otherwise there is no guarantee that $\tilde{\beta}_1$ is small in the region $E_{\text{beam}} > k_l^0 > m_e$.

* Otherwise $\tilde{\beta}_1$ would be infrared singular at $k^0 \rightarrow 0$.

$$\Delta_S^{(2)} = \frac{1}{2} \left(\frac{\alpha}{\pi} \right)^2 L^2 + \left(\frac{\alpha}{\pi} \right)^2 \left\{ - \left(-\frac{13}{16} + \frac{3}{2} \zeta(2) - 3 \zeta(3) \right) L - \frac{16}{5} \zeta(2)^2 + \frac{51}{8} \zeta(2) + \frac{13}{4} - \frac{9}{2} \zeta(3) - 6 \zeta(2) \ln(2) \right\} \quad (3.9)$$

is the exact $O(\alpha^2)$ contribution calculated using the $O(\alpha^2)$ vertex correction F_1 of ref. [1]. Note that the resulting $\Delta_S^{(2)}$ is remarkably shorter than the original $O(\alpha^2)$ vertex function F_1 .

The $O(\alpha^2)$ $\tilde{\beta}_1$ is obtained from

$$\tilde{\beta}_1^{(2)}(k) = \rho_1^{(2)}(k) - \bar{S}(k) \tilde{\beta}_0^{(1)}. \quad (3.10)$$

where one substitutes $O(\alpha^2)$ distribution $\rho_1^{(2)}$ which includes contributions from one real and up to one virtual photons[†] and the $O(\alpha)$ $\tilde{\beta}_0^{(1)}$. The new object in $O(\alpha^2)$ exponentiation is now

$$\tilde{\beta}_2^{(2)}(k_1, k_2) = \rho_1^{(2)}(k_1, k_2) - \bar{S}(k_1) \tilde{\beta}_1^{(1)}(k_2) - \bar{S}(k_2) \tilde{\beta}_1^{(1)}(k_1) - \bar{S}(k_1) \bar{S}(k_2) \tilde{\beta}_0^{(0)} \quad (3.11)$$

which is constructed using the tree level distributions only[‡] (no virtual photons). The use of reduction procedure which eliminates all but two photons is understood implicitly.

3.4 COMPARISONS, NUMERICAL RESULTS

We are now in position to compare the ad hoc inclusive and YFS exclusive exponentiations and to answer in a more detail some “unanswerable” questions from the first part of our talk, illustrating the discussion with numerical Monte Carlo results.

As already indicated, the question “Where are the real soft photons?” referring to the matching, in the ad hoc exponentiation, the exclusive ρ_{hard} of region (A) and the inclusive ρ_{soft} of region (B) is in the present Yennie-Frautschi-Suura exclusive exponentiation *not really relevant* because there is no exclusive region (A) – hard photon can always be accompanied in this case with some real soft photons. There is also no mysterious transition region (C) any more.[°] Nevertheless the distribution $v\rho(v)$ which was the object of common exponentiation in the previous Section can be obtained from the Monte Carlo representing the exclusive Yennie-Frautschi-Suura exponentiation and we may check that indeed the Monte Carlo result is rather close to the result of the ad hoc exponentiation/interpolation procedure. In Fig. 2 compare the $O(\alpha)$ and $O(\alpha^2)$ results from the Monte Carlo program YFS2 [6] with the “exponentiated distributions” from the previous Section. The agreement is very good and the analogous excellent agreement on the total cross section, see ref. [6], is a direct consequence of the agreement for $\rho(v)$. Let us also note that once the exponentiation-type Monte Carlo is set up then we may easily produce “exponentiated” distribution not only for v but for any other variable (acollinearity etc.).

[†] The distribution $\rho_1^{(2)}$ is not available in the literature and in [6] we used the leading/subleading approximation. The contributions of $O((\alpha/\pi)^2)$ are neglected in this way.

[‡] In ref. [6] we use the leading/subleading logarithmic approximation for $\rho_2^{(2)}$.

[°] The role of interpolation by “continuity principle” is now, to some extent, played by the more complicated but better defined “reduction procedure”. As it was already explained, it is a natural consequence of the process of extracting the infrared real photon singularities.

Fig. 2 The histogram represents the distribution $v\rho_{MC}(v) = (d\sigma/d\ln v)/\sigma^{\text{Born}}(s(1-v))$, from the Monte Carlo sample of $4 \cdot 10^5$ events produced with the help of the YFS2 program [6]. The continuous lines are the same as in Fig. 1 (b) and represent the $O(\alpha^2)$ "common exponentiation". This distribution depends very little on $\sqrt{s}$. We have used here $\sqrt{s} = 40\text{GeV}$ and we cut $v < 0.8$.

It is also clear that in the presented exclusive exponentiation there is *no sharp distinction*, for instance in a form of a cut-off, among the hard and soft real photons. The only distinction in the phase space is such that in some part of it the YFS matrix element of eq. (3.1) is less precise and in another one more precise than the actual (experimental) precision requirement. The precision depends on the actual perturbative order of the calculation and we shall try to illustrate this point by looking into $O(\alpha^0)$, $O(\alpha^1)$ and $O(\alpha^2)$ results and their differences.

In Fig. 3 a) we present the $O(\alpha^0)$ and $O(\alpha^1)$ Monte carlo results for $v\rho(v)$ and their difference. The difference between the two is most visible at $v > 0.1$ and it comes from the fact that the soft photon approximation is not very adequate the real hard photon – it could be wrong by the factor two (at most). In technical terms, the $O(\alpha^0)$ calculation does not include the $\tilde{\beta}_1$ contribution. It should be stressed however that the difference persists at $v \rightarrow 0$ limit and this is due to the lack of the virtual correction Δ_1 in the $O(\alpha^0)$ version of the $\tilde{\beta}_0$. For these reason the differences between $O(\alpha^0)$ and $O(\alpha^1)$ total cross sections may be of order 5%. This can be actually seen directly in Fig. 3 where we indicate a rectangle which corresponds roughly to 1% of the integrated cross section.

In Fig. 3b) we examine the analogous difference between $O(\alpha^1)$ and $O(\alpha^2)$ Monte Carlo result for $v\rho(v)$. The difference is, as expected, much smaller. According to

• We say roughly because $\rho(v)$ is weighted with the Born cross section $\sigma(s(1-v))$.

Fig. 3 The distribution $v\rho_{MC}(v) = (d\sigma/d\ln v)/\sigma^{\text{Born}}(s(1-v))$, from the Monte Carlo sample of $4 \cdot 10^5$ events produced with the help of the YFS2 program [6]. In the case (a) the upper solid line shows $O(\alpha^0)$ result and the dashed line represents $O(\alpha^1)$ result. The lower solid line shows the difference of the previous two. The case (b) includes the corresponding $O(\alpha^1)$ and $O(\alpha^2)$ results. We have used here $\sqrt{s} = 40\text{GeV}$ and we cut $v < 0.8$.

what was said in the beginning of this Section the difference between $O(\alpha^1)$ and $O(\alpha^2)$ may be regarded as a precision estimate of the $O(\alpha)$ calculation. In Fig. 3 b) we see that, as far as the integrated cross section is concerned, the precision of the $O(\alpha)$ exponentiated calculation is below 1% level. To estimate the precision of the $O(\alpha^2)$ calculation we would need the $O(\alpha^3)$ Monte Carlo calculation which is not yet available (although some attempts were made in this direction [14]). One may only guess that it is of order $\sim 0.1\%$. In the following part of this Section we shall show the result of some additional Monte Carlo exercises which demonstrate more clearly where the higher order contributions are located in the phase space.

In the $O(\alpha)$, we truncate the series on the $\bar{\beta}_1$ term and the rates of events with two hard photons may be distorted – some sort of “double counting” of $\bar{\beta}_1$ occurs. If we go to $O(\alpha^2)$ then the inclusion of $\bar{\beta}_2$ improves the two hard photon distribution (the problem remains still for three hard photons). As a result, in Fig. 3 b) we find a pronounced peak in the plot of the difference between $O(\alpha^2)$ and $O(\alpha)$ at high v . It comes mainly from $\bar{\beta}_2$ i.e. from the situations when we have two hard photons in the sample. Let us check this conjecture with the Monte Carlo exercise.

In Fig. 4 we plot the energy distribution of the photon which is not the hardest one but the second in the energy order. It means that if this photon has an energy $E_\gamma^{(2)} = 0.1E_{\text{beam}}$ then in fact we have two photons with the energy $E_\gamma \geq 0.1E_{\text{beam}}$.

Fig. 4 The probability distribution of the energy of the second photon in the energy order. Plotted is $O(\alpha^2)$ result and the difference with the $O(\alpha^1)$ result.

The difference between the $O(\alpha^2)$ and $O(\alpha^1)$ results for this distribution is also shown in the Fig. 4 and we see that, as expected, it is well concentrated at the high v region.

One could argue that the higher order effect (in this case the $O(\alpha^2)$) could be at least partly eliminated by cutting off both in the Monte Carlo and the experiment the region with the cut of the type $E_\gamma^{(2)} < 0.1 E_{beam}$ which eliminates events with the simultaneous emission of two hard photons. This however may appear not to be really necessary because the gain in the precision is not substantial and it may also happen that the change of the cross section due to such a cut is in fact of order of the actual experimental precision requirement anyway.

In order to illustrate this point let us plot the change of the total cross section due to elimination of the events with three hard photons by means of the cut $E_\gamma^{(3)}/E_{beam} < x^{(3)}$. (Such a cut would hopefully eliminate some of $O(\alpha^3)$ effects.) As we see in the Fig. 5 in the vicinity of the Z resonance it does not really matter if we introduce such a cut or not because it affect the total cross section below 0.1%, even if we use $x_{min}^{(3)} \simeq 10^{-2}$. This is a simple reflection of the fact that events with many hard photons are more and more rare when we go to higher number of hard photons. In this case the third photon in the energy order has a distribution which is rather strongly suppressed at high energy end of its spectrum. This can be seen in Fig. 6 where we plot the energy distribution for the first three photons in the energy order. Of course, such a study is not really complete before we are able to calculate the $O(\alpha^3)$ corrections. and examine the difference between the $O(\alpha^3)$ and $O(\alpha^2)$ results.

Fig. 5 The Monte Carlo result for the probability of getting three hard photons with the energy above $x_{min}^{(3)} E_{beam}$ limit. This probability is plotted as a function of $\ln x_{min}^{(3)}$ at $\sqrt{s} = 40 \text{ GeV}, M_Z, M_Z \pm 2 \text{ GeV}$. The continuous smooth line represents a simple estimate $(\gamma^3/6)|\ln x_{min}^{(3)}|^3$.

Fig. 6 The energy distribution of the photon which has respectively the highest energy $E_{\gamma}^{(1)}$, then the second and third highest energy $E_{\gamma}^{(i)}, i = 2, 3, \dots$. Plotted are the probability distributions $dp/d\ln x^{(i)}$ where, $x^{(i)} = E_{\gamma}^{(i)}/E_{beam}$, at $\sqrt{s} = 40 \text{ GeV}$. The continuous lines represent a simple analytical estimate $p_i(x) = e^{\gamma \ln x} |\gamma \ln x|^{i-1} / (i-1)!$. Monte Carlo samples of $4 \cdot 10^5$ event were generated with $\epsilon = 10^{-8}$ and $v < 0.8$.

SUMMARY

We have discussed extensively what is the difference between the common inclusive exponentiation and the exclusive Yennie-Frautschi-Suura exponentiation. The discussion concentrated around the problem of the phase space limits for soft real photons and on the question where the higher order correction are located in the phase space.

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