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TORONS, CHIRAL SYMMETRY BREAKING AND THE U(1) PROBLEM
IN THE σ -MODEL AND IN GAUGE THEORIES

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ABSTRACT

A novel class of self-dual solutions in σ -models and in SU(2) gauge theories is considered. The solution is defined on a manifold with a boundary and has topological charge $Q = \frac{1}{2}$. The contribution of the corresponding fluctuations to the chiral condensate is calculated. This contribution has a finite non-zero value.

The APS (Atiyah, Patodi, Singer) theorem for a manifold with a boundary is discussed for the O(3) σ -model. The necessity of imposing non-local boundary conditions for the Dirac operator is explained. The toron effects in the supersymmetric two-dimensional O(3) σ -model and four-dimensional supersymmetric gluodynamics (SYM) may be reconducted to fermionic zero modes (ZM). In the gauge theories (QCD and SQCD for example), containing the fields in the fundamental representation (quarks), the situation is quite different. The unbound resonances of the continuum at $\lambda \rightarrow 0$ play a crucial role in this case.

The contribution of toron configurations to chiral condensates $\langle \bar{\psi}\psi \rangle$, $\langle \lambda^2 \rangle$ in SQCD is calculated and it is consistent with the Konishi anomaly. For the fermion condensate in QCD (with $N_f = N_c = 2$), we find $\langle \bar{\psi}\psi \rangle = -\pi^2 \exp\{5/12\} 2^4 \Lambda^3$, $\Lambda^3 \equiv M_0^3 g^{-4} e^{-4\pi^2/g^2}$. The U(1) problem and the θ -periodicity puzzle in QCD are also discussed.

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1. - INTRODUCTION

At this time the best known example of non-perturbative fluctuations is the instanton [1,2]. The integral nature of the topological charge Q is in that case related to the compactification of the space to a sphere, i.e., with the identification of all infinitely distant points. The choice of other boundary conditions could result in fractional topological charges. In particular, in gluodynamics with the $SU(N)$ gauge group, the introduction of so-called twisted boundary conditions [3] permits new solutions of the classical equations-torons [4,5] with $Q = K/N$, $K = 0, 1, \dots$ and with the action $S = (8\pi^2/g^2 N)K$.

What physical effects arise due to fluctuations with fractional Q ? These effects appear most glaringly in supersymmetric variants of a theory. In particular, in supersymmetric Yang-Mills theory (SYM) with an $SU(2)$ gauge group, torons ensure spontaneous breaking of discrete chiral symmetry. Indeed, the model possesses naive $U(1)$ chiral symmetry with respect to the transformations $\lambda^a \rightarrow \exp(i\alpha)\lambda^a$ (λ^a is a gluino field) which is broken by the anomaly $\partial_\mu a_\mu \sim G\tilde{G}$. However, under this transformation the discrete symmetry $Z_2 \times Z_2$ is conserved. As was shown some years ago, the 't Hooft torons [4] generate the condensate $\langle \lambda^2 \rangle$ and break this symmetry down to Z_2 [6]. However, the standard quasi-classical calculations of $\langle \lambda^2 \rangle$ [6], based on the solution of Ref. [4], are unreliable because the solution of Ref. [4], defined in a box of size L_μ , exists only if the ratios of the sizes L_μ satisfy certain relations and the constant $g(L \rightarrow \infty) + \infty$ becomes too large for the correctness of the quasi-classical calculations. Moreover, the introduction of fields in the fundamental (rather than adjoint) representation is rather difficult because of special (twisted) boundary conditions. Therefore, the 't Hooft solution can be considered as only an illustrative example with fractional charge.

Nevertheless, we believe that solutions with a fractional number may play an important role in the theory, but these solutions should be formulated in some way other than the 't Hooft solution. As will be shown below, our solution can be formulated on a manifold with boundary and may be understood as the point defect. This solution admits the introduction of the fields in the fundamental representation. So, our solution can be used for the consideration of interesting physical theories (like QCD) as well as of SYM.

Let us recall that instantons give zero contribution to $\langle \lambda^2 \rangle$ in $SU(2)$ SYM and can ensure non-zero values only for the correlator $\langle \lambda^2(x), \lambda^2(0) \rangle$ [7,8]. Indeed, in this model there are four fermion zero modes per instanton (ZM). It means that one instanton transition is always accompanied by the emission of four fermion fields, and thus $\langle \lambda^2(x), \lambda^2(0) \rangle \neq 0$. By clustering, this relation implies a non-vanishing $\langle \lambda^2 \rangle$ condensate, in agreement with the value of the Witten index [9],

which equals two, and in agreement with the rigorous results of Ref. [10]. However, $\langle \lambda^2 \rangle_{\text{instanton}} = 0$ because we have four (not two) zero modes. It is obvious that we would get $\langle \lambda^2 \rangle \neq 0$ for solutions which have two ZM. But the toron solution with $Q = \frac{1}{2}$ has just two ZM, so it is the main reason for considering fractional Q in supersymmetric theories. In ordinary (non-supersymmetric) theories (like QCD), the solution with fractional topological number should play an important role (in the solutions of the $U(1)$ problem and θ -periodicity puzzle) too, as was shown some years ago [11,12].

The paper is organized as follows. In Section 2, an $O(3)$ σ -model is formulated in terms of various fields, the unit vector field n^a , $a = 1, 2, 3$, $n^a n^a = 1$, the complex field ϕ and the unit complex spinor u_a , $a = 1, 2$, $u^\dagger u = 1$. The various formulations help to understand different aspects of the configurations with fractional Q . In this section the toron solution is formulated and the interpretation of this solution as the point defect is discussed. In Section 3, the problems of quantum fluctuations surrounding the classical solution and the requirements for the selection of "right" modes are discussed. The APS theorem for a manifold with a boundary is formulated and the necessity of imposing non-local boundary conditions is explained. In Section 4 the stability of the toron configuration with $Q = \frac{1}{2}$ is proved and the corresponding contribution to $\langle \psi\bar{\psi} \rangle$ is calculated. Then, in Sections 5 and 6, the corresponding construction is generalized to gauge theories. For this purpose the self-dual equation for gauge theories will be formulated in a language analogous to the Cauchy-Riemann condition for the $O(3)$ σ -model. The contribution of the toron configuration to the gluino condensate $\langle \lambda^2 \rangle$ in SYM is calculated. In Sections 7 and 8, the very important question concerning the introduction of the fields in the fundamental representation (quarks) of gauge theories is considered. A well-understood supersymmetric QCD (SQCD) model presents a perfect theoretical laboratory which helps to understand the role of fundamental representation fields in gauge theories. In this case a lot of various results are known (dependence of condensates on m and g , the Konishi anomaly, and so on ...). Our approach is in agreement with these general results.

After that, in Sections 9-11 we pass on to the analysis of toron calculation in QCD. The contribution of the corresponding configurations to the chiral condensate is calculated and is equal to $\langle \psi\bar{\psi} \rangle = -\pi^2 \exp[5/12] 2^{1/3} \Lambda^3$ at $N_f = N_c = 2$. It is our main result. The non-zero value for $\langle \psi\bar{\psi} \rangle$ is obtained because of quasi-zero modes of quarks.

This property of the spectrum of the Dirac operator is an important feature of the configurations with fractional topological number. In Section 10, the pseudo-scalar correlation function is considered and the pion pole is obtained in agreement with the Goldstone theorem. In Section 11 the $U(1)$ problem, the anomalous Ward identities and the θ -periodicity puzzle are considered from the point of view of

the toron calculation. As is well known, in any consistent mechanism for chiral breaking all these problems must be solved in an automatic way. The calculation of the corresponding correlators will provide a valuable consistency check for our approach.

Thus, in Section 2-4 we consider a simple two-dimensional $O(3)$ σ -model as an illustration of a general approach to fractional Q. Then, in Sections 5-8 we test our calculations in supersymmetric theories, where many different results are known independently. Gaining some experience with toron calculation in a simple model, we consider in Sections 9-11 the theory of hadrons, QCD.

2. - TORONS* IN THE $O(3)$ σ -MODEL

Before describing the toron solution, we discuss the duality equations and the Lagrangian for the ordinary (non-supersymmetric) $O(3)$ σ -model. The modification due to the introduction of fermions will be considered later. The action, the topological charge and the equations of duality have the following form in terms of the fields n^a [1]:

$$\begin{aligned} S &= \frac{1}{4f} \int d^2x (\partial_\mu n^a)^2, \quad n^a n^a = 1, \quad a = 1, 2, 3; \quad f = 1, 2 \\ Q &= \frac{1}{8\pi} \int d^2x \epsilon^{abc} \epsilon_{\mu\nu} n^a \partial_\mu n^b \partial_\nu n^c \\ \partial_\mu n^a &= -\epsilon^{abc} n^b \epsilon_{\mu\nu} \partial_\nu n^c \end{aligned} \quad (1)$$

Here f is the bare coupling constant.

To avoid complications due to the constraint $n^a n^a = 1$, one often introduces (see, e.g., the review in Ref. [13]), in place of the three fields n^a which live on the unit sphere, two independent fields ϕ_1 and ϕ_2 by means of a stereographic projection:

* We keep the term "toron", introduced in Ref. [4], for the self-dual solution in the $O(3)$ σ -model and in gauge theories (Section 5) as well. By this means we emphasize the fact that the solution minimizes the action and carries the topological $Q = \frac{1}{2}$, i.e., it possesses all the characteristics ascribed to the toron [4]. We keep the term "toron" despite the fact that our solution is formulated in principle in another way than in Ref. [4].

$$\begin{aligned} \varphi_1 &= \frac{n_1}{1+n_3} & \varphi_2 &= \frac{n_2}{1+n_3} \\ n_1 &= \frac{2\varphi_1}{1+\varphi_1^2+\varphi_2^2} & n_2 &= \frac{2\varphi_2}{1+\varphi_1^2+\varphi_2^2} & n_3 &= \frac{1-\varphi_1^2-\varphi_2^2}{1+\varphi_1^2+\varphi_2^2} \end{aligned} \quad (2)$$

Next one combines ϕ_1 and ϕ_2 into one complex field $\phi = \phi_1 + i\phi_2$ and introduces the complex variable $z = x_1 + ix_2$, and then reformulates Eqs. (1) as follows:

$$\begin{aligned} S &= \frac{2}{f} \int d^2x \frac{1}{(1+\bar{\phi}\phi)^2} \left(\left| \frac{\partial\phi}{\partial z} \right|^2 + \left| \frac{\partial\phi}{\partial \bar{z}} \right|^2 \right) \\ Q &= \frac{1}{\pi} \int \frac{d^2x}{(1+\bar{\phi}\phi)^2} \left(\left| \frac{\partial\phi}{\partial z} \right|^2 - \left| \frac{\partial\phi}{\partial \bar{z}} \right|^2 \right) \\ \frac{\partial\phi}{\partial \bar{z}} &= 0, \quad z = x_1 + ix_2, \quad \bar{z} = x_1 - ix_2, \quad \phi = \varphi_1 + i\varphi_2 \\ \partial &\equiv \frac{\partial}{\partial z} = \frac{1}{2} (\partial_1 - i\partial_2), \quad \bar{\partial} \equiv \frac{\partial}{\partial \bar{z}} = \frac{1}{2} (\partial_1 + i\partial_2) \end{aligned} \quad (3)$$

As can be seen from Eqs. (3), the duality equations have their simplest form in the ϕ language. So, the analytical function ϕ_{cl} (2) is the solution of the duality equations (3).

Regarding topological ideas, it turns out that for the closest analogy with gauge theories we need another formulation of the $O(3)$ σ -model in which local gauge invariance is present. Namely, we define the action, the topological charge and the equations of duality of the CP^1 theory, equivalent to the $O(3)$ σ -model, as follows [14]:

$$\begin{aligned} S &= \frac{1}{f} \int d^2x |D_\mu n_a|^2, \quad D_\mu = \partial_\mu + iA_\mu, \quad A_\mu = -i\bar{n}_2 \partial_\mu n_2 \\ Q &= \frac{1}{4\pi} \int d^2x \epsilon^{\mu\nu} F_{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad \mu = 1, 2; \quad a = 1, 2 \\ (D_\mu n_a)/d. &= i\epsilon_{\mu\nu} (D_\mu n_a)/d., \quad \bar{n}_2 n_2 = 1 \end{aligned}$$

Here n_a is a two-component complex spinor and A_μ is an auxiliary gauge field. In terms of (4), local gauge invariance has the obvious form

$$n'_\alpha = \exp(i\theta/n_\alpha); \quad A'_\mu = A_\mu + \partial_\mu \theta \quad (5)$$

Equivalence with the original formulation is verified with the help of the relations

$$n^\alpha = \bar{n}_\alpha (\sigma^\alpha)_{\alpha\beta} n_\beta, \quad \varphi = \frac{n_2}{n_1}, \quad n_\alpha = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \quad (6)$$

where σ^α are the usual Pauli matrices. We note that the duality equations are written in the spinor language very simply:

$$\bar{\partial} \begin{pmatrix} n_2 \\ n_1 \end{pmatrix} = 0 \quad (7)$$

That is why the classical solutions in the spinor language

$$(n_\alpha)_{cl.} = \frac{P_\alpha(z)}{|P_\alpha(z)|}, \quad |P_\alpha|^2 = \bar{P}_\alpha P_\alpha \quad (8)$$

are determined by any analytical function $P_\alpha(z)$.

As can be seen from the definition of Q , Eq. (4), the topological charge is determined by the change of spinor phase around the large contour:

$$Q = \frac{1}{4\pi} \int d^2x \epsilon_{\mu\nu} F_{\mu\nu} = \frac{1}{2\pi} \oint A_\mu d\gamma_\mu = \frac{1}{2\pi} \oint \partial\theta \quad (9)$$

$|x| = \infty$

$$n(x \rightarrow \infty) = n_0 \exp(i\theta), \quad n_0 = \text{const.}, \quad A_\mu(x \rightarrow \infty) = \partial_\mu \theta$$

The standard instanton solution in this language takes the form

$$P_\alpha(z) = u_\alpha \cdot \rho + v_\alpha (z-a) \quad u_\alpha \bar{u}_\alpha = 0$$

$$|\rho| = (|\rho|^2 + |z-a|^2)^{1/2} \quad \bar{v}_\alpha \partial_\alpha = \bar{v}_\alpha u_\alpha = 1 \quad (10)$$

Here v_α and u_α are the unit constant spinors, and " ρ " and " a " are four free parameters associated with translation and scale invariance. Because $n_2/n_1 = \rho_2/\rho_1$ depends only on z , according to (7), the duality equation is satisfied. Furthermore, after traversing a large contour the spinor acquires the phase 2π which, in accordance with (9), corresponds to $Q = 1$.

We pass now to the analysis of the toron solution. It is not hard to see that the action is invariant not only with respect to global $SU(2)$ transformations, but also with respect to local $U(1)$ transformations (5). However, the group transfor-

mation is not simply $SU(2) \times U(1)$, but $G = SU(2) \times U(1)/Z_2$, so that $\pi_1(G) \sim Z_2$. This last circumstance is connected with the fact that a simultaneous transformation from the $SU(2)$ group of the form $\exp(i\alpha_3)$ and rotation $\exp(i\pi)$ by an angle π from the $U(1)$ group leaves the form of the fields unchanged. Consequently the corresponding transformations should be identified with unity. This in turn means that the theory admits $Q = \frac{1}{2}$, and consequently (as will be seen below), multivalued functions appear in the description of the classical solution. A geometric interpretation of this fact is given below.

As is easily verified, the toron solution with $Q = \frac{1}{2}$ is a double-valued function. Indeed, as was discussed above, we admit a larger class of solutions; namely, upon completion of the contour of the large radius, we allow the appearance of an overall factor $\exp(i\alpha_3) = -1$. Taking into account that a factor (-1) arises due to analytic functions of the type $z^{1/2}$, we arrive at the following form of the toron solution:

$$n_{cl.} = \lim_{a \rightarrow b} \left(\frac{1}{z-a} + \frac{1}{z-b} \right)^{-1/2} \begin{pmatrix} (z-a)^{1/2} \\ (z-b)^{1/2} \end{pmatrix}, \quad \varphi_{cl.} = \lim_{a \rightarrow b} \left(\frac{z-b}{z-a} \right)^{1/2} \quad (11)$$

This solution is defined on two Riemann sheets; real physical space corresponds to only one of them. Further, it is easily seen that the duality equation $\bar{\partial}(n_2/n_1) = 0$ is automatically satisfied.

We note further that the solution (11) is defined in the limit $a \rightarrow b$. In terms of the field $\phi = n_2/n_1$, Eq. (6), the solution (11) corresponds to the function $\phi = \lim_{a \rightarrow b} (z-b/z-a)^{1/2}$, with a cut tending to zero as $a \rightarrow b$, i.e., in terms of the field ϕ , the limit $a \rightarrow b$ means re-establishment of the single-valuedness on one physical sheet.

If one sets $a = b$ from the very beginning, then

$$n_{cl.} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \exp(i\theta) \quad (12)$$

corresponding, according to Eq. (5), to the gauge rotation of the empty vacuum solution. At first sight this suggests that such a solution cannot lead to physical effects. As will be shown below, this is not so. In particular, in the supersymmetric 0(3) σ -model the solution (11) ensures a non-zero value of the chiral condensate. Analogous behaviour arises in the calculation of the 't Hooft toron contribution to the gluino condensate in supersymmetric gluodynamics. Although a field strength $G_{\mu\nu} \sim L^{-2}$, which tends to zero everywhere in the system with

increasing size $L \rightarrow \infty$, corresponds to the toron solution [4], the condensate turns out to be finite [6].

We return to the analysis of the solution (11). To this end, instead of Eq. (11), corresponding to the boundary condition (12) ($\phi = n_2/n_1 + 1$) as $|z| \rightarrow \infty$, we consider the solution

$$\psi_d = \lim_{\Delta \rightarrow 0} \left(\frac{\Delta}{z-a} \right)^{1/2}, \quad \eta_d = \lim_{\Delta \rightarrow 0} \frac{1}{\sqrt{\Delta/1+z-a}} \left(\frac{z-a}{\Delta} \right)^{1/2} \quad (13)$$

satisfying the standard boundary conditions $\phi(z \rightarrow \infty) = 0$ and differing from the original by an overall rotation.

The toron solution (13) is defined on the manifold with a boundary, Fig. 1. We can consider the conformal mapping

$$z = \frac{\tilde{z}^2}{\tilde{z}} \quad (14)$$

In this case, the physical space is half-plane with a boundary $\text{Im } \tilde{z} = 0$, Fig. 2. If we consider the conformal mapping

$$\omega = R \frac{\Delta + i\tilde{z}}{\Delta - i\tilde{z}} \quad (15)$$

from the half-plane to the disk with radius R , Fig. 3, the physical space corresponds to this disk R . We may then define the theory on a disk R and take the limit $R \rightarrow \infty$ in the final stage of the calculation. The toron solution in this interpretation is smeared over all space with size $R \rightarrow \infty$, and in this sense reminds us of 't Hooft's solution [4]. On the other hand, we may define the theory on the exterior of a small circle Δ , Fig. 4, and take the limit $\Delta \rightarrow 0$ in the final stage of the calculation. In this language the toron is the point defect at $\Delta \rightarrow 0$. In any case, the toron action equals $S_{cl} = \pi/f$ and does not depend on dimensional parameters R, Δ due to the classical conformal symmetry.

Thus, the description of the toron solutions with fractional topological number on a manifold with a boundary is not so hard. The only problems which arise are: is this solution stable? Is the toron contribution to $\langle \bar{\psi}\psi \rangle$ finite after the limit $\Delta \rightarrow 0$ ($R \rightarrow \infty$)? To answer these questions, we have to calculate the toron measure and condensate (Sections 3 and 4). However, before calculating the toron measure, we would like to note that the solution with any fractional number can be described on a manifold with a boundary, but only $Q = \frac{1}{2}$ is stable at the quantum

level [for the $SU(2)$ group]. The correct renormalization-group dependence is restored just for this value.

There is an alternative point of view on the solution with fractional number. It is connected with the consideration of the orbifolds [15] or ordinary non-singular manifolds [16]. We will not discuss this question in more detail below because our starting manifold, Fig. 1, is a more suitable one from a technical point of view. However, we would like to note that the parameter $\Delta \rightarrow 0$, which is present in the definition of the toron solution (13), is interpreted as a regularization ("blowing up" in the literature) of the fixed points of the orbifold (see the Appendix of Ref. [15]). The analysis of the toron solution on a disk R , Fig. 3, is discussed in Appendix A of this paper.

To conclude this section, I would like to describe the beautiful analogy between the toron solutions and dislocations in solid state physics. If we denote by $U_1(x)$ a displacement of the atom placed at the point x of the crystal, and by L a loop surrounding the point x_0 (the place of dislocation), then

$$\oint_L dU_1 = \oint \frac{\partial U_1}{\partial x_k} dx_k = -b_1 \quad (16)$$

Here b_1 is a lattice vector (see, e.g., Ref. [17]). For such $U_1(x)$ it is said that we have a dislocation located at x_0 . This means that the multivalued functions appear in the description of the dislocation. Therefore we have to account for jumps in $U_1(x)$. However, the physical values (the tensors of strength and deformations) are single-valued. Usually, for the description of dislocations, certain fictitious δ -like singularities are introduced. They ensure the necessary jump in $U_1(x)$ [17].

In the present toron calculation we prefer to describe the solution in two Riemann sheets without introducing fictitious δ -like singularities. The gauge-invariant values (in particular, the action density) must be single-valued. Namely, this requirement singles out the value $Q = \frac{1}{2}$ as compared to other fractional values (see next section).

It is obvious that the situation described above for the dislocation corresponds to a line defect which is determined by the vector b_1 (16). In our case the vector analogous to b is orthogonal to physical space and so the toron defect is the point defect with finite full action.

Really, the evaluation of the action (3) in terms of the ϕ -field for the solution (13) is quite simple:

$$\begin{aligned}
\varphi_{cl.} &= \left(\frac{\Delta}{z-a} \right)^{1/2}, \quad S_{cl.} = \frac{2}{f} \int \frac{dx}{(1+\bar{\varphi}\varphi)^2} \left| \frac{\partial \varphi}{\partial \bar{z}} \right|^2 = \\
&= \frac{2\pi}{f} Q = \frac{\Delta}{2f} \int dx \frac{1}{(z-a)(\bar{z}-a)(1+\bar{\varphi}\varphi)^2} = \\
&= \frac{\pi\Delta}{f} \int_0^{\infty} \frac{\rho d\rho}{(\Delta+\rho)^2} = \frac{\pi}{f}, \quad S'_{cl.} = \pi/f. \quad (17)
\end{aligned}$$

As expected, the classical action has decreased in comparison with the instanton value by a factor 2.

The analogy between the toron solution and dislocation allows us to interpret the Δ -regularization from the lattice point of view. In this case, as is well known [18], in order to account for the periodicity of the action one has to introduce a set of vortices into the system (i.e., just our point-like singularities). To describe these singularities directly in the continuum theory, one needs some regularization procedure. In particular, we can consider the lattice regularization or some generalization of the starting theory^{*}.

If we use the lattice regularization, then a cut in the complex plane appears in the continuum limit with the branch point at the position of the vortex [18]. It corresponds exactly to our Fig. 1.

We prefer to use the Δ -regularization, which preserves the duality equation at finite values of Δ (not only in the limit $\Delta \rightarrow 0$).

It may be useful to bear in mind another analogy related to the consideration of two-dimensional conformal theories. As is known, in this case, for calculating correlation functions involving twist fields, multivalued functions should be considered. A cut which appears in the description of our toron solution (13) exactly corresponds to a cut with the branch point at the position of the vortex in two-dimensional conformal theory. The physical spectrum of the conformal theory should consist only of operators that do not see the string of the vortex, so that the theory is local (this requirement is called "modular invariance" in two-dimensional conformal theory). In our case this requirement singles out the value $Q = \frac{1}{2}$.

^{*}) For example, the compact QED can be obtained from non-Abelian gauge theory. All heavy fields can be viewed as a regularization of the photon theory which replaces the lattice regularization [18,19].

3. - EVALUATION OF THE TORON MEASURE

We pass now to the analysis of quantum fluctuations surrounding the classical solution (13). As usual, for the quasiclassical calculation it is necessary to expand the field $n_\alpha(x)$ in the neighbourhood of $n_{cl.}^\alpha(x)$ (13), keeping only the quadratic terms in the action. The resultant bilinear form reduces to the following expression [20-22]:

$$\begin{aligned}
S &= S_{cl.} + \int dx^2 \delta \bar{n}_\alpha M_{\alpha\beta} \delta n_\beta \\
&- M_{\alpha\beta} = 1/\rho \partial_{1/\rho^2} (\delta \varphi_\beta - \frac{\rho \bar{\rho}}{1/\rho^2} \bar{\varphi}/\rho) \\
\delta \bar{n}_\alpha \rho_\alpha &= 0, \quad n_\alpha = (n_{cl.})_\alpha + \delta n_\alpha
\end{aligned} \quad (18)$$

Here δn_α is the small quantum fluctuation and $P_\alpha(z)$ is the classical toron solution (13). The supplementary condition $\delta \bar{n}_\alpha P_\alpha(z) = 0$ is due to the constraint $\bar{n}_\alpha n_\alpha = 1$. This supplementary requirement can be satisfied with the help of the vector t_α which is orthogonal to the classical solution [23]:

$$\begin{aligned}
\delta n_\alpha &= t_\alpha F(z, \bar{z}); \quad \alpha = 1, 2; \quad t_\alpha = \frac{1/\alpha \bar{\alpha} - \alpha \bar{\alpha}^{1/2}}{1/\alpha} \\
\bar{t}_\alpha t_\alpha &= 1; \quad \bar{t}_\alpha \rho_\alpha = 0; \quad \bar{t}_\alpha (n_{cl.})_\alpha = 0
\end{aligned} \quad (19)$$

Without loss of generality we set $a = 0$, $\Delta = 1$. In the final relations the corresponding dependence can easily be restored by dimensional consideration.

For further analysis the following change of variables is crucial:

$$\zeta = \frac{1/\bar{z} - 1}{1/\bar{z} + 1}, \quad \varphi = \arctg \frac{x_1}{x_2}, \quad -1 \leq \zeta \leq 1, \quad 0 \leq \varphi < 2\pi \quad (20)$$

The meaning of η , ϕ is obvious - they are corresponding co-ordinates of the sphere obtained with the help compactification of the complex plane z with boundary (Fig. 1) into the sphere S with the cut joining the north and south poles. In terms of the variables η , ϕ the bilinear form (18) becomes the well-known equation for d -functions. Really, substituting the expressions (19) and (20) for Eq. (18), we obtain the following bilinear form:

$$S' = S_{cl.} + \int d^3x \int_0^{2\pi} \bar{\psi} \left[- (1-\epsilon^2) \frac{\partial^2}{\partial z^2} + 2\epsilon \frac{\partial}{\partial z} \right] \psi - \frac{1}{1-\epsilon^2} \left[4 \frac{\partial^2}{\partial \varphi^2} - 4i(1-\epsilon^2) \frac{\partial}{\partial \varphi} - 2(1-\epsilon^2) \frac{\partial}{\partial \varphi} \right] \psi \quad (21)$$

Let us note that the standard requirement consists of single-valuedness of the physical modes on the manifold without boundary (sphere). Now this is not the case because our manifold has a boundary, Fig. 1, and the lines $\phi = 0$ and $\phi = 2\pi$ are not identified. However, as can be seen from the expression (13), the toron solution is defined on the two Riemann sheets with corresponding identification. That is why the polar angle dependence must be the following:

$$F \sim \exp \{ i m \varphi / 2 \}, \quad m = 0, 1, \dots \quad (22)$$

As will be shown below, exactly this behaviour ensures regular solutions. The dependence (22) can be arrived at in another way. We can require the regularity of the modes F and this requirement is satisfied precisely by Eq. (22). Taking (22) into account, we obtain the following equation for the eigenvalues:

$$\left\{ - (1-\epsilon^2) \frac{\partial^2}{\partial z^2} + 2\epsilon \frac{\partial}{\partial z} + \frac{1}{1-\epsilon^2} \left[(m^2 - 2m + 2) - 2(2m - 2) \right] \right\} F = \lambda_F F \quad (23)$$

The regular solutions of Eq. (23) are well known (see, e.g., Ref. [24]) and are as follows:

$$F \sim \exp \{ i m \varphi / 2 \} d_{-m,1}^{j+1}(z), \quad \lambda_F = j(j+3); \quad m, j = 0, 1, \dots \quad (24)$$

As usual, similar calculations should be carried out for the vacuum field, i.e., for $(n_{cl.})_\alpha = v_\alpha$ (for definiteness). In that case the equation for the eigenvalues is determined by the standard angular momentum operator:

$$\begin{aligned} & \left[- (1-\epsilon^2) \frac{\partial^2}{\partial z^2} + 2\epsilon \frac{\partial}{\partial z} + \frac{m^2}{1-\epsilon^2} \right] \delta n_\alpha = \lambda \delta n_\alpha \\ & \delta n_\alpha = v_\alpha \exp \left(\frac{i m \varphi}{2} \right) P_{j,m}(z) \\ & \lambda = j(j+1); \quad j = 0, 1; \quad \delta \bar{n}_\alpha (n_\alpha)_{cl.} = \bar{v}_\alpha v_\alpha = 0. \end{aligned} \quad (25)$$

Anticipating events, we also formulate equations for fermionic modes in the SUSY 0(3) σ -model. In the quasiclassical calculation the addition to the action is determined by the expression [20]:

$$\begin{aligned} \Delta S_f &= \int d^3x \bar{\psi}_\alpha L_{\alpha\beta} \psi_\beta; \quad \bar{\psi}_\alpha (n_\alpha)_{cl.} = 0 \\ L_{\alpha\beta} &= i \left(\delta_{\alpha\beta} - \frac{P_\alpha \bar{P}_\beta}{P^2} \right) \left(0 \quad 1/P \frac{\partial}{\partial P} \right) \quad (26) \end{aligned}$$

Here $\psi_\alpha = (\psi_\alpha^1 / \psi_\alpha^2)$ is a two-component spinor belonging to the fundamental representation of the global $SU(2)$ group.

We shall not try to solve Eq. (26). Instead, we shall demonstrate that each bosonic mode (with $\lambda \neq 0$) is necessarily accompanied by two degenerate fermionic modes. Taking into account the structure $\psi_\alpha \sim t_\alpha$ orthogonal to the classical solution $(n_\alpha)_{cl.}$, we obtain the following equation for the fermion eigenfunctions:

$$\begin{aligned} \psi_\alpha &= t_\alpha \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \\ & - 1/P^2 \frac{\partial}{\partial P} \left(\frac{1}{P^{1/2}} F_2 \right) = \lambda F_1 \\ & \frac{1}{P^{1/2}} \frac{\partial}{\partial P} \left(P^{1/2} F_1 \right) = \lambda F_2 \end{aligned} \quad (27)$$

It has two solutions - this fact immediately stems from Eq. (23). Indeed, the first solution

$$F_1 \sim F, \quad F_2 \sim \frac{1}{\sqrt{\lambda_F}} \frac{1}{P^{1/2}} \left(\frac{\partial}{\partial P} P^{1/2} F \right)$$

corresponds to $\lambda = +\sqrt{\lambda_F}$, while the second one

$$F_1 \sim F, \quad F_2 \sim -\frac{1}{\sqrt{\lambda_F}} \frac{1}{P^{1/2}} \frac{\partial}{\partial P} P^{1/2} F$$

corresponds to $\lambda = -\sqrt{\lambda_F}$. The two-fold degeneracy of the fermion modes is the consequence of the γ_5 invariance of the model; the boson-fermion degeneracy reflects the supersymmetry. Hence it is no surprise that the fermion contribution coincides with the boson one and that they cancel each other. As a result, all the non-zero modes combine to give unity in the functional integral.

Up to now we have considered the equation for the eigenvalues only and our discussion of the eigenvalue (24) was independent of supplementary requirements applicable to them because of the existence of the boundary. But it is obvious

that only some of them must be taken into account. Indeed, the different eigenfunctions (24) are non-orthogonal to each other on one Riemann sheet (on the physical space with boundary), and they will be mutually orthogonal only on two sheets. Another explanation is the following: since the eigenvalues of λ for the toron (24) and the instanton solution [23] coincide, the degree of degeneracy must be different in the toron and instanton cases^{*}. So, only some (not all) eigenfunctions must be taken into account. It is obvious that all these problems are connected with the boundary of our manifold.

We now want to formulate the criterion for the selections of modes (24) which must be taken into account. Besides that, we consider only the zero modes due to the cancellation (of fermion and boson non-zero modes) which is mentioned above.

The simplest way to understand the requirements applicable to modes is to consider the $O(3)$ σ -model in terms of an unconstrained ϕ field (3). In this case any supplementary requirements on the modes $\delta\phi$ due to the constraint are absent, and the ϕ field describes exactly the physical degree of freedom (the local gauge invariance in this language is absent). It is well known [13] that the normalized zero modes satisfy the equation which is just the Cauchy-Riemann condition:

$$\bar{\partial} \delta\phi = 0, \quad \int d^2x \frac{|\delta\phi|^2}{(1 + \bar{\phi}_a \phi_a)^2} = 1 \quad (28)$$

However, an arbitrary analytical function is not yet a zero mode; only the functions also satisfying the finiteness condition [25]

$$\frac{|\delta\phi|^2}{(1 + \bar{\phi}_a \phi_a)^2} \Big|_{z \rightarrow 0} = \text{const.} \quad (29)$$

are acceptable.

In particular, for the instanton this requirement is satisfied precisely by two complex modes (four real) [13]:

$$\delta\phi \sim z^{-1}, \quad \delta\phi \sim z^{-2} \quad (30)$$

in accordance with four (two complex) free parameters (10) ρ, a , associated with the collective co-ordinates of the instanton. We also note that the normalization

^{*}) If it were not so, the toron and instanton measures would coincide and we would obtain the wrong answer.

integral (28) diverges logarithmically for the mode $\delta\phi_0 \sim z^{-1}$ for large z . However, as noted in Ref. [25], this fact has no effect on the physical content of the theory. We shall run into an analogous behaviour also in the case of torons. Usually, the infra-red regularization is achieved by introduction of the factor [20-22] Q :

$$\Omega = \left(1 + \frac{x^2}{R^2}\right)^{-1}, \quad \int \frac{|\delta\phi|^2 \Omega^2 d^2x}{(1 + \bar{\phi}_a \phi_a)^2} = 1 \quad (31)$$

In this case two extra normalizable modes $\delta\phi_0 \sim 1$ would arise. However, the vacuum amplitude would simultaneously acquire the same two zero modes. Since the instanton amplitude is always normalized to the vacuum amplitude (see above), the effect of the two extra zero modes would cancel out. Consequently the number of non-trivial zero modes in the instanton equals $(4+2)-2 = 4$, as before (30). The number of non-trivial fermion zero modes in the SUSY $O(3)$ σ -model in the instanton field coincides with the number of boson ones and equals four. This is because the fermion zero modes satisfy the same equations (28). Moreover, the number of the fermion zero modes is known beforehand, from consideration of the axial anomaly and equals four (two complex zero modes).

Collecting all the factors together, we obtain the following expression for the instanton density in the supersymmetric $O(3)$ σ -model [13,20]:

$$Z_{\text{inst.}} \sim \exp\left(-\frac{2\pi}{f}\right) M_0^4 d\rho d^2a \frac{d^2\epsilon_1}{M_0} \frac{d^2\epsilon_2}{M_0} \quad (32)$$

In obtaining (32) we took into account the fact that the non-zero mode contributions cancel between bosons and fermions. The factor $\exp(-2\pi/f)$ is connected with the classical action; d^2ad^2p corresponds to integration over the four collective variables mentioned above, and M_0^4 is the regulator contribution corresponding to these four zero modes. Furthermore, each complex fermion zero mode is accompanied by the corresponding collective integral $d^2\epsilon$ and regulator contribution M_0^{-1} . As was expected, in Eq. (32) there appears the renormalization-invariant combination

$$m^2 = M_0^2 \exp\left(-\frac{2\pi}{f(M_0)}\right) \quad (33)$$

We have on purpose analyzed in detail the instanton zero modes and the requirements applicable to them. In the following, the corresponding criteria will help us to choose the "correct" zero modes in the case of the toron.

We return to the analysis of the zero modes in the toron background (13). In this case, zero modes satisfy Eq. (28) as well. However, the only function satisfying the finiteness condition (29) is

$$\delta\varphi \sim Z^{-1} \quad (34)$$

in accordance with two free parameters "a" (13), associated with translations^{*}.

Only this function is acceptable. Although this mode, as in the case of the instanton, is logarithmically divergent for $Z \rightarrow \infty$, this fact has no bearing on the physical content of the theory [see Eq. (31) and corresponding discussion]. In the fermionic sector, as explained above, the number of non-trivial modes coincides with the boson number and equals two in the toron field. This number - two (one complex mode) - is in agreement with the axial anomaly condition and is known beforehand. Indeed, decreasing the action (and topological number) by two, we see that the multiplicity of admissible zero modes also decreases twice, as it should.

If we were interested in the $O(3)$ σ -model only, this would be the end of the story, because we can now find the toron measure and calculate the chiral condensate $\langle \bar{\psi}\psi \rangle$. The result turns out to be finite (see next section), because the toron solution with $Q = \frac{1}{2}$ changes the chiral charge by two units and has two zero modes. Therefore the corresponding vacuum transition is necessarily accompanied by the production of a $\bar{\psi}\psi$ pair, as will be demonstrated by an explicit calculation.

However, our main purpose is to find a criterion for the selection of modes in the formulation with local gauge invariance and fictitious degrees of freedom.

For empty space the zero modes correspond to $j = 0$ (25) and have the multiplicity $g = 2$ (two real modes). For the toron, zero modes correspond to $j = 0$, $m = 0, 1, 2$ (24). However, only two of them ($m = 0, 2$, $g = 4$ real modes) are orthogonal to each other on the physical space

$$\delta n_a^{(1)} \sim t_a d_{11}' \sim t_a (1-z)$$

(35a)

^{*}) We recall that the parameter Δ is the regulator but it does not have the sense of a collective co-ordinate. The analogous situation is seen in 't Hooft's solution [4], where parameters L_i (the sizes of the box) are only regulators but are not connected with the collective co-ordinates.

$$\begin{aligned} \delta n_a^{(2)} &\sim t_a d_{11}' \sim t_a (1+z) \exp(i\varphi) \\ \int_{-\pi}^{\pi} d\varphi \delta n_a^{(1)} \delta n_a^{(2)} &= 0; \quad \int_{-\pi}^{\pi} d\varphi \delta n_a^{(2)} = 0 \end{aligned} \quad (35b)$$

[On the two Riemann sheets, three complex modes ($m = 0, 1, 2$) are orthogonal.]

Consequently, we choose from the solutions (24) the orthonormal set of eigenfunctions. Roughly speaking, for large j this reduces the degree of degeneracy by a factor two as compared to the instanton case, and this fact automatically ensures the correct renormalization-group dependence (see Ref. [15]). The other requirement of regularity is satisfied by solutions (24) in the form of d -functions.

Collecting all the results together, we obtain that the number of non-trivial zero modes in the toron field equals $4-2 = 2$ [for the instanton $6-2 = 4$, see discussion after Eq. (31)], in agreement with the number obtained before, and in accordance with two free parameters associated with translations.

The forms of zero modes in ϕ -language and in n_a -language are in agreement with each other; this can be checked with the help of Eq. (6). Upon substitution of $n_a = (n_a)_{cl.} + \delta n_a$ and $\phi = \phi_{cl.} + \delta\phi$ we arrive at the following connection between the modes δn_a in terms of the n_a -field (4) and the modes $\delta\phi$ in terms of the ϕ -field (3)

$$\delta\phi = \frac{(\delta n_a)(n_a)_{cl.} - (\delta n_a)(n_a)_{cl.}}{(n_a)_{cl.}} \quad (36)$$

When we substitute an explicit expression for (δn_a) , from (35a) we obtain $\delta\phi_0 \sim 1$ and from (35b) we obtain $\delta\phi_0 \sim Z^{-1}$, in agreement with the results obtained before.

Our problem, however, was not the derivation of toron zero modes by yet another method. Rather, we wanted to demonstrate the important criterion for the selection of modes, which must be accounted for in the formulation with local gauge invariance. Namely, the mode $\delta\phi_0 \sim Z^{-1}$ in ϕ -language is single-valued, but the same mode in (35) is not single-valued (because of the factor $Z^{\frac{1}{2}}$ in t_a). However, the gauge-invariant value $\bar{\delta n}_a \cdot \delta n_a$ is single-valued.

The lesson from this is as follows. Only gauge-invariant values in the theories with an extra degree of freedom must be single-valued; namely, these modes have a physical sense. The fermion modes in n_a -language can be obtained in the same way and they satisfy the same requirement.

Moreover, this result can be understood from a different viewpoint, i.e., from the APS theorem for a manifold with a boundary [26]. The necessity of imposing non-local boundary conditions for the calculation of the zero-mode number is well known [26,27]. The global boundary conditions must be imposed so that the Dirac operator $L_{\alpha\beta}$ (26) is the self-adjoint operator on the manifold with boundary. Since $L_{\alpha\beta}$ is the first-order differential operator, the self-adjoint condition has the following form for arbitrary $\psi_\alpha^1, \psi_\alpha^2$ [26,27]:

$$\psi_\alpha^1 = \left(\begin{array}{c} \psi_\alpha^1 \\ \psi_\alpha^2 \end{array} \right) \Big|_Y \int_Y dy \bar{\psi}_\alpha^1 \psi_\alpha^2 + H.C. = 0 \quad (37)$$

Here $\int_Y dy$ is the integral over the boundary. In particular, for the manifold of Fig. 1*, the global boundary condition has the form:

$$\int_0^\infty dk \left(\bar{\psi}_\alpha^1 \psi_\alpha^2 \right) \Big|_{|q=2\pi} - \int_0^\infty dk \left(\bar{\psi}_\alpha^1 \psi_\alpha^2 \right) \Big|_{|q=0} + H.C. = 0 \quad (38)$$

If the gauge-invariant value $\bar{\psi}_\alpha^1 \psi_\alpha^2$ on the upper edge of the cut coincides with the value on the lower edge, then Eq. (38) is satisfied. But precisely this requirement is formulated above for the gauge-invariant values from a different viewpoint.

Thus, from the various points of view (APS index theorem, analysis of the CP¹ model in terms of the unconstrained ϕ -field), the criteria for the choice of the "correct" modes in the toron field was formulated. It turns out that these requirements can be satisfied only for $Q = \frac{1}{2}$. Besides, a few questions, such as the single-valuedness of the gauge-invariant values, the self-adjointness of the operator of (18) and the orthonormality of the set of eigenfunctions are interconnected in the formulation of the theory on a manifold with boundary. Moreover, the configuration with $Q = \frac{1}{2}$ is stable at the quantum level, as will be shown below.

4. - THE STABILITY OF THE TORON SOLUTION AND THE CALCULATION OF THE CHIRAL CONDENSATE IN THE SUSY Q(3) σ -MODEL

In this section we prove that the toron configuration with $Q = \frac{1}{2}$ is stable under quantum fluctuations and the toron topological charge is conserved. With the above consideration taken into account (concerning the stability of the solution, the number of zero modes and so on ...), the toron measure will be obtained and the

* The analysis of the manifold of Fig. 3 (disk R) is discussed in Appendix A.

corresponding contribution to $\langle \bar{\psi}\psi \rangle$ will be calculated. The degenerate ground states arising from non-anomalous discrete chiral symmetry (see introduction) will be discussed. As is well known, instanton calculations yield the average vacuum expectation value of a product of operators over these degenerate vacua [28]. The toron calculations yield the non-vanishing value of $\bar{\psi}\psi$ itself.

We return to the analysis of the definition of the topological charge Q (4) and rewrite it in the following form:

$$\begin{aligned} Q &= \frac{1}{4\pi} \int d^2x \epsilon_{\mu\nu} F_{\mu\nu} = -\frac{i}{2\pi} \int d^2x \bar{D}_\mu n \cdot D_\nu n \epsilon_{\mu\nu} = \\ &= \frac{1}{\pi} \int d^2x \left(|Dn|^2 - |\bar{D}n|^2 \right); \quad D \equiv \frac{1}{2} (D_1 - i D_2) \end{aligned} \quad (39)$$

Upon substitution of

$$\begin{aligned} n_\alpha &= \frac{P_\alpha}{|P|} \sqrt{1 - |\delta n_\alpha|^2} + \delta n_\alpha \\ \bar{n}_\alpha n_\alpha &= 1, \quad \bar{P}_\alpha \delta n_\alpha = 0 \end{aligned} \quad (40)$$

where $P_\alpha(Z)$ is the classical solution (13) and δn_α is any quantum mode orthogonal to the toron solution, we arrive with the help of (39) at the following expression for δQ :

$$\begin{aligned} Q &= \frac{1}{2} + \delta Q; \quad \delta Q \sim \int d^2x \left\{ -\bar{\partial} \left[\frac{\delta n_\alpha}{|P|} \partial (|P| \delta n_\alpha) \right] - \right. \\ &\quad \left. - \partial \left[\frac{|P|}{\delta n_\alpha} \bar{\partial} (\delta n_\alpha |P|) \right] + 2 \partial \left[\delta n_\alpha \bar{\partial} (\delta n_\alpha) \right] \right\}. \end{aligned} \quad (41)$$

As it should be, δQ is a full derivative and hence reduces to a surface integral. The latter in turn vanishes for the modes which satisfy the requirements discussed above. Indeed, because $\delta n_\alpha \sim t_\alpha F$ (19), δQ can be rewritten as:

$$\delta Q \sim \int d^2x \left\{ -\bar{\partial} \left[\frac{F}{|P|} \partial (|P| \bar{F}) \right] - \partial \left[\frac{F}{|P|} \bar{\partial} (|P| F) \right] + 2 \partial (F \bar{\partial} F) \right\} \quad (42)$$

Now it is obvious that $\delta Q = 0$ because the non-single-valued structure $\sim t_\alpha \sim \sqrt{Z}$ is absent in the expression (42) and the functions F are well defined.

We pass to the discussion of the toron measure of the supersymmetric 0(3) σ -model. As we know, supersymmetric models differ conveniently from ordinary ones in that only zero modes need be considered. In the bosonic and fermionic sectors we found two non-trivial modes, written in the form of one complex mode (34). Collecting all the factors together, we obtain the following expression for the toron density in the model:

$$Z_{\text{toron}} \sim M_0^2 d^2 a \frac{d^2 \tilde{\epsilon}}{M_0} \exp\left(-\frac{\pi}{f(M_0)}\right) = m d^2 d\tilde{\epsilon} \quad (43)$$

Here the factor $M_0^2 d^2 a$ is due to the single complex bosonic zero mode, $d^2 a$ is the corresponding integral over the collective variable, the factor $d^2 \tilde{\epsilon}/M_0$ is connected with the single complex fermion zero mode, and lastly, $\exp(-\pi/f)$ is the contribution of the classical toron action. As in the instanton case (32), the expression (43) for the toron measure has precisely the renormalization-invariant form. It is easy to trace this phenomenon; while the action is decreased by a factor two, the number of zero modes is decreased by the same factor, which exactly restores the correct renormalization-invariant relation (33).

Now all is ready for the calculation of the chiral condensate $\langle \bar{\psi}\psi \rangle$ in the model. Substituting the zero modes in place of ψ , and recalling that integration over the collective fermionic variables exactly satisfies $\int \tilde{\epsilon}^2 d^2 \tilde{\epsilon} = 1$, we verify that

$$\langle Q_5 = 2/\bar{\psi}\psi / Q_5 = 0 \rangle = m C \int d^2 a \bar{\psi}(x-a) \psi(x-a) = m C. \quad (44)$$

where "C" is the calculable numerical constant. In the last step we used the value of the normalization integral.

We note that the instanton can only ensure a non-vanishing value for the correlator $\langle \bar{\psi}\psi(x), \bar{\psi}\psi(0) \rangle$ [13,20], in accordance with the fact that the solution with $Q = 1$ changes the chiral charge ΔQ_5 by four units. The toron solution with $Q = \frac{1}{2}$ changes the chiral charge by two units and has two zero modes. Therefore the corresponding vacuum transition is necessarily accompanied by the production of a $\bar{\psi}\psi$ pair, as the explicit calculation of (44) also demonstrated. Because the transition amplitude (44) is non-zero, and because the toron tunnelling process changes the chiral charge Q_5 by two units, the true physical states $|Q_1\rangle$ must be the superposition of the states $|Q_5 = 0\rangle$, $|Q_5 = 2\rangle$ [29; 9, page 297]:

$$\begin{aligned} |Q_0\rangle &= \frac{1}{\sqrt{2}} (|Q_5=0\rangle + |Q_5=2\rangle) \\ |Q_1\rangle &= \frac{1}{\sqrt{2}} (|Q_5=0\rangle - |Q_5=2\rangle) \end{aligned} \quad (45)$$

The construction of vacuum states $|Q_1\rangle$ from $|Q_5\rangle$ eigenstates is very similar to the standard $|\theta\rangle \sim \int \exp(i\theta n) |n\rangle$ vacuum consideration, but now there is only a finite number of vacua.

Now the quantum numbers of vacuum states $|Q_5\rangle$ are suitable, so that appropriate linear combinations of the states $|Q_1\rangle$ (45) would be the two vacua of spontaneously broken discrete chiral symmetry:

$$\langle R_k / \bar{\psi}\psi / R_k \rangle = \exp(i\pi k) \langle Q_5 = 2 / \bar{\psi}\psi / 0 \rangle = m \exp(i\pi k / 46) \quad (46)$$

As is well known [9,13], the non-vanishing of the condensate (46) indicates spontaneous breaking of the discrete chiral symmetry $\psi \rightarrow i\gamma_5 \psi$, which does not take place in any order of perturbation theory. This agrees with the value of the Witten index, which equals two [9]. We would like to recall that the instanton calculation gives an average over these vacua [28]:

$$\Theta(x) = \langle \bar{\psi}\psi(x), \bar{\psi}\psi(0) \rangle_{\text{inst.}} = \frac{1}{2} \sum_{k=0}^1 \langle R_k / \bar{\psi}\psi / R_k \rangle^2 \quad (47)$$

Moreover, each vacuum gives one and the same contribution to (47). That is why the non-vanishing result in the instanton calculation can be obtained only for a special form of the Green function, in particular for (47). In this case the result (46) can be obtained by the extraction of the square root.

Let us now discuss the θ -dependence of the $\langle \bar{\psi}\psi \rangle$ condensate. The θ -term in the Lagrangian has the form

$$\mathcal{L}_\theta = \frac{1}{4\pi} \theta \cdot \epsilon_{\mu\nu} F_{\mu\nu} \quad (48)$$

Let us now rescale the field ψ and introduce a new field

$$\psi'_{R,L} = e^{\pm i\alpha} \psi_{R,L}$$

Then, due to the anomaly relation

$$\partial_\mu Q_5 = 4 \cdot \frac{1}{4\pi} \epsilon_{\mu\nu} F_{\mu\nu}, \quad q_\mu = \pi \frac{1}{4\pi} \epsilon_{\mu\nu} F_{\mu\nu}$$

the θ -term in the Lagrangian for ψ' will disappear provided that $\alpha = -\theta/4$. Hence, the θ -dependence of the starting condensate (46) is

$$\langle \bar{\psi} \psi \rangle_\theta = \exp(i \frac{\theta}{2}) / \langle \bar{\psi} \psi \rangle_{\theta=0} \quad (49)$$

Now the θ evolution from $\theta = 0$ to $\theta = 2\pi$ according to the law (49) renumbers two degenerate states (46) with spontaneous breaking of discrete chiral symmetry. An analogous situation has been previously observed in QCD with N_f flavours [11,12] and in SYM in Ref. [10], and will be discussed in the next sections.

To conclude this section we would like to note that the corresponding construction can also be generalized to Cp^{N-1} theories. For a supersymmetric variant of the Cp^{N-1} theory, one can also calculate the $\langle \bar{\psi} \psi \rangle$ condensate, in complete analogy with the calculations discussed above in the $O(3)$ σ -model [30]. Moreover, it can be shown (by the method which will be described for gauge theories in Sections 8 and 9) that the constant C (46) equals one, in agreement with the instanton results [20]. Anticipating events, we also note that the analogous calculations in gauge theories (SYM, SQCD) do not coincide with instanton results [28], and the difference is equal to $1/\sqrt{5}$ (see Section 8).

5. - TORONS IN SU(2) SUPERSYMMETRIC GLUODYNAMICS

We pass now to the analysis of the gauge theories and to the corresponding toron solutions. For this purpose the self-dual equation for the gluodynamics will be formulated in a language analogous to the Cauchy-Riemann condition for the $O(3)$ σ -model. The corresponding criteria, which are discussed above for the $O(3)$ σ -model in terms with local gauge invariance, will help us to choose the "correct" zero modes and to calculate the toron measure and chiral condensate (see next section) in the supersymmetric gluodynamics case.

We begin with the formulation of the self-dual equation in Witten's ansatz [31]:

$$\begin{aligned} A_0^a &= n^a A_0(z, \bar{z}) \\ A_i^a &= \epsilon^{iak} n^k \frac{1 + \bar{z} z}{2} + (\delta^{ai} - n^a n^i) \frac{P_i}{z} + n^a n^i A_i \\ A_\mu &= (A_0, A_i) ; \quad n^a = x^a/z, \quad z = (x_i x_i)^{1/2}, \quad \mu = 0, i \end{aligned} \quad (50)$$

Here i, j, k refer to the three spatial dimensions and a is the isospin index. The precise definitions of $P_{1,2}$ and $A_{0,1}$ are chosen for future convenience. These

functions depend only on z and $\bar{z}$. Given (50), one readily calculates the field tensor $G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - \epsilon^{abc} A_\mu^b A_\nu^c$. The self-dual equations $G_{\mu\nu}^a = \tilde{G}_{\mu\nu}^a$ or, equivalently, $G_{0i}^a = \frac{1}{2} \epsilon_{ijk} G^{jk a}$ now become

$$\begin{aligned} D_\mu \Phi &= i \epsilon_{\mu\nu} D_\nu \Phi, \quad D_\mu \Phi = (\partial_\mu + i A_\mu) \Phi, \quad \Phi = \Phi_1 - i \Phi_2 \\ \partial_0 \partial_0 \ln \frac{1}{z} &= \frac{1}{z^2}, \quad \mu = 0, i \end{aligned} \quad (51)$$

Here ∂_0 denotes $\partial/\partial t$ and ∂_i denotes $\partial/\partial x_i$. We see that ansatz (50) leads to a two-dimensional Abelian Higgs system. The form of (51) suggests that I regard Φ as a charged scalar interacting with the two-dimensional Abelian gauge field A_μ , with covariant derivative $D_\mu \Phi$.

In this case it is more convenient to study the solutions of Eq. (51) by considering the complex plane and defining

$$\begin{aligned} z &= z + i\bar{z}, \quad \partial = \partial/\partial z = \frac{1}{2}(\partial_1 - i\partial_0), \\ \bar{z} &= z - i\bar{z}, \quad \bar{\partial} = \partial/\partial \bar{z} = \frac{1}{2}(\partial_1 + i\partial_0); \quad A = A_1 - iA_0 \end{aligned} \quad (52)$$

It can easily be verified that the most general solution of (51) is [31]

$$\begin{aligned} \Psi &= \ln \frac{z + \bar{z}}{1 - \bar{z}z}, \quad f = \frac{dg}{dz}, \quad g = g(z), \quad \Phi = f e^\Psi \\ A &= A_1 - iA_0 = -2i\partial \Psi = -2i\partial \ln \bar{\Phi} \end{aligned} \quad (53)$$

where $g(z)$ is an analytic function. For ψ to be non-singular, we require

$$|g| = 1 \text{ for } z=0; \quad |g| < 1 \text{ for } z > 0 \quad (54)$$

Thus the expressions (53) for $\Phi = \Phi_1 - i\Phi_2$ and for $A = A_1 - iA_0$ with the extra requirements (54) solve our problem. Namely, the self-dual solution is formulated in terms of the analytical function $g(z)$ analogous to the $O(3)$ σ -model (3). The extra requirement (54) reflects the fact that our theory is defined only on the half-plane $z + \bar{z} > 0$, while the $O(3)$ σ -model is of course defined on the full complex plane.

To determine the solution with topological charge $Q = \frac{1}{2}$, we have to express the four-dimensional topological charge in terms of 2d fields [31]:

$$Q = \frac{1}{32\pi^2} \int d^4x \, G_{\mu\nu}^a \tilde{G}_{\mu\nu}^a = \frac{1}{8\pi} \int_0^\infty \int_{-\infty}^\infty dt \int_{-\infty}^\infty dx \, \frac{1}{2} \epsilon_{\mu\nu} F_{\mu\nu} + i \epsilon_{\mu\nu} \partial_\mu (\tilde{\phi} \partial_\nu \phi) = \frac{1}{8\pi i} \oint_{\partial S} ds \, \frac{d}{ds} \ln f, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (55)$$

It is easy to see from (55) that the topological charge is determined by the change of phase function $f(z)$ around the contour which encloses the region $\text{Re } z > 0$.

Now all is ready for the description of the self-dual solutions in complete analogy with the consideration discussed above in the $O(3)$ σ -model (Sections 2 and 3). The solution $g(z) = (a-z)/(\bar{a}+z)$ satisfies the finiteness condition (54) for any complex number a with $\text{Re } a > 0$. In this case an easy calculation shows that this solution is a gauge transformation of the vacuum; the solution

$$g(z) = \prod_{i=1}^Q \left(\frac{a_i - z}{\bar{a}_i + z} \right), \quad \text{Re } a_i > 0$$

describes the instanton [2] (but in a different gauge) with $Q = 1$. The toron solution with $Q = \frac{1}{2}$ (by analogy with the $O(3)$ σ -model (11)) is described by the function

$$g(z) = \lim_{\Delta \rightarrow 0} \left(\frac{a - z}{\bar{a} + z} \right)^{3/2}, \quad a = \Delta + i\epsilon_0, \quad \text{Re } a = \Delta > 0 \quad (56)$$

The solution $g(z)$ (56) turns out, in a sense, to be between the vacuum ($Q = 0$) and instanton ($Q = 1$) solutions described above. Here $\Delta = \frac{1}{2}(a + \bar{a}) + 0$ is a regulator analogous to the dimensional parameter Δ (13) in the $O(3)$ σ -model. This solution, like in the σ -model, is defined on two Riemann sheets; real physical space corresponds only to one of them and can be understood as a manifold with a boundary. If one sets $\Delta = 0$ from the very beginning, then $g(z) = 1$, corresponding to the empty vacuum solution.

However, the solution (56) is non-trivial. As will be shown in the next section, the solution (56) ensures a non-zero value of the gluino condensate in SYM. An analogous behaviour, as was discussed above, arises in the SUSY $O(3)$ σ -model. Obviously, the solution (56) satisfies the finiteness condition (54) for $\Delta > 0$. The imaginary part of the parameter "a" ($\text{Im } a = \epsilon_0$) determines the location of the toron along the time axis, and without loss of generality we take in what follows $\Delta = 1, \epsilon_0 = 0$. Moreover, upon completion of the large circle in the phys-

cal space ($\text{Re } z > 0$), the function $f(z)$ acquires a phase π which, according to (55), corresponds to $Q = \frac{1}{2}$. In the next section we calculate the value Q by using a gauge-invariant approach. As in the σ -model, the limit of the solution (56) with a cut, tending to zero as $\Delta \rightarrow 0$, means re-establishment of single-valuedness on one physical sheet for the gauge field A_μ^a . The gauge-invariant values (like $G_{\mu\nu}^2$) are single-valued at any finite Δ (not only in the limit $\Delta \rightarrow 0$). So the solution (56) can be understood as a point defect with the Δ -regularization which is preserved by the duality equation at finite value Δ .

Thus Eq. (53) with the analytic function $g(z)$ (56) provides the solution of $G_{\mu\nu} = \tilde{G}_{\mu\nu}$ with finite action and with fractional topological charge. However, the explicit expressions (53) for A_μ, ϕ are not so useful for future calculations because of the complexity of their forms. So we would like to find the appropriate gauge transformations and make changes in the variable for the next analysis.

The rest of the section is a series of technical comments concerning some implications of Eqs. (53) and (56). First we consider the explicit expression for $G_{\mu\nu}^2$. Substituting (50)-(53) in $G_{\mu\nu}^2$, we obtain

$$\begin{aligned} G_{\mu\nu}^2 &= \epsilon_{\mu\nu} \tilde{G}_{\mu\nu} = \frac{4}{z^2} |\partial_\mu \phi|^2 + F_{\mu\nu}^2 + \frac{2}{z^2} (1 - |\phi|^2)^2 \\ |\partial_\mu \phi|^2 &= 2 |\phi|^2 \partial \ln |\phi| / \partial \bar{z} \ln |\phi| / \partial z = \frac{1 - |\phi|^2}{z^2} |\phi|^2 \partial \bar{\partial} |\phi|^2 / \partial z^2 \\ F_{\mu\nu}^2 &= 2 (-4 \partial \bar{\partial} \psi)^2 = 2 \left(\frac{1 - |\phi|^2}{z^2} \right)^2 \end{aligned} \quad (57)$$

It is easy to see that $G_{\mu\nu}^2$ depends only on the absolute value of the field ϕ . So we make the following change of variables:

$$z = -i c t \phi \bar{\omega}, \quad \bar{\omega} = u + i v, \quad \partial = -\frac{i}{1 - z^2} \partial_\omega = -i \sin^2 \omega \cdot \partial_\omega \quad (58)$$

Here $(-v)$ ($0 < (-v) < \infty$) plays the role of the measure of length and (u) ($-\pi/2 < u < \pi/2$) is the angle variable; see Fig. 5. In terms of the variables u, v , the absolute value of $|\phi|$ depends on "v" only and not on the angle variable "u" [this is just the reason for the change of variables (58)]:

$$|\phi| = \frac{3}{2} \frac{\delta h \partial v}{\delta h \partial u}, \quad |\phi|_{v \rightarrow 0} = 1 - \frac{5}{6} v^2 \rightarrow 1, \quad |\phi|_{v \rightarrow -\infty} = \frac{3}{2} e^v \rightarrow 0 \quad (59)$$

We are now ready to describe the dependence of $G_{\mu\nu}^2$ on r, t at $r \rightarrow 0$ and $|z| \rightarrow \infty$. In this case the parameter v tends to zero:

$$\hbar \partial V = \partial V = -\frac{2\gamma}{1+\gamma^2}, \quad \gamma \rightarrow 0 \quad (60)$$

Substituting (58)-(60) for the expression for $G_{\mu\nu}^2$ (57), we obtain

$$G_{\mu\nu}^2(\gamma \rightarrow 0) \sim \frac{1}{(1+\gamma^2)^4}, \quad G_{\mu\nu}^2(\gamma \rightarrow \infty) \sim \frac{1}{\gamma^2} \quad (61)$$

The solution (56) possesses finite action, but in our gauge the four-dimensional gauge field A_μ^a is actually singular at $r = 0$. It is necessary to perform a gauge transformation on the solution to satisfy the regularity condition; such a transformation always exists because $|\phi|^2 = 1$ at $r = 0$, see Eq. (59). We shall discuss the corresponding gauge below. Now we shall consider the behaviour of $G_{\mu\nu}^2$ on the cut. It is easy to see that the gauge-invariant value $G_{\mu\nu}^2$ on the upper edge of the cut coincides with the value on the lower edge and has an integrable singularity as z tends to 1 ($v \rightarrow -\infty$):

$$G_{\mu\nu}^2(z \rightarrow 1) \sim \frac{1/\phi^2}{(1-z^2)(1-\bar{z}^2)} \sim \frac{e^{2V}}{1-z^2} \sim \frac{1}{\sqrt{(1-z)(1-\bar{z})}} \quad (62)$$

The origin of this physical singularity is connected with the fractional topological number of our solution. This singularity plays an important role because it will help us choose the "correct" zero modes in the toron background.

Using (57) and (59), we can calculate the action and topological charge in a more direct, gauge-invariant, way:

$$Q = \frac{S_0^2}{8\pi^2} = \frac{1}{32\pi^2} \int d^4x G_{\mu\nu}^2 = \frac{1}{32\pi^2} \int 4\pi r^2 dr dt \left[\frac{4}{z^2} \frac{1-\phi^2}{\gamma^2} \right] + \frac{8}{\gamma^2} \partial \bar{\partial} / \phi^2 + \frac{4}{\gamma^4} (1-\phi^2)^2 = \frac{1}{2\pi} \int dr dt \left[\frac{1-\phi^2}{z^2} + 2 \partial \bar{\partial} / \phi^2 \right]. \quad (63)$$

The second term is a total divergence, and so can be written as a boundary integral which vanishes because, for solution (56), $\partial_\mu \phi = 0$ at the boundary of the space volume at $v \rightarrow 0$ (59). From expression (59) for $|\phi|$ I find that the first term becomes

$$Q = \frac{S_0^2}{8\pi^2} = \frac{1}{2\pi} \int dr dt \frac{1-\phi^2}{z^2} = \frac{4}{2\pi} \int_{-\frac{1}{2}}^{\frac{1}{2}} du \int_{-\frac{1}{2}}^{\frac{1}{2}} dv \frac{1-\phi^2}{zh^2v} = \frac{1}{2} \int dv \left[\frac{4}{gh^2v} - \frac{9}{gh^3v} \right] = \frac{1}{2}. \quad (64)$$

in agreement with the result described above.

We would now like to find the gauge transformation which allows us to write down the solution (56) in the superpotential form:

$$A_\mu^a = -\bar{\Sigma}_{\mu\nu}^a \partial_\nu \ln P(z, \bar{z}), \quad G_{0123} = 1$$

$$\Sigma_{0i}^a = \delta_i^a, \quad \Sigma_{i0}^a = -\delta_i^a, \quad \Sigma_{ij}^a = -\epsilon^{aij}, \quad \bar{\Sigma}_{\mu\nu}^a = -\frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \bar{\Sigma}_{\alpha\beta}^a \quad (65)$$

Here $\bar{\Sigma}_{\mu\nu}^a$ are the 't Hooft symbols [32]* and P is the superpotential, depending only on r and t . For this let us recall that the duality equation (51) possesses a remaining gauge invariance with any analytic function $h(z)$ [31]:

$$f(z) = f(z)/h(z), \quad \varphi' = \varphi \cdot \frac{h(z)}{|h|} = \varphi \exp(i\alpha) \quad (66)$$

Using complex variables, Eq. (65) can be rewritten as

$$\frac{\varphi' - i}{z} = A' = 2i \partial \ln P(z, \bar{z}) \quad (67)$$

It can easily be verified that (65) and (66) are solved by

$$h(z) = [i + g(z)]^{-2}, \quad \varphi' = \frac{z + \bar{z}}{1 - g\bar{g}} \frac{1 + g}{1 + g} = e^{i\alpha} = e^{i\alpha}/\gamma^2. \quad (68)$$

In fact, substituting (68) for (53) and (66), we obtain the following relations:

* The difference with the standard definition [32] is connected with the non-standard expression for $G_{\mu\nu}^2$ (50).

$$\begin{aligned}
 A' &= -2i\partial\psi - 2i\partial\ln(1+q) = -2i\partial\psi - 2i\frac{\partial q}{1+q} = \\
 &= -\frac{2i}{2} + 2i\left[\frac{-q}{1+q} - \frac{1}{1+q}\right]\partial q = -\frac{i}{2} + 2\frac{1+q}{(1+q)^2}\partial q = \\
 &= -\frac{i}{2} + \left(\frac{fe^q}{1+q}\right)\frac{1+q}{(1+q)^2} = -\frac{i}{2} + \frac{1}{2}\phi' \quad (69)
 \end{aligned}$$

It is easy to see that relation (69) is precisely the superpotential solutions (65) and (67). Moreover, as will be shown below, the solution in the superpotential form (65) is regular at $r = 0$; because $\phi'(r=0) = 1$, and hence $\phi_1'(r=0) = 0$, $\phi_2'(r=0) = -1$ and A_μ^a (50) is regular at $r = 0$. In the following relations the superpotential solution (65) will be used only and corresponding primes will be removed.

It is instructive to check the result (65) with superpotential P (68) in a more direct way. As is well known, the self-duality equation for the gauge (65) can be written as follows [33]:

$$P' \square P = 0 \quad (70)$$

Here $\square$ is the four-dimensional Laplacian. By algebraic manipulations this can be proved using the relations (57) and (66):

$$\begin{aligned}
 P' \square P &= P'(\partial_0^2 + \partial_1^2 + \frac{2}{2}\partial_1/P = 4P' \partial \partial P + \frac{2}{2}\partial \ln P + \frac{2}{2}\partial \ln P = \\
 &= \frac{1}{2} \left(1 - \frac{1}{1+q}\right) + \frac{(\phi_-')(\phi_+')}{2x} + \frac{\phi_-'}{1+q} - \frac{\phi_+'}{1+q} = 0 \quad (71)
 \end{aligned}$$

To conclude this section, I would like to analyse the superpotential P and field ϕ in the gauge (65). Let us consider the P, ϕ functions in the u -variable. From (58), (66) and (68) we have

$$P' = \frac{2i \sin(\omega - \bar{\omega})}{\sin \frac{3}{2}(\omega - \bar{\omega})} \cdot \frac{\sin(\frac{3}{2}\bar{\omega})}{\sin \omega \cdot \sin \bar{\omega}}$$

$$\phi = \frac{2}{2} \cdot \frac{\sin(\omega - \bar{\omega})}{\sin \frac{3}{2}(\omega - \bar{\omega})} \cdot \frac{\sin \bar{\omega}}{\sin \bar{\omega}} \cdot \frac{\sin(\frac{3}{2}\bar{\omega})}{\sin(\frac{3}{2}\bar{\omega})} \quad (72)$$

Let us summarize some important properties of these functions. The discussion of these properties will facilitate the choice of the "correct" zero modes in the toron background in the next section. First we note that the P -function is a regular, single-valued function, tending to constant as $|z| \rightarrow \infty$. Then, as is easily verified from the explicit expression (72), the ϕ -function tends to (1) when v tends to zero. As mentioned above, the gauge potential A_μ^a is therefore regular at $r = 0$. Besides that, the imaginary part of ϕ ($\text{Im} \phi = -\phi_2$) is a single-valued function and the real part ($\text{Re} \phi = \phi_1$) has a different sign on the different edges of the cut. Thus the expression

$$\bar{A}A = \left(\frac{\phi_-'}{2}\right) \left(\frac{\phi_+'}{2}\right) = \frac{1}{2x} [\bar{\phi}_1^2 + (1 + \phi_2)^2] \quad (73)$$

is a regular and single-valued function.

The properties of the functions discussed above, $G_{\mu\nu}^2$, P , ϕ are very important for further analysis.

6. - ZERO MODES, TORON MEASURE AND GLUINO CONDENSATE IN SYM

As we know, supersymmetric models differ conveniently from ordinary ones in that only zero modes need be considered. This is because the non-zero modes cancel between bosons and fermions, and the measure is defined by the zero modes only.

Usually the explicit determination of zero modes in any self-dual field is a simple exercise because they can be expressed in terms of the field strength tensor $G_{\mu\nu}^a$. Indeed, in the $D_{\mu}^{cl, a}$ -gauge, zero modes satisfy the following equation [32]:

$$[(-D^2)^{ac} g_{\mu\nu} + 2\epsilon^{abc} G_{\mu\nu}^b] q_c = 0$$

$$D_\mu^a q_\mu^c = 0$$

$$D_\mu^a \equiv \delta^{ac} \partial_\mu - \epsilon^{abc} A_\mu^b; [\bar{D}_\mu, D_\nu]^a = -\epsilon^{abc} G_{\mu\nu}^b \quad (74)$$

Here A_μ^a is the classical field and a_μ^a is the small quantum fluctuation, μ being the four-dimensional index. It is easy to prove that four translation modes, three gauge modes and one conformal mode can be expressed in terms of $G_{\mu\nu}^a$, and they satisfy Eq. (74)

$$\begin{aligned} q_\mu^a(\lambda) &\sim G_{\mu\lambda}^a, \quad \lambda = 0, 1, 2, 3 \\ q_\mu^a(d) &\sim G_{\mu\nu}^a \sum_{\nu\lambda} x_\lambda, \quad d = 1, 2, 3 \\ q_\mu^a &\sim G_{\mu\nu}^a x_\nu \end{aligned} \quad (75)$$

In particular, in the instanton field $G_{\mu\nu}^a \sim \eta_{\mu\nu}^a (1/(1+x^2)^2)$, we obtain eight well-known zero modes [32]

$$q_\mu^a(\lambda) \sim \sum_{\nu\lambda} \frac{1}{(1+x^2)^2}, \quad q_\mu^a(d) \sim \sum_{\nu\lambda} \frac{x_\lambda}{(1+x^2)^2}, \quad q_\mu^a \sim \sum_{\nu\lambda} \frac{x_\nu}{(1+x^2)^2} \quad (76)$$

These modes are normalizable and regular, and thus they satisfy all requirements which were discussed above (see Section 3).

In the toron case under consideration with topological charge equal to 1, the zero modes (75) still satisfy Eq. (74). They do not, however, satisfy the regularity condition and so they are inadmissible modes. In fact, because $G_{\mu\nu}^a$ (62) has a singularity at $z \rightarrow 1$, then the same singularity has zero modes (75). Thus they are forbidden in the toron case.

This situation was expected beforehand from the corresponding analysis of the $O(3)$ σ -model. Let us recall that the zero modes (75) are derivatives of the classical solution up to a gauge transformation. In the instanton case this is a correct way to obtain zero modes, but not in the toron case, as can be seen from consideration of the zero modes in the $O(3)$ σ -model. In this case the toron solution is as follows: $\phi_{cl} \sim (z-a)^{-\frac{1}{2}}$ (13). The "natural" zero mode which is the derivative of ϕ_{cl} with respect to the collective co-ordinate "a" is $\delta\phi_0 \sim 3/3a\phi_{cl}$

$(z-a)^{-3/2}$. Such a function satisfies the corresponding equation for zero modes $\bar{\partial}(\delta\phi_0) = 0$ (28). However, this function does not satisfy the regularity condition (29) and hence is not acceptable. As we know, the correct zero mode is $\delta\phi_0 \sim z^{-1}$ (34).

From this, we may conclude that the "natural" zero modes satisfy the corresponding equations, but they are forbidden because of the regularity requirement. To find the correct zero modes, it is necessary to solve the corresponding equation (74) explicitly and in such a way that it satisfies the regularity requirement.

However, as is well known [34], the vector field equations for the small fluctuations about a self-dual field are equivalent to the Dirac equation for a spinor with unit isospin and with definite chirality. The problem of counting the number of the modes of massless excitations of the vector field is thus reduced to that of counting the number of massless modes of the Dirac field. However, the number of the fermion zero modes is known beforehand, from consideration of the axial anomaly; so, for each solution of the spinor field equation there are precisely two linearly independent solutions of the vector field equation [34].

For instance, for the instanton we expect four fermion zero modes, in accordance with the fact that the solution with $Q = 1$ changes the chiral charge ΔQ_5 by four units and can ensure a non-vanishing value for the correlator $\langle \bar{\psi}\psi(x), \psi\psi(0) \rangle$ [7,8]:

$$\psi_{1,2}^a \sim \sigma_\mu^+ G_{\mu 0}^a \cdot \epsilon, \quad \psi_{3,4}^a \sim \sigma_\mu^+ G_{\mu\nu}^a x_\nu \cdot \epsilon \quad (77)$$

Thus, from [34] we expect eight real vector zero modes in agreement with the explicit calculation (75).

For the toron we expect two (real) fermion zero modes with unit isospin and so we expect four (real) vector zero modes in agreement with four collective co-ordinates associated with translations of the toron solution. We recall that the parameter Δ is the regulator, but it does not have the sense of a collective co-ordinate. Because the toron solution with $Q = \frac{1}{2}$ changes the chiral charge by two units and has (as will be shown explicitly below) two gluino zero modes, the corresponding vacuum transition is necessarily accompanied by the production of a $\bar{\psi}\psi$ (pair). We thus expect that the toron can ensure a non-vanishing value for the condensate $\langle \bar{\psi}\psi \rangle$.

We now return to the analysis of Eq. (74). As was discussed above, these equations can be rewritten in terms of the spinor field with isospin one. To make this construction explicit, we note that if

$$\psi = \begin{pmatrix} a \\ b \end{pmatrix}$$

is a solution of the spinor field equation

$$\sigma_\mu^- \partial_\mu \psi = 0, \quad \sigma_\mu^+ = (\pm i \vec{\sigma}), \quad \sigma_\mu^- \sigma_\nu^+ = g_{\mu\nu} - i \vec{\sigma}_{\mu\nu} \cdot \vec{\sigma}^a \quad (78)$$

then two real vector modes

$$a_\nu = -\frac{1}{2} \vec{\sigma}_\nu \cdot (\vec{\sigma}_\nu^+ \psi), \quad \varphi'' = \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix}, \quad \varphi' = \begin{pmatrix} i a & i b^* \\ i b & -i a^* \end{pmatrix} \quad (79)$$

satisfy the starting equation (74), see Ref. [34].

We are now ready to find the explicit solution of the Dirac equation (78). For this we consider the toron solution in the gauge (65), $A_\mu^a = -\vec{\eta}_{\mu\nu}^a \ln P(r, t)$, and look for the solution of (78) in the corresponding form

$$\psi^c = \sum_{\mu} \sigma_\mu^+ \partial_\mu P \cdot f(P) \cdot \varepsilon \quad (80)$$

Here ε is the constant Lorentz spinor, P is the superpotential (68), and lastly $f(P)$ is some function which will now be found.

Substituting the expression (80) for the Dirac equation (78), we obtain the following equation:

$$[\partial_\mu \delta^{ac} + \varepsilon \vec{\sigma}_{\mu\nu} \partial_\nu \ln P] \sigma_\mu^- \sigma_\nu^+ \vec{\sigma}^c \partial_\nu P \cdot f(P) \cdot \varepsilon = 0 \quad (81)$$

Using the properties of the σ_μ^a matrix (78) and the $\vec{\eta}_{\mu\nu}^a$ symbols [32], Eq. (81) can be rewritten in the following form:

$$[P \partial_\mu f + \partial_\mu P] \left[\frac{2}{\partial P} + \frac{2f}{P} \right] = 0$$

Because the P -function satisfies Eq. (70), the f -function is equal to

$$f(P) = P^{-2}$$

Substituting this solution in (80), we obtain the following expression for two gluino zero modes:

$$\psi^a \sim \vec{\sigma}_{25}^a \sigma_1^+ (\partial_\sigma \ln P) P^{-1} \cdot \varepsilon \sim \sigma_\mu^+ A_\mu^a P^{-1} \cdot \varepsilon$$

$$\varepsilon = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad |\psi^a|^2 \sim P^{-2} A_\mu^a A_\mu^a = 3 P^{-2} \vec{A} \cdot \vec{A}. \quad (82)$$

In obtaining (82), we took into account the expression for the toron solution (65) and the expression for the potential in complex notation (52). Now we are ready to prove that the solution (82) is a regular, single-valued and square-integrable zero mode. For this purpose let us recall that P is the regular function tending to constant for $|Z| \rightarrow \infty$. Then $|\langle \Phi-1 \rangle / r|^2$ is the regular function at the critical points $r = 0$, $Z = 1$. Moreover, the gauge-invariant value $|\psi^a|^2$ is a single-valued function because of the property (73); lastly, as can be seen from Eq. (72), the $|\psi|^2$ tends to zero like $|Z|^{-6}$ at $|Z| \rightarrow \infty$, and so $|\psi|^2$ is square-integrable. Thus both the fermion zero modes satisfy all the requirements discussed above and must be taken into account. In the following relations the normalization

$$\int |\psi^a|^2 d^4x = 1 \quad (83)$$

is assumed, although not explicitly indicated.

Now we would like to note that the instanton solution can be expressed in the form (65) as well (the so-called "singular gauge"). In this case the superpotential "P" and the gauge field A_μ^a are respectively equal to

$$P = \frac{x^2 + 1}{x^2}, \quad A_\mu^a = 2 \vec{\sigma}_{\mu\nu}^a \frac{x_\nu}{x^2} \frac{1}{1+x^2}$$

The zero modes (82) are now

$$\psi^a \sim \sigma_\mu^+ \vec{\sigma}_{\mu\nu}^a \frac{x_\nu}{(1+x^2)^{3/2}} \cdot \varepsilon$$

and coincide with expression (77). In the toron case the solution (82) does not coincide with the expression (77) and only the solution (82) satisfies all our requirements.

As we mentioned above, the number of the "correct" fermion zero modes is known beforehand from a consideration of the axial anomaly, and we wanted to demonstrate just the realization of this general index theorem. The explicit calculation (82) confirms our general consideration. Substituting (82) for two gluino zero modes in Eq. (79), we obtain four gluon zero modes satisfying all requirements. This comes from the fact that the gauge-invariant values for bosonic and fermionic zero modes coincide, as can be seen from (79):

$$1/4 a_1^2 \sim g_4^2 q_4^2$$

An explicit form of these gluon zero modes a_μ^a is inessential for our purpose; however, the number of the admissible vector zero modes is a very important question. This number is equal to four, in agreement with the general discussion, and in accordance with the existence of four translation collective co-ordinates.

With the above consideration taken into account, the toron measure in SYM acquires the following form:

$$Z_{\text{toron}} = C \frac{M_0^4}{g^4} d^4 x_0 \frac{d^2 \tilde{\epsilon}}{M_0} \exp\left(-\frac{4\pi^2}{g^2}\right) = C \frac{1}{g^4} \Lambda_{1\text{-loop}}^3 d^4 x_0 d\tilde{\epsilon} \quad (84)$$

$$\Lambda_{1\text{-loop}}^3 \equiv M_0^3 \exp\left(-\frac{4\pi^2}{g^2(M_0)}\right).$$

Here the factor $g^{-4} M_0^4 d^4 x_0 \sim (\sqrt{\text{Sci.}} M_0 dx_0)^4$ is due to the four bosonic zero modes mentioned above; $d^4 x_0$ is the corresponding integral over the collective variables, the factor $d^2 \tilde{\epsilon} M_0^{-1}$ is connected with two fermionic zero modes (82); lastly, $\exp(-4\pi^2/g^2)$ is the contribution of the classical toron action (84), and "C" is some calculable constant.

As in the case of the instanton calculation [7,8], the expression (84) for the toron measure has precisely the renorm-invariant form. It is easy to trace this phenomenon, starting from the instanton density [7,8,28]:

$$Z_{\text{inst.}} = \frac{M_0^8}{g^8} d^4 x_0 \frac{d^2 \tilde{\epsilon}}{(2\pi^2)^2} \frac{4}{\pi^2} (4\pi^2)^4 d^2 \tilde{\epsilon}_1 d^2 \tilde{\epsilon}_2 \exp\left(-\frac{8\pi^2}{g^2}\right) \quad (85)$$

Here the factor $\exp(-8\pi^2/g^2)$ is connected with the instanton action, $M_0^8 g^{-8}$ is the regulator contribution corresponding to the eight bosonic zero modes (76), and the factor $(d^2 \tilde{\epsilon}_1/M_0)(d^2 \tilde{\epsilon}_2/M_0)$ is connected with the four fermionic zero modes (77). While the action decreases by a factor two, the number of admissible zero modes decreases by the same factor, which ensures the correct renorm-invariant behaviour. Let us recall that an analogous situation occurs in the σ -model, see Section 4.

Now all is ready for the calculation of the gluino condensate in SYM. Substituting in place of ψ their zero modes (82), we verify that

$$\langle g^2 \lambda^2 \rangle = 2C \frac{1}{g^2} \Lambda_{1\text{-loop}}^3 \quad (86)$$

In the last step we used the value of the normalization integral (83) and changed the notation of the gluino field from ψ^a to λ^a , because in the next section we

shall consider the model with a fundamental representation of fields (quarks). Namely, for quark fields we reserve the ψ -notation.

The non-vanishing of the transition amplitude (86) means the non-vanishing of the corresponding condensate, analogous to the σ -model consideration (Section 4). In the case which we look at here, as in the σ -model, the non-zero value of the gluino condensate indicates the spontaneous breaking of discrete chiral symmetry, which does not take place in any order of perturbation theory. The toron solution with $Q = \frac{1}{2}$ changes the chiral charge by two units and has two admissible zero modes (82). Therefore the corresponding vacuum transition is necessarily accompanied by the production of a λ^2 pair, as the explicit calculation of (86) also demonstrated. As in the σ -model, the correct physical states are the linear superpositions (45) of the states with the definite chiral charge, and we shall not dwell on this issue.

To conclude this section we would like to discuss the possibility of the extraction of the constant C in Eq. (84) from the instanton formula (85). For this, let us consider two torons at positions Z_1 and Z_2 . For small toron separation we suppose that $\frac{1}{2}(Z_1 + Z_2) = X_0$ can be interpreted as the position of the instanton, and $(Z_1 - Z_2) = \rho \rightarrow 0$ as the size of this instanton.

Each of these systems (two torons and one instanton) has the same values of the action, topological charge and the numbers of fermionic and bosonic zero modes. Moreover, this interpretation is supported by consideration of the expressions for the toron (84) and the instanton (85) densities. In fact, one can interpret $d^4 x_0 d^2 \tilde{\epsilon}$ from (85) as the integral over two toron collective co-ordinates (translations Z_1 and Z_2), because $d^4 x_0 d^2 \tilde{\epsilon} \rho = d^4 Z_1 d^2 Z_2^*$. Now the instanton measure (85) can be understood as the square of the toron measure (84) and so

*) As will be shown below, this interpretation is still valid in other theories, such as SQCD, QCD, and so on. Moreover, our conjecture is confirmed by consideration of the arbitrary gauge group G. In this case, as is well known, the number of bosonic zero modes in the instanton field is defined by the quadratic Casimir operator $C(G)$ and equal $4C(G)$. In particular for the $SU(N)$ gauge group, $C(SU(N)) = N$ and $4N$ instanton zero modes can be interpreted as translations of N torons. We think that in this case the admissible value for the toron topological charge is $Q = 1/N$ [29], as in CP^{N-1} theories [30]. Besides that, this conjecture is in agreement with the number of vacuum states (equal to N) in SYM and with $2N$ gluino instanton zero modes, which are necessarily present in this model. So the instanton measure can be understood as $Z_{\text{inst}} \sim (Z_{\text{toron}})^N$. This formula, as can be shown, ensures the correct renorm-invariant dependence of Z_{toron} and a non-zero value for $\langle \lambda^2 \rangle$ in analogy with the result (84) described above for the $SU(2)$ gauge group.

$$Z_{inst.} = \frac{1}{g!} Z_{toron}(z_1) \cdot Z_{toron}(z_2) \quad (87)$$

where the combinatorial factor $(2!)^{-1}$ is necessary to avoid double counting. From (84), (85) and (87), we obtain

$$C = 2^5 \pi^2 \quad (88)$$

Now we are ready to compare our direct calculation of $\langle g^2 \lambda^2 \rangle$ (86) with the strong coupling instanton computation [8,28] and with an indirect method [10] which allows one to compute $\langle g^2 \lambda^2 \rangle$ exactly. From the expression (86) for the constant "C" it can be seen that our result (86) is in agreement with [10] and disagrees with [8,28] by a factor $\sqrt{4/5}$. We do not yet understand the origin of this difference because our method, in a sense, is the strong coupling computation analogous to the method of Refs. [8] and [28].

Ending this section, let us stress that in our approach we have considered the contribution to $\langle g^2 \lambda^2 \rangle$ just due to the classical toron solution; the instanton solution can ensure a non-vanishing value only for a certain correlator $\langle \lambda^2, \lambda^2 \rangle$. Besides that, we have found two vacua in SYM (45) with non-vanishing $\langle \lambda^2 \rangle$ in an explicit way, in agreement with the Witten index and with the discrete symmetry breaking in this model. The instanton calculation gives only an average over these vacua.

7. - THE TORON MEASURE IN SUPERSYMMETRIC QCD (SQCD)

This model, besides the SYM part we have discussed before, contains the N_f matter fields in the fundamental representation. We shall follow the notation of Ref. [28] and consider the toron measure in SQCD.

In this case, in comparison with the toron measure in SYM (84), we have additionally the two factors d_B and d_f connected with bosonic and fermionic matter fields and with the corresponding quadratic functional integration around the toron [32]:

$$Z_{SQCD} = Z_{SYM}(d_f)^{N_f} (d_B)^{N_f} \quad (89)$$

$$d_f = \text{Det} \left[\begin{matrix} -i\hat{D} - im \\ -i\hat{D} - im \end{matrix} \right] \quad d_B = \text{Det} \left[\frac{-\hat{D}^2 + m^2}{-\hat{D}^2 + m^2} \right]$$

Here the regulator contribution is taken into account implicitly. Formal manipulation of d_f allows us to write [35]:

$$m \frac{d}{dm} \ln d_f = \text{Tr} \left[\frac{m^2}{-\hat{D}^2 + m^2} - \frac{m^2}{-\hat{D}^2 + m^2} \right] =$$

$$= \text{Tr} \left[\frac{m^2}{-\hat{D}^2 + m^2} - \frac{m^2}{-\hat{D}^2 + m^2} \right] - \text{Tr} \left[\frac{m^2 \delta_5}{-\hat{D}^2 + m^2} \right]. \quad (90)$$

The symbol Tr denotes a trace over space-time, Dirac and colour indices. In obtaining (90) we took into account the identity

$$\hat{D}^2 \left(\frac{1 + \delta_5}{2} \right) = \hat{D}^2 \left(\frac{1 + \delta_5}{2} \right)$$

which is valid for any self-dual fields. It is well known [34,36] that the last term in (90) is connected with the index of the Dirac operator, is actually independent of "m" and is equal to a topological charge Q of the background field. This is established most easily from the formal expression of the last term in (90):

$$- \text{Tr} \left[\frac{m^2 \delta_5}{-\hat{D}^2 + m^2} \right] \equiv \text{Tr} (m^2) \quad (91)$$

Differentiating $\Gamma(m^2)$ with respect to m^2 , we conclude that $\partial/\partial m^2 \Gamma(m^2) = 0$. Thus, $\Gamma(m^2)$ is a constant which can be evaluated by the large m^2 limit. In this limit, $-\hat{D}^2$ can be replaced by p^2 , where the momentum p_μ has the differential operator realization $-i\partial_\mu$. Thus, the calculation in this case is a very simple one and yields $\Gamma(m^2) = Q$ [34,36].

The first term in (90) is connected with non-zero modes and cancels with the contribution of d_B corresponding to bosonic non-zero modes. So, from Eq. (89) we have

$$m \frac{d}{dm} \ln Z_{SQCD} = Q N_f, \quad Z_{SQCD} = Z_{SYM} \left(\frac{m}{N_0} \right)^{Q N_f} \quad (92)$$

In particular, in the instanton case with $Q = 1$, the expression (92) corresponds simply to the zero-mode contribution, i.e., $Z_{SQCD} \sim m^{N_f}$, as it should. So the $m^2 \rightarrow 0$ limit of $\Gamma(m^2)$, at integer value Q, receives contributions only from zero modes, and each normalized zero mode gives an exactly unity contribution to Γ (91).

In the toron case with fractional topological charge, formula (92) is still correct but the quark's admissible zero modes are absent. The puzzle now is as follows: the right-hand side of (91) need not be an integer (in our case $\tau = \frac{1}{2}$), and the left-hand side of (91) is usually determined by zero modes with the integer contribution to Eq. (91). The paradox is solved in the following way.

Formula (91) is still correct because of the existence of contributions to the left-hand side which come from the continuum. Indeed, let us call $C(\lambda, r)$ the spectral density of the corresponding Dirac operator [37]. The parameter $r \rightarrow \infty$ in the $C(\lambda, r)$ has a sense of infra-red regularization and tends to infinity in the final answer [37]. In this case, Eq. (91) can be rewritten as follows:

$$\lim_{z \rightarrow \infty} \int d\lambda C(\lambda, z) \frac{m^2}{\lambda^2 + m^2} = Q = \frac{1}{2} \quad (93)$$

and at $m \rightarrow 0$ we have

$$\lim_{z \rightarrow \infty} C(\lambda, z) \rightarrow Q \delta(\lambda) \quad (94)$$

This is exactly the result of Ref. [37], where it was argued that the continuum contribution is zero unless there are bound states or unbound resonances at $\lambda = 0$ in the spectrum of the Dirac operator. So we can conclude that in the presence of topologically non-trivial gauge fields [$Q = \frac{1}{2} \neq 0$ in Eq. (93)], the spectrum of the Dirac operators must include an unbound resonance at $\lambda = 0$. In either case, when the system is placed in a box of radius R_0 , the lowest eigenvalue will approach zero faster than R_0^{-1} . The argument given by 't Hooft [32] and formula (92) allows us to conclude that the functional integral tends to zero as $Z \sim m^Q \sim m^{\frac{1}{2}}$ when $m \rightarrow 0$. We will call these modes "quasi-zero modes" (QZM), because they belong to the continuum.

As will be shown below, the Dirac spectrum with the aforementioned properties (the absence of a gap between the zero eigenvalue and the continuum) plays a very important role in the forming of the chiral condensates. On the other hand, these properties are exactly the features of the solutions with fractional topological number. Let us recall that in the instanton case with integer $Q = 1$, the left-hand side of (91) is determined by the zero mode and the continuum is separated by the gap.

From a mathematical point of view, this phenomenon is connected with the definition of our solution on the manifold with boundary and the index theorem should be modified (for more details see Refs. [26] and [27]). Equations (91) and (93) can be understood from another point of view, namely, from an extension of

Levinson's theorem of potential scattering, from the analysis of the Jost ratio, and so on [38].

In any case, the result (92) does not depend on our interpretation, and thus we can write the toron measure in SQCD:

$$Z_{SQCD} = 2\pi^2 \frac{1}{g^4} M_0^4 d\lambda_0 \frac{d^2 \vec{m}}{M_0} \left(\frac{m}{M_0} \right)^{N_f/2} \exp(-g^2). \quad (95)$$

In obtaining (95) we take into account Eqs. (89) and (92) and the expression for the toron measure for SYM (84) with constant C (88).

The expression (95) implies that each fermion contribution equals $1/m$, and so the formulation of the theory with $m = 0$ from the beginning is incorrect. This fact was noticed in Refs. [28] and [39], from another consideration. Thus, for the small (but non-zero) values of m , the theory is well defined and the chiral limit will be understood just in the sense of the limit $m \rightarrow 0$.

8. - CALCULATION OF CONDENSATES IN SQCD

Because we have an explicit expression for the toron measure (95), we can calculate the gluino condensate $\langle \lambda^2 \rangle$ in SQCD. As usual, substituting in place of λ their zero modes (82), we verify, as in SYM, that $\langle \lambda^2 \rangle$ is non-zero and equal to:

$$(96)$$

As in the case of the calculation of $\langle \lambda^2 \rangle$ in SYM (86), the expression (96) for $\langle \lambda^2 \rangle$ in SQCD has precisely the renorm-invariant form. Moreover, the value (96) we obtain for $\langle g^2 \lambda^2 \rangle$ satisfies the expected mass dependence $\langle \lambda^2 \rangle N_f^{-1} \sim 1/m$, $\langle \lambda^2 \rangle N_f^{-2} \sim m$, [28]) and differs only by a numerical factor $\sqrt{4/5}$ from the value obtained in Refs. [28] and [39], and coincides with the result of Ref. [40].

Let us note that the possibility of instanton calculations [40] is limited to the cases with $N_f < N = 2$. Our result (96) is still valid for all N_f and in this sense is a new one. It is more important, however, that in our approach we can find the $\langle \lambda^2 \rangle$ itself; in instanton calculations, the non-zero contribution can be obtained only for some Green function. The condensates in this approach are obtained by extraction of the root. Besides that, the mass-dependence of (96) has a very unnatural form from the instanton point of view, because instanton calculation gives an integer degree of (m) (the correct answer, as mentioned above, is obtained by extraction of the root). In toron calculation the dependence $m^{\frac{1}{2}}$ is a

natural one, because in the toron background the QM ensure exactly this dependence (92) on the toron measure (95).

In a more general case of the $SU(N)$ gauge group, one may think that the admissible value for the toron topological charge equals $1/N$ (see page 34). In this case, one can obtain that the dependence of $\langle \lambda^2 \rangle$ on " m " equals $(m)^{N_f/N_c}$, in agreement with the general results of the supersymmetry and Ward identities [28].

Another important difference with the instanton calculation is related to the fact that in our approach, analogously to the σ -model results, we have degenerate vacua (45) with spontaneously broken symmetry. The instanton calculation gives an average over these vacua [28], and we do not dwell on this issue.

At this point one can make use of the Konishi anomaly [41]

$$\frac{1}{32\pi^2} \langle g^2 F^2 \rangle = m \langle \tilde{\varphi} \varphi \rangle \quad (97)$$

and find the value of the scalar field condensates. However, an independent determination of them is possible and will provide a valuable consistency check of our approach.

The computation of $\langle \tilde{\varphi} \varphi \rangle$ is a more difficult problem, for two reasons. The first is that $\langle \tilde{\varphi} \varphi \rangle$ is zero to lowest order in g because $\phi = 0$ at the toron saddle point. In order to get a non-zero contribution to $\langle \tilde{\varphi} \varphi \rangle$ we have to bring down the two vertices from $\exp\{-S_{\text{Yukawa}}\}$, see Fig. 6. Secondly, no closed form exists for the inverse of the operator $(D^2 - m^2)$ for non-vanishing masses in the toron (as in the instanton) background. This difficulty will be circumvented by going to a large m -limit, as was discussed in Refs. [28] and [39]. Using the methods employed in these works, we obtain the following expression for the condensate:

$$\langle \tilde{\varphi} \varphi(x) \rangle = 2^5 \pi^2 \frac{1}{g^4} \Lambda^{3-\frac{N_f}{2}} m^{\frac{N_f}{2}} d\tilde{E}(-2g^2 m) \int d\tilde{x}_0 d\tilde{y} d\tilde{y}' \cdot \int d\tilde{x} \left(\frac{1}{D^2 - m^2} \right)_{14} \lambda(\tilde{y}) \left(\frac{1}{D^2 - m^2} \right)_{44'} \lambda_0(\tilde{y}') \left(\frac{1}{D^2 - m^2} \right)_{4'x} \cdot \quad (98)$$

In obtaining Eq. (98), we take into account the expression (95) for the toron measure. Here λ_0 is the gluino zero mode (82) and $(D^2 - m^2)^{-1}$ is the corresponding Green function in the toron field, which is localized at the point x_0 . A more general case of the different values of the mass can be reached by the trivial replacement in the measure

$$(m)^{N_f/2} \rightarrow \prod_{i=1}^{N_f} \sqrt{m_i}$$

As in the instanton calculations [28,39], we can replace the Green function $(D^2 - m^2)^{-1}$ by the free propagator in the limit $m \rightarrow \infty$. The evaluation of Eq. (98) is now straightforward and gives

$$\langle \tilde{\varphi} \varphi \rangle = 2^5 \pi^2 \Lambda^{3-\frac{N_f}{2}} (m)^{N_f/2} \frac{1}{m^{3/2}} \quad (99)$$

in accordance with the Konishi anomaly (97) and with the value of the gluino condensate (96). The possibility of two independent calculations, hence, is the non-trivial check of our approach.

It is not difficult to understand this result, at least from a technical point of view. In fact, the contribution surviving in the large m limit is identical to what one finds for the contribution of the regulators when the Pauli-Villars method is used in the derivation of the anomaly equation.

Of course, thanks to supersymmetry, the $\langle \tilde{\varphi} \varphi \rangle$ condensate at the small m limit is the same, and is equal to (99). However, an explicit check of the mass dependence of Eq. (98) at $m \rightarrow 0$ is still lacking.

We end the discussion of the calculations in massive SQCD with a few remarks.

First, we expect that the integral (98) at the small m -limit is equal to the value (99) and thus has a singularity (m^{-2}) at $m \rightarrow 0$. This means that the QM, playing an important role in the toron measure [see the discussion of Eq. (93)], should automatically give a correct expression for the integral (98) and correspondingly for the condensate (99).

As a second remark, we note that our results exhibit a discontinuous behaviour when going from the small mass limit of the massive theory to the strictly massless one from the beginning. Indeed, at $N_f = 2$, we have the finite value for $\langle \tilde{\varphi} \varphi \rangle \sim \Lambda^2$ (99). However, starting from the measure for the massless theory [in this case $Z_{\text{SQCD}}(m=0) = 0$], we obtain the wrong result (we recall that the zero modes of quarks are absent in the toron background). We think that such behaviour is related to the fact that the massless theory is not well defined.

number, in accordance with Eq. (92). The second term in Eq. (101), $(-1/3 \ln M_0) N_f$, is related to the regulator contribution of the non-zero modes and is analogous to the contribution of the spinless field. This correspondence is in agreement with the expression (90), where the first term coincides with the spinless field contribution and the last term ensures the QZM contribution.

Collecting all factors together, we obtain the following expression for the toron density in QCD:

$$Z_{QCD} = K \frac{M_0^4}{g^4} d^4 X_0 \left(\frac{m}{M_0} \right)^{N_f/2} \exp\left(-\frac{4\pi^2}{g^2}\right) \exp\left(-\frac{2-N_f}{6} \ln M_0 \Delta\right). \quad (102)$$

Here the standard factor $M_0^4 g^{-4} d^4 X_0$ is due to the four translation zero modes; the factor $(m/M_0)^{N_f/2}$ is connected with QZM; lastly, the factor $\exp(-(2-N_f)/6 \ln M_0 \Delta)$ is the contribution of the gauge (100) and the fermion (101) modes, and "K" is the constant. In obtaining (102) we took into account the fact that the contribution of the non-zero modes which related to the first term of Eq. (90) is small in the limit $m \rightarrow 0$ ($\sim m^2 \ln m$, see Appendix B). Let us recall that in SQCD this contribution cancels with the corresponding boson determinant d_B .

As it should, the expression (102) for the toron measure has precisely the renorm-invariant form. It is easy to trace this phenomenon: while the action decreases by a factor two, the number of admissible zero modes decreases by the same factor and the contribution of the non-zero modes (100), (101) is smaller by a factor of two also (because of a factor Q).

The parameter Δ which is present in the expression (102) is the regulator of our toron solution. As discussed above, it may be understood as the point defect with size $\Delta \rightarrow 0$. The important difference with the supersymmetric case is that in SUSY theories the regulator Δ in the expression for the toron measure is absent because of the cancellation of the non-zero mode contributions. In the QCD case the dependence of the toron density (102) on Δ appears.

In particular, for $N_f = 1$ we have $Z_{QCD}(N_f = 1) \sim d^4 X \Delta^{-1/6}$. It means that the toron density increases when the regulator parameter Δ tends to zero. But the semiclassical expression which was used in obtaining the expression (102) is correct only for the small densities ($Z \sim d^4 X \Delta^{-1/6} \ll 1$). This means that the semiclassical approximation (or, in other words, the dilute-gas approximation) is broken down in this case and the toron interaction should be taken into account.

At the moment, this interaction cannot be taken into account in gauge theories in a self-consistent way, but this can be done in the usual $O(3)$ σ -model. It turns

9. - THE CHIRAL CONDENSATE IN QCD

In the previous section we saw that the spectrum of the Dirac operator for the fundamental representation of the field is very unusual. Namely, the gap in the spectrum is absent and the QZM appear. Exactly these properties ensure the correct results in the well-known supersymmetric theories. In particular, we saw the correct mass dependence of condensates, the correct renorm-invariant relation for the measure, and so on. All these results are in agreement with the general relations of the supersymmetry, Ward identities and Konishi anomaly.

In this section we shall discuss the toron calculation in the physically interesting theory of QCD with $SU(2)$ gauge group. In this case, in comparison with the toron measure in SQCD (95), we have the following distinctions:

- a) the factor $d^2 \epsilon/M$ which is related to the gluino zero modes is now absent;
- b) the factor $(d_B)^{-N_f}$ relating to the scalar matter fields is also absent;
- c) the non-zero modes do not cancel between bosons and fermions and should be taken into account.

Let us begin with the case of the non-zero gauge modes. As is well known [42], up to logarithmic accuracy, the total contribution of the non-zero modes can easily be calculated with the help of the usual Feynman diagrams, as was done for gauge theories in Ref. [42] and for the σ -model in Ref. [15]. The effective addition to the action is determined by Fig. 7 and equals [42]:

$$S' = S_{cl.} + \Delta S_g + \Delta S_f, \quad S_{cl.} = 4\pi^2/g^2 \quad (100)$$

$$\Delta S_g = \frac{2}{3} \frac{g^2}{16\pi^2} \ln M_0^2 \int d^4 X \left[\frac{1}{4} G_{\mu\nu}^2 \right] = \frac{2}{3} Q \ln M_0$$

$$\exp(-\Delta S_g) = \exp\left(-\frac{2}{3} \frac{1}{2} \cdot \ln M_0\right)$$

Analogously, we can find the effective addition to the action related with fermions. The result is determined by Fig. 8 and equals [42]:

$$\Delta S_f = \frac{3N_f}{16\pi^2} \ln M_0^2 \left(1 - \frac{1}{3}\right) \int d^4 X \left[\frac{1}{4} \bar{\psi} \psi \right] = Q N_f \ln M_0 \left(1 - \frac{1}{3}\right) \quad (101)$$

We have deliberately separated the fermion contribution (101) into two parts. One of them, $Q N_f (1 - 1/3) \ln M_0$, is related to the regulator contribution of zero modes at integer Q and to the regulator contribution of QZM at fractional topological

out that this interaction of the point defects (torons and antitorons at $\Delta \rightarrow 0$) is Coulomb-like and the result is independent of the regularization size Δ . Moreover, the temperature T of this classical Coulomb system is exactly equal to $T = 1$, in full agreement with the results of Ref. [44] (in this paper the Coulomb particles were called "instanton quarks"). For more details about this subject, see the conclusion (Section 12) and the following publication.

We return to the analysis of QCD. In this case, we cannot take into account the interaction in the general case. But we can consider the two-toron system with a very small separation and show that the results do not depend on the regularization size Δ . So the divergences (at $\Delta \rightarrow 0$) which occur in the dilute-gas approximation should disappear in the exact solution. As was explained in Section 6, the two-toron system can be understood as the instanton with size $\rho = Z_1 - Z_2$ and with position $X_0 = \frac{1}{2}(Z_1 + Z_2)$. Now we shall find the toron interaction and verify that this interaction is very essential and it cancels the Δ -dependence of the density. The small toron separation arises in place of Δ . Indeed, we define the toron interaction energy analogously to the instanton case [43] as follows:

$$Z_{\text{2torons}} \sim d^4 Z_1 d^4 Z_2 \frac{1}{\Delta^{\frac{2-N_f}{6}}} \exp(-W_{\text{int}}) \quad (103)$$

Here Z_1 and Z_2 are the toron positions while the $\exp(-W_{\text{int}})$ takes into account the toron interaction. The dilute-gas approximation for the two-toron contribution is simply the product of the two single toron densities (102), and is independent of the toron separation.

We can find the $\exp(-W_{\text{int}})$ now by subtracting the non-interacting two-toron contribution from the instanton one. As in supersymmetric theories one can interpret $d^4 X_0 d^4 \rho$ from the instanton measure as the integral over two toron collective co-ordinates (the translations Z_1 and Z_2). The interaction is defined only by the non-zero modes

$$\exp(-W_{\text{int}}) = \exp \left\{ - \left[\frac{2-N_f}{3} \right] \ln M_0 \rho - 2 \left(\frac{2-N_f}{6} \right) \ln M_0 \Delta \right\} \\ = \exp \left\{ - \left(\frac{2-N_f}{3} \right) \ln \frac{|X_1 - X_2|}{\Delta} \right\}. \quad (104)$$

Here the factor $\exp[-(2-N_f)/3 \ln M_0 \rho]$ is related to the instanton non-zero modes [32] and the two factors $\exp[-(2-N_f)/6 \ln M_0 \Delta]$ are related to the non-zero modes (102) of the two torons.

Substituting Eq. (104) for Eq. (103), we verify that the Δ -dependence disappears from the expression for density; in place of Δ arises the toron separation

$|Z_1 - Z_2|$ accounting for the interaction. In particular, at $N_f = 1$ we have

$$W_{\text{int}}(Z_1 \rightarrow Z_2) = \frac{1}{3} \ln |Z_1 - Z_2| \rightarrow -\infty$$

which means that there is a logarithmic attraction which gives an increase of the toron density. It is in agreement with the qualitative aforementioned notes. For $N_f > 3$, Eq. (104) gives a logarithmic repulsion in the interaction energy^{*}. This effect can be qualitatively understood as being due to Fermi statistics.

The case of QCD with $N_f = 2$ (in a more general case $N_f = N_c$) calls for particular attention^{**}. In this case the toron density (102) has a finite limit at $\Delta \rightarrow 0$. From the technical point of view, this singling out is related to the cancellation of non-zero modes, as in the supersymmetric models.

Summarising, we saw that the way to find the toron contribution to the different physical values in the general case is not possible in the semiclassical approximation. This is because the dilute-gas approximation is broken down and the toron interaction should be taken into account.

In the case with $N_f = 2$ (in general $N_f = N_c$), the toron preserves its duality and can give a finite contribution to the physical values. It so happens that this class of theories is very interesting from the physical point of view because in Nature we have $N_c = 3$ and N_f (the number of light flavours with $m = m_d = m_s = 0$) = 3. Here and in what follows we consider the case $N_f = N_c = 2$ only.

The toron measure (104) in this case takes the form

$$Z_{\text{QCD}}(N_f = 2) = K \frac{M_0^4}{g^4} d^4 X_0 \left(\frac{m}{M_0} \right) \exp \left(-\frac{4\pi^2}{g^2} \right) = \\ = K m \Lambda^3 d^4 X_0, \quad \Lambda^3 \equiv M_0^3 g^{-4} \exp \left(-\frac{4\pi^2}{g^2} \right). \quad (105)$$

We can find the constant "K" in the way described for obtaining the constant "C" (88) in the supersymmetric theories. From (87) and (105) [32], we obtain

^{*} The instanton interaction has the same qualitative properties, in particular $W_{\text{inst.int.}} \sim (2-N_f) \ln |Z_1 - Z_2|$ [43].

^{**} This point of view was intensively discussed early in Ref. [29].

$$K = \pi^2 g^5 \left(\frac{e^{5/6}}{4} \right)^{1/2} \quad (106)$$

We are now ready to calculate the chiral condensate in QCD. As in SQCD, the QZM play an important role in this calculation. Exactly these modes can cancel out the small mass factor $\sim m$ (105) and ensure the final result in the limit $m \rightarrow 0$.

By definition, taking into account Eqs. (105) and (106), we have

$$\begin{aligned} \langle \bar{\psi}\psi \rangle_{\text{Minkowsky}} &= -i \langle \psi^* \psi \rangle_{\text{Euclid.}} = \\ &= -i \pi^2 g^4 e^{5/12} m \int d^4 x_0 \int d^4 x_1 \frac{e^{-im}}{[-iD + m^2]}. \end{aligned} \quad (107)$$

Here ψ is the field of any light flavour (u, d). In obtaining (107) we replace $\psi\psi^\dagger$ by the Green function in the toron background. Let us note that the integral obtained in (107) was considered earlier when we calculated the fermion determinant in SQCD; see Eq. (90). We do not know the massive Green function, but integrals like (107) we know exactly in the limit $m \rightarrow 0$! Indeed,

$$\int d^4 x_0 \int d^4 x_1 \frac{m^2}{(-D^2 + m^2)} = \int d^4 x_0 \int d^4 x_1 \left(\frac{m^2 (1+D^2)}{(-D^2 + m^2)} \right) - \int d^4 x_0 \int d^4 x_1 \frac{m^2 D^2}{(-D^2 + m^2)} = Q \quad (108)$$

Here, as in Eq. (90), we have taken into account the identity $\hat{D}^2(1+\gamma_5) = D^2(1+\gamma_5)$, which is correct for any self-dual fields. Besides that, we take into account the fact that the first term of Eq. (108) is small in the limit $m \rightarrow 0$ ($m^2 \ln m$), see Appendix B, and the last term in (108) is related to the index of the Dirac operator, actually independent of "m" and equal to the topological charge $Q = \frac{1}{2}$ of the background field [36]. The antitron gives the same contribution and so:

$$\langle \bar{\psi}\psi \rangle_{\text{Mink.}} = -\pi^2 g^4 \exp(5/12) / \Lambda^3 \quad (109)$$

It is the main result of this work.

As in supersymmetric theories the result (109) is defined by QZM and the chiral limit is understood only in the sense $m \rightarrow 0$ of the massive theory. As a second remark, we note that the mechanism of spontaneous chiral symmetry breaking under consideration recalls that of Ref. [45]. In both cases $\langle \bar{\psi}\psi \rangle \neq 0$ because of

QZM at $\lambda \rightarrow 0$. The difference is that now the QZM are an inherent property of configurations with fractional Q. In Ref. [45] this effect is due to the instanton interaction.

10.- SCALAR AND PSEUDOSCALAR TWO-POINT CORRELATION FUNCTIONS AND THE GOLDSTONE THEOREM

In this section we would like to check the Goldstone theorem saying that there should be a light ($\sim m$) pseudoscalar boson if the chiral symmetry is spontaneously broken and the condensate $\langle \bar{\psi}\psi \rangle$ is non-zero. Of course, we are not doubting a general theorem, but rather we are eager to learn how it works: what is the mechanism which ensures the singular behaviour of the pseudoscalar correlator (because of the light Goldstone boson) and does not exhibit the singular behaviour of the scalar one?

Besides that, in the singlet pseudoscalar channel there should not be a Goldstone particle, owing to the solution of the U(1) problem (for a review see Ref. [12]). On the other hand, the U(1) problem arises with the condensate (109). So, in any consistent mechanism for chiral breaking, the U(1) problem must be solved automatically, and the pole in the corresponding singlet correlator must be absent.

With this discussion taken into account, we would like to consider the following three correlation functions:

$$P_5 \equiv i \int d^4 x e^{iqx} \langle 0 | T \{ \bar{u}(x) d(x), \bar{d}(x) u(x) \} | 0 \rangle \quad (110)$$

$$P_{\text{scalar}} \equiv i \int d^4 x e^{iqx} \langle 0 | T \{ \bar{u} d(x), \bar{d} u(x) \} | 0 \rangle \quad (111)$$

$$P_{\text{singlet}} \equiv i \int d^4 x e^{iqx} \langle 0 | T \{ \bar{u} i \gamma_5 u(x), \bar{u} i \gamma_5 u(x) \} | 0 \rangle \quad (112)$$

(we work temporarily in Minkowsky space). As is well known (see, e.g., Ref. [12]), for the pseudoscalar correlator P_5 there is a famous Ward identity

$$P_5 = - \frac{\langle \bar{\psi}\psi \rangle}{m} \quad (113)$$

which means that in this channel there should be a light ($\sim m$) Goldstone boson (π -meson) and that this correlator tends to infinity when $m \rightarrow 0$. The other two correlators (111) and (112) should be regular ones. Now we would like to verify all these properties and the calculation of the corresponding correlators will provide a valuable consistency check of our approach.

Returning back to the Euclidean space and using the expression (105) for the toron measure, we have (see Fig. 9)

$$P_5 = K m \Lambda^3 (\Delta_1 + \Delta_2), \quad K = \pi^2 \exp(\sqrt{5}/12) 2^4$$

$$\Delta_1 = \text{Tr} \frac{m}{-D^2 + m^2} \frac{m}{-D^2 + m^2}$$

$$\Delta_2 = \text{Tr} \vec{D} \frac{1}{-D^2 + m^2} \delta_5 \frac{1}{-D^2 + m^2} \vec{D} \delta_5 \quad (114)$$

Here we employ the usual operator notation and the Tr arises because of the factor $d^4 x_0$ in the expression (105). Besides that, we preserve the eigenfunctions with negative chirality $\gamma_5 \psi_R = -\psi_R$ only because for the functions with positive chirality we have $\vec{D}^2(1+\gamma_5) = D^2(1+\gamma_5)$ and the corresponding terms, like in Eqs. (90) and (108) are small in the limit $m \rightarrow 0$. This fact is implicitly taken into account in relations (114). So the corresponding terms here and in what follows will be omitted because they cannot ensure the singular behaviour $\sim m^{-1}$ of the correlators.

The Δ_1 -term in Eq. (114) consists of the quark Green function proportional to m . This part of the correlator has the same sign for the scalar and pseudoscalar correlators because

$$\delta_5 \frac{1}{D^2 - m^2} \delta_5 \frac{1}{D^2 - m^2} = \frac{1}{D^2 - m^2} \cdot \frac{1}{D^2 - m^2}$$

The Δ_2 contribution consists of the quark Green function proportional to $D_{\mu} \gamma_{\mu}$, and so this part of the correlator has an opposite sign for the scalar and pseudoscalar correlators.

Now we will show that Δ_1 and Δ_2 contributions have the same values and tend to infinity like m^{-2} for $m \rightarrow 0$. In this case $P_5 \sim m^{-1}$ and $P_{\text{scal}} \sim 1$, as they should.

We do not know the closed form for the massive Green function. Fortunately, the evaluation of the integral (114) reduces to the well-known expression (108) (up to $m^2 \epsilon m \rightarrow 0$ accuracy), which is actually independent of m and equals $\frac{1}{2}$ in the limit $m \rightarrow 0$.

To show this, let us consider the relation (108) from the spectral representation point of view:

$$\lim_{m \rightarrow 0} \text{Tr} \frac{m^2}{-D^2 + m^2} = \lim_{m \rightarrow 0} \text{Tr} \sum_k \frac{m^2}{k^2 + m^2} \psi_k(x) \psi_k^+(x) = Q \quad (115)$$

Here k^2 is the eigenvalue of the operator $\hat{D}^2$:

$$(-D^2 + m^2)\psi = (-D^2 + i\sigma_{\mu\nu} \tilde{G}_{\mu\nu} \frac{1}{2} + m^2)\psi = (k^2 + m^2)\psi \quad (116)$$

Besides that it is easy to see that if ϕ is a scalar mode function, $(-D^2 + m^2)\phi = (k^2 + m^2)\phi$, then $\phi \sim \gamma_{\mu} D_{\mu} \phi \cdot \epsilon$ satisfies the spinor mode equation (116), and its normalization differs from that of the scalar mode function ϕ by a factor of the eigenvalue $(k^2 + m^2)$. Thus we have

$$\begin{aligned} \text{Tr} \frac{1}{-D^2 + m^2} &= \text{Tr} \sum_k \frac{\psi_k \psi_k^+}{k^2 + m^2} = -2 \text{Tr} \vec{D} \sum_k \frac{\psi_k \psi_k^+}{k^2 + m^2} \vec{D} = \\ &= -2 \text{Tr} \vec{D} \frac{1}{-D^2 + m^2} \times \frac{1}{-D^2 + m^2} \vec{D} \end{aligned} \quad (117)$$

In obtaining the factor two in Eq. (117), we took into account that for each solution of the scalar field equation there are precisely two linearly independent solutions of the spinor field equation because we have two orthogonal spinors ϵ_{α} . As we know, the left-hand side of Eq. (117) equals Q/m^2 in the limit $m \rightarrow 0$, see Eq. (108). But the right-hand side of (117) is just what is needed for the calculation of the Δ_2 contribution. Thus, from (114) and (117) we have

$$(118)$$

For the calculation of the Δ_1 contribution a little more work is needed. Let us rewrite Eq. (114) in the following form:

$$\Delta_1 = \text{Tr} \sum_k \frac{m}{k^2 + m^2} \cdot \frac{1}{2} \cdot \frac{m}{k^2 + m^2} \psi_k(x) \psi_k^+(x) = \frac{1}{2}$$

The toron contribution to P_5 was calculated beforehand; the last term of (122) is exactly the square of the index operator (108). As usual, preserving only a singular term in P_{singlet} , and using Eqs. (114), (118) and (120), we have

$$P_{\text{singlet}} = \kappa \Lambda^3 \left(\frac{1}{2m} \right) - \frac{1}{2} \kappa \Lambda^3 m \left(\frac{1}{m} \right) \left(\frac{1}{m} \right) = 0. \quad (123)$$

As it should, the mass singularity in the singlet channel is absent.

Formulae (121) and (123) demonstrate the complete consistency of our understanding of the chiral symmetry breaking in the toron vacuum.

11.- THE U(1) PROBLEM AND THE θ -PERIODICITY PUZZLE IN QCD

In the previous section we saw that in the singlet channel the mass singularity is absent and so the U(1) "Goldstone boson" must have a finite mass in the chiral limit $m \rightarrow 0$. But this is not the end of the story, because the solution of the U(1) problem means that the relevant anomalous Ward identities (WI) should be fulfilled [11,12]. In this section we prove that the corresponding WI are satisfied automatically, as they should be, because the U(1) problem arises together with spontaneous breaking of chiral symmetry, i.e., with the condensate $\langle \bar{\psi}\psi \rangle$ (109).

Let us consider the following WI [11]:

$$i \int dx \langle 0 | T \left\{ \frac{\partial^2 \bar{\psi} \psi}{32 \pi^2} (x), \frac{\partial^2 \bar{\psi} \psi}{32 \pi^2} (0) \right\} | 0 \rangle = \frac{1}{2} m \langle \bar{\psi} \psi \rangle \quad (124)$$

$$i \int dx \langle 0 | T \left\{ \frac{\partial^2 \bar{\psi} \psi}{32 \pi^2} (x), \frac{\bar{u} \gamma_5 u + \bar{d} \gamma_5 d}{12} (0) \right\} | 0 \rangle = -\frac{1}{12} \langle \bar{\psi} \psi \rangle \quad (125)$$

As in the previous section, we write the WI in Minkowski space, in a standard notation. We took into account the fact that the P_{singlet} (123) contribution is a regular one, and so the corresponding term was omitted in the WI.

As is well known, it is very difficult to satisfy the WI in the standard instanton picture (see Ref. [12] and references therein). For instance, the instanton dilute-gas approximation shows [11,12] that the left-hand side of Eq. (124) is of order m^2 and not of order m , like the right-hand side. Furthermore,

$$\begin{aligned} &= \lim_{m \rightarrow m'} \frac{1}{2mm'} T_2 \sum_k \frac{m^2 m'^2}{m^2 m'^2} \left(\frac{1}{k^2 m'^2} - \frac{1}{k^2 m^2} \right) \psi_k \psi_k^* \\ &= \lim_{m \rightarrow m'} \frac{1}{2mm'} T_2 \left[\frac{m^2}{m^2 m'^2} \left(\frac{m'^2}{\Delta_1^2 + m'^2} \right) - \frac{m'^2}{m^2 m'^2} \left(\frac{m^2}{\Delta_1^2 + m^2} \right) \right] \quad (119) \end{aligned}$$

Here the factor $\frac{1}{2}$ is related to the fact that only functions with negative chirality give a main contribution ($\sim m^{-2}$) to the Δ_1 . Besides that, the expression (119) is understood in the sense of the limit ($m \rightarrow m'$). This formal manipulation of Δ_1 allows us to write Δ_1 in terms of the index operator (108). From (108) and (119), we obtain

$$\Delta_1 = \lim_{m \rightarrow m'} \frac{1}{2mm'} \left(\frac{1}{2} \frac{m^2}{m^2 m'^2} - \frac{1}{2} \frac{m'^2}{m^2 m'^2} \right) = \frac{1}{4mm^2} \quad (120)$$

$\Delta_1 = \Delta_2$ as it should, and the correlation P_5 (114) equals (taking into account that the antitron gives the same contribution to P_5):

$$P_5 = \pi^2 \exp\left(\frac{5}{12}\right) \frac{1}{2} m \Lambda^3 \left(\frac{1}{4m^2} + \frac{1}{4m'^2} \right) \frac{1}{2} = \frac{1}{2} \frac{\pi^2 \exp\left(\frac{5}{12}\right) \Lambda^3}{m} \quad (121)$$

$$P_{\text{scalar}} = 0. \quad (121)$$

which reproduces the result of the Ward identity (113) with the expression (109) for the chiral condensate.

We pass now to the analysis of the singlet channel. In this case, besides the diagram Fig. 9, we should bear in mind the diagram Fig. 10. So, from (112) we have the following expression for P_{singlet} in the toron background:

$$P_{\text{singlet}} = P_5 - \frac{1}{2} \kappa m \Lambda^3 T_2 \left(\frac{2m\delta}{\Delta_1^2 + m^2} \right) T_2 \left(\frac{2m\delta}{\Delta_1^2 + m^2} \right). \quad (122)$$

Here the first term is related to the standard contribution of Fig. 9. The last term (Fig. 10) corresponds to the factorization into a product of two averages. The minus sign is due to Fermi statistics.

the right-hand side depends on θ as $\exp\{i\theta/2\}$, and it is hard to see how the integer topological charge can give such a dependence on θ .

The standard approach to this problem is related to the assumption [46] that the left-hand side of Eq. (124) is non-zero in pure YM theory. In this case, as was shown by Veneziano [47] in QCD with light flavour fields, the left-hand side should be of order m^1 , in agreement with WI. This approach is a phenomenological one, and cannot explain the dynamical questions, such as what kind of vacuum fluctuations are responsible for the solution of the U(1) problem and corresponding WI.

Because for $N_f = 2$ ($N_f = N_c$) we have some consistent mechanism for chiral breaking and can find the condensate $\langle\bar{\psi}\psi\rangle$ (109), the WI should be fulfilled by the same vacuum fluctuations. Indeed, returning to the Euclidean space and using expression (105) for the toron measure, we have the following expression for the left-hand side of Eq. (124):

$$K m \Lambda^3 i \int d^4x \frac{g^2 \theta \tilde{G}}{32\pi^2}(x) \int d^4x_0 \frac{g^2 \tilde{G}}{32\pi^2}(x_0) + \text{antit.} = -K m \Lambda^3 \left(\frac{1}{4}\right). \quad (126)$$

In agreement with WI (124) and with the value of the condensate (109). Here the minus sign is due to Euclidean space (remember $\tilde{G}_M \rightarrow i\tilde{G}_E$). It is absolutely crucial that the correlator (124) be negative (in our notation); otherwise the phenomenological formula [47] would result in the wrong mass² sign.

Analogously, we have the following expression for the left-hand side of Eq. (125)

$$K m \Lambda^3 \int d^4x \frac{g^2 \tilde{G}}{32\pi^2}(x) \int d^4x_0 \text{tr} \frac{-m \tilde{E} \delta_5}{-D^2 + m^2} + \text{antit.} = \frac{1}{16} K \Lambda^3. \quad (127)$$

In agreement with WI (125). As in the previous case, the calculation corresponds to the factorization into a product of two averages. In obtaining Eq. (127), we took into account the expression (108) for the index operator.

We pass now to the analysis of the θ -dependence of the condensates. We start from the following WI [11,12,47]:

$$\langle \bar{\psi} \psi \rangle_\theta = \exp\left(-\frac{i\theta}{2}\right) \langle \bar{\psi} \psi \rangle_{\theta=0} \quad (128)$$

We differentiate the Lagrangian with respect to θ at $\theta = 0$ and obtain

$$\frac{d}{d\theta} \langle \bar{\psi} \psi \rangle_{\theta=0} = -i \int d^4x \langle \frac{g^2 \tilde{G}}{32\pi^2}(x), \bar{\psi} \psi \rangle = -\frac{1}{2} \langle \bar{\psi} \psi \rangle_{\theta=0} \quad (129)$$

In the last step we return to the Euclidean space and carry out the standard toron calculation as was discussed before. Here the factor $(\frac{1}{2})$ has a clear meaning. Indeed, because the calculation corresponds to the factorization into a product of two averages, the term $\int d^4x (g^2 \tilde{G}/32\pi^2)$ gives exactly the topological charge $(\frac{1}{2})$ of the toron. It is a general result: when we differentiate the Lagrangian n times with respect to θ at $\theta = 0$, we obtain the factor

$$\left(-i \int d^4x \frac{g^2 \tilde{G}}{32\pi^2} \right)^n = \left(-\frac{1}{2} \right)^n \quad (130)$$

in agreement with WI (128). Analogously we can show that the θ -dependence of the $\langle \tilde{G} \tilde{G} \rangle_\theta$ is equal to

$$\left\langle \frac{g^2 \tilde{G}}{32\pi^2} \right\rangle_\theta = -\sin \frac{\theta}{2} \left\langle \bar{\psi} \psi \right\rangle_{\theta=0} \quad (131)$$

in agreement with the Veneziano approach [47]. We should emphasize that (non-trivial) θ -dependence is through $(\theta/2)$. An analogous situation has been discussed in the σ -model in (49) and in SYM in Ref. [10]. This general consequence of WI can be easily understood from a dynamical point of view. Indeed, each differentiation with respect to θ gives the factor (130) which, in the toron background with fractional topological number, is just what is needed.

We end the discussion of the calculations in QCD with a few remarks. We saw that the very unusual but well-known properties of QCD [like θ -dependence, the U(1) problem, the Goldstone theorem, the counting of the discrete number of vacuum states at a fixed value of θ^* , and so on] are fulfilled in a very simple manner. On the other hand, it is very difficult to satisfy all these properties in another approach. From my point of view any consistent mechanism for chiral symmetry breaking should ensure all these properties automatically, as they are strongly

* Remember that this number in general equals N_f . In our case $N_f = 2$, and these states are analogous to two vacua, as discussed in the σ -model (45) and in SYM, and are related to the discrete chiral symmetry breaking of the $U(1)_A$ -symmetry, $\psi \rightarrow \exp(i \pi/2 \gamma_5) \psi$. We do not dwell on this issue.

interconnected. So, our valuable check of these properties confirms the consistency of the approach described above.

12.- CONCLUSIONS

The main point of this work is an analysis of the physical consequences of the existence of a fractional charge in the σ -models and, especially, in the physically interesting theory, QCD. It is shown that the corresponding fluctuations ensure spontaneous breaking of the chiral symmetry and give a non-zero contribution to the chiral condensate.

It is very important that the toron solution be determined on the manifold with a boundary. In this case, many questions arise, such as global boundary conditions, the stability of the solution, self-adjointness of the Dirac operator, single-valuedness of the physical values, and so on. These questions are interconnected and turn out to be self-consistent only for the special choice of the topological number $[Q = \frac{1}{2}]$ for $SU(2)$. Only for this value of Q is the correct renormalization invariant dependence restored.

Besides that, it is shown that in the Dirac spectrum of quarks, the gap between zero and the continuum is absent. This property plays a very important role in the forming of the chiral condensates in SQCD and QCD. Exactly this property ensures the correct dependence of condensates on "m", "g" and "g" in the well-understood SQCD model, and gives exactly two vacua [for $SU(2)$] in agreement with general results. The instanton calculation gives only an average over these vacua.

In QCD, this property ensures the non-zero value of $\langle \bar{\psi}\psi \rangle$ and the singular behaviour of the pseudoscalar correlator (because of the light π -meson). Moreover, because the $U(1)$ problem arises together with spontaneous breaking of chiral symmetry, the corresponding W should be satisfied automatically. We have demonstrated this, and besides that we have checked that the θ -dependence of the condensates agrees with the W (so-called θ -puzzle).

All these results confirm the consistency of our approach. From my point of view this description does not contradict the old idea of Ref. [48], according to which the instanton is the superposition of two objects with half-integer topological charge. In Ref. [48] such an object with $Q = \frac{1}{2}$ was the meron [49], possessing infinite action. In a certain sense, our solution is similar to the meron: both have zero size. There is also a difference: the toron has finite action, the meron infinite. This interpretation can be confirmed by consideration of the arbitrary gauge group G in other theories, such as SYM and SQCD. In this case this conjecture

is in agreement with numbers of the vacuum states and zero modes. Besides that, this conjecture ensures the correct renormalization-invariant dependence of the toron measure and the correct form of the axial anomaly.

So far we have discussed the models (supersymmetric theories and QCD with $N_f = N_c$) where the toron density turns out to be finite. In these cases the dilute-gas approximation gives a self-consistent picture and can be used for the calculation of different "topological" characteristics (like chiral condensates, correlators of a special form, and so on). But for another purpose, as was briefly discussed in Section 9, the interactions play an important role and should be taken into account. At present we do not know the solution of this problem in gauge theories, but we do know it in the $O(3)$ σ -model. However, we believe that the $O(3)$ σ -model presents a perfect theoretical laboratory for helping to understand the role of the toron interaction in the analysis of more complex gauge models.

It so happens that the problem of the toron interaction in the $O(3)$ σ -model reduces to the analysis of the classical Coulomb system (CCS) [51]. One can check that the temperature of this Coulomb system exactly equals $T = 1$ and so this system is in the plasma phase*. The same result was obtained in Ref. [44] from the instanton point of view. In this paper it was shown that the instanton individuality is lost and that they "melt". Thus the dilute instanton-gas approximation is not reasonable because the Coulomb particle (called "instanton quark" in Ref. [44]) is in the plasma phase. Now we can identify our torons (point defects at $\Delta \rightarrow 0$, which have only translation collective co-ordinates) with instanton quarks from Ref. [44].

Of course, we can reproduce some Green functions from Ref. [44] and obtain new ones [51], using the correspondence between the CCS, massive Thirring model and sine-Gordon model. But from my point of view the calculation of any Green function itself is not so important; it is more important that we can do this starting from the initial objects (torons, instanton quarks) which preserve their individuality and so are good quasiparticles. Besides that, the calculation with the toron gas is much simpler than in the instanton case because the toron is the point quasiparticle and has only translation collective co-ordinates.

From the formal point of view, the problem of calculation of the toron gas contribution can be reduced [51] to the study of the vortex contribution in the 2d XY model (Berezinsky-Kosterlitz-Thouless model). As we know [18], in order to

* Let us recall that there exists a phase transition in the CCS: this system is in the plasma phase for $T > T_{cr}$, and in the molecular phase for low temperatures: $T < T_{cr}$, $T_{cr} = \frac{1}{2}$.

account for the periodicity of the XY action [compactness of the $U(1)$ group], one has to introduce a set of singular vortices into the system.

It so happens that our torons exactly reduce to these singular vortices; our regularization of the toron solution ($\Delta \rightarrow 0$) corresponds to some regularization of the vortex and a cut which appears in the description of our toron solution (see Fig. 1) exactly corresponds to a cut with the branch point at the position of the vortex in the XY model. As is known, the exact shape of the cut is irrelevant because its change is just a symmetry transformation. Besides that, the structure of the core of the vertex in the XY model depends on the detailed form of regularization, but fortunately it appears to be irrelevant [18]. So it is no big surprise that in our toron calculation the results do not depend on the parameter Δ , which is as it should be.

To end this section, let us stress that a lot of problems, such as chiral breaking, the $U(1)$ problem, the number of discrete vacuum states, the θ -puzzle, low-energy theorems and so on, are strongly interconnected. So our check of these properties confirms the self-consistency of the approach described above.

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APPENDIX A

The main goal of this Appendix is the formulation of the toron solution in the CPI theory on a disk. The disk, like the manifold with a boundary, is intensively discussed in the literature [27] for the analysis of the APS theorem [26]. So this manifold is well understood and our description is a particular example only of the general approach [27].

We find that the introduction of the non-local boundary condition ensures the existence of exactly one zero complex mode, in agreement with the result of Section 4.

We begin with the consideration of the Dirac operator (26). Let us write the starting action in the following form:

$$\int d\bar{z} d\bar{z} (\bar{F}_1 \bar{F}_2) \begin{pmatrix} 0 & L^+ \\ L & 0 \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \int d\bar{z} d\bar{z} (\bar{F}_1 L^+ F_2 + \bar{F}_2 L F_1) \\ L = \bar{\partial} + \bar{\partial} \ln |P|^2, \quad L^+ = -\partial + \partial \ln |P|^2, \quad |P|^2 = 1 + |\bar{z}|^2. \quad (A1)$$

In this case it is more convenient to study Eq. (A1) by considering another complex variable ω (ω is the co-ordinate of the disk, Fig. 2) and defining a new spinor (f_1, f_2) :

$$\omega = R \frac{1+i\bar{z}}{1-i\bar{z}}, \quad \bar{z} = \frac{\bar{\omega}}{R}, \quad f_1 = F_1 \left(\frac{d\bar{z}}{d\bar{\omega}} \right)^{1/2}, \quad f_2 = F_2 \left(\frac{d\bar{z}}{d\bar{\omega}} \right)^{1/2}$$

It can easily be verified that the action (A1) can be rewritten in new variables in a standard Dirac form:

$$\int d^2\omega (\bar{f}_1 L_\omega^+ f_2 + \bar{f}_2 L_\omega f_1), \quad |P|^2 = 1 + |\frac{\omega - R}{\omega + R}|^2 \\ L_\omega^+ = -\partial_\omega + \bar{\partial}_\omega \ln |P|^2, \quad L_\omega = \bar{\partial}_\omega + \partial_\omega \ln |P|^2 \\ \bar{\omega} = U + iV = \exp(i\alpha), \quad 0 \leq \alpha < 2\pi, \quad 0 \leq r \leq R, \quad \partial_\omega = \frac{1}{2}(\partial_u - i\partial_v). \quad (A2)$$

with gauge field A_μ , $\mu = u, v$:

$$A_\mu = \epsilon_{\mu\nu} \partial_\nu \ln |P|^2, \quad A_u - iA_v = 2i \partial_\omega \ln |P|^2. \quad (A3)$$

Let us note that the form (A2) is still unacceptable for the use of the general relations of Ref. [27] because the gauge field does not satisfy the "radial gauge", where

$$A_u = \frac{V}{z} A, \quad A_v = -\frac{u}{z} A$$

The gauge transformation with the function Λ :

$$e^{i\Lambda} = \frac{\partial - P}{\partial + R}, \quad A'_\mu = A_\mu - \partial_\mu \Lambda, \quad f'_{12} = e^{i\Lambda} f_{12} \quad (A4)$$

provides us with just the form which is needed:

$$A'_u = \frac{V}{z} A, \quad A'_v = -\frac{u}{z} A, \quad A = \frac{R}{v^2 R^2}, \quad v^2 = u^2 + v^2 \quad (A5)$$

The number of zero (complex) modes in this case is defined by the general relation of Ref. [27] and equals

$$n_+ - n_- = [\bar{z} A(v)]_{z=R} = 1 \quad (A6)$$

where $[x]$ denotes the largest integer less than or equal to x . As we should, we have exactly one complex zero mode at any R , in agreement with the result of Section 4.

APPENDIX B

The goal of this Appendix is to find the Green function in the toron background and to calculate the value

$$m^2 \int d^4x \, \bar{\psi} \psi \left(\frac{1}{-\partial^2 + m^2} - \frac{1}{-\partial^2 + m^2} \right) \quad (B1)$$

at $m \rightarrow 0$. As discussed above, see Eq. (108), this value is related to the calculation of the toron measure (102) and chiral condensate in QCD. We shall show that the value (B1) tends to zero in the chiral limit and so the corresponding contribution in Eq. (108) can be omitted.

We start from the following representation of the toron solution (65) in the superpotential form

$$A_\mu^a = -\bar{\partial}_{\mu\nu}^a \partial_\nu \ln P, \quad P = \frac{1}{z} (G + \bar{G})$$

$$G = \frac{i-g}{i+g} \quad g(z) = \left(\frac{1-z}{1+\bar{z}} \right)^{3/2} \quad z = v + i\bar{v} \quad (B2)$$

Here $G(z)$ is the analytic function and P is the superpotential satisfying the equation $\square P = 0(70)$. The form (B2) is just that which was used in the instanton case [50]. So we can use all the methods of this paper. With Ref. [50], we will look for the solution of the equation

$$-\bar{\partial}_x^2 \Delta(x, y) = \delta^4(x, y), \quad \bar{\partial}_\mu = \partial_\mu + i A_\mu^a \frac{\tau^a}{2} \quad (B3)$$

in the form

$$\Delta(x, y) = P^{-1/2}(x) \frac{F(x, y)}{4\pi^2(x-y)^2} P^{-1/2}(y) \quad (B4)$$

where the function $F(x, y)$ must obey the condition

$$F(x, x) = P(x)$$

This is because the propagation function must have the same short-distance singularity as the free propagation function

$$\Delta(x \rightarrow y) = \frac{1}{4\pi^2(x-y)^2}$$

Inserting the decomposition (B4) into the Green function equation (B3) and using the facts that

$$P^{-1/2}(x) \delta(x-y) P^{1/2}(y) = \delta(x-y), \quad \sigma_\mu^T = (\pm i, \vec{\sigma})$$

$$\sigma_\mu^- \partial_\mu \frac{\sigma_\mu^+ (x-y)}{2\pi^2 [(x-y)^2]^2} = \delta(x-y), \quad \sigma_\mu^- \partial_\mu \sigma_\nu^+ \partial_\nu = \partial_\lambda^2$$

we can write Eq. (B3) as follows:

$$\sigma_\mu^+ \partial_\mu F(x,y) - \frac{2\sigma_\mu^+ (x-y)}{(x-y)^2} [F(x,y) - P(x)] = 0 \quad (B5)$$

To find the solution of this equation, let us introduce the complex notations

$$x_\mu = (z_1, \vec{n}_1 z_1), \quad z_1 = z_1 + i t_1, \quad \bar{z}_1 = z_1 - i t_1, \quad G_1 = G(z_1)$$

$$y_\mu = (z_2, \vec{n}_2 z_2), \quad z_2 = z_2 + i t_2, \quad \bar{z}_2 = z_2 - i t_2, \quad G_2 = G(z_2) \quad (B6)$$

It is easy to assert that Eq. (B5) is satisfied if $F(x,y)$ is given by

$$F(x,y) = \frac{1}{4} \left\{ \frac{(1 + \bar{\sigma} \vec{n}_1)(1 + \bar{\sigma} \vec{n}_2)}{z_1 + \bar{z}_2} (G_1 + \bar{G}_2) - \frac{(1 - \bar{\sigma} \vec{n}_1)(1 + \bar{\sigma} \vec{n}_2)}{\bar{z}_2 - \bar{z}_1} \cdot \right.$$

$$\left. \cdot \frac{(\bar{G}_1 - \bar{G}_2)}{z_2 + \bar{z}_1} - \frac{(1 + \bar{\sigma} \vec{n}_1)(1 - \bar{\sigma} \vec{n}_2)}{z_2 - z_1} (G_1 - G_2) + \right.$$

$$\left. + \frac{(1 - \bar{\sigma} \vec{n}_1)(1 - \bar{\sigma} \vec{n}_2)}{z_2 + \bar{z}_1} (G_2 + \bar{G}_1) \right\}. \quad (B7)$$

The proof rests on the use of properties of projectional operators ($1 \pm \vec{\sigma} \vec{n}$) and analytical properties of the function $G(z)$.

In particular, for the instanton $G(z)$ equals $z + z^{-1}$ and formula (B7) passes to the well-known instanton solution. Now we are ready to calculate the integral (B1). The leading term of this trace for small m can be calculated from Eqs. (B4) and (B7). From these equations one finds that $\text{Tr}(-1/D^2 + 1/\partial^2)$ is logarithmically divergent at large x . A natural cut-off at the corresponding trace is provided by the mass m . So we can find the integral (B1) with logarithmic accuracy by cutting the integral off from above by the value m^{-1} and using the massless Green function.

Taking into account the asymptotic behaviour of the Green function from (B2) and (B7),

$$G(z \rightarrow \infty) \rightarrow \frac{2}{3} + \frac{1}{2z}, \quad P(z \rightarrow \infty) \rightarrow \frac{2}{3}$$

$$F(x,y) \rightarrow \frac{2}{3} + \frac{1}{2} (\sigma_\mu^+ x_\mu) (\sigma_\nu^+ x_\nu) \frac{1}{x^2 y^2}$$

we have at $m \rightarrow 0$

$$m^2 \int_0^{1/m} d^4 x \, t_2 \left[\frac{1}{-D^2} - \frac{1}{4\pi^2 (x-y)^2} \right] \sim \frac{3}{2} m^2 \ln m. \quad (B8)$$

The result (B8) means that the corresponding contributions in the toron measure and in the expression for the chiral condensate can be omitted in the limit $m \rightarrow 0$.

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Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6

Fig. 7

Fig. 8

Fig. 9

Fig. 10

..... scalar ψ
 ----- gluino λ
 ----- gluon a_μ
 x external field