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A NOTE ON SUPER W-ALGEBRAS

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ABSTRACT

We show that the $m = 12$ superconformal theory ($c = \frac{10}{7}$) with the (E_6, D_8) modular invariant exhibits a bigger symmetry algebra. It is argued that this is a super W_3 -algebra, and that this latter can be realized (after a projection) from the coset $\hat{su}(3)_1 \otimes \hat{su}(3)_3 / \hat{su}(3)_4$. We also argue that one has some chance to realize general super W -algebras from cosets $\hat{g}_k \otimes \hat{g}_l / \hat{g}_{k+l}$ with $l = h_c$, the dual Coxeter number of the (simply-laced) Lie algebra g .

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Coset constructions¹⁾ have proved to be a very powerful tool in constructing (rational unitary) conformal field theories. It is well known that all unitary theories of the standard minimal $c < 1$ series can be obtained from the coset¹⁾

$$(A_1^{(1)})_k \otimes (A_1^{(1)})_1 / (A_1^{(1)})_{k+1}, \quad c = 1 - \frac{6}{m(m+1)}, \quad m = k+2 \quad (1)$$

In a similar way, the unitary superconformal theories with $c < \frac{3}{2}$ are given as¹⁾

$$(A_1^{(1)})_k \otimes (A_1^{(1)})_2 / (A_1^{(1)})_{k+2}, \quad c = \frac{3}{2} \left(1 - \frac{8}{m(m+2)} \right), \quad m = k+2 \quad (2)$$

On the other hand, unitary minimal W -algebras²⁾ are realized by replacing $A_1^{(1)}$ by a bigger Kac-Moody algebra³⁾. The so-called W_{n+1} algebras are obtained from the cosets

$$(A_n^{(1)})_k \otimes (A_n^{(1)})_1 / (A_n^{(1)})_{k+1}, \quad c = n \left(1 - \frac{(n+1)(n+2)}{m(m+1)} \right), \quad m = k+n+1 \quad (3)$$

In the above cases, as k (or m) $\rightarrow \infty$, $c \rightarrow 1, \frac{3}{2}, n$, respectively, and the corresponding theories can be realized by 1 free boson, 1 free boson and 1 free fermion, and n free bosons, respectively. For general m , one can obtain the correct central charge by adding an appropriate background charge^{4,3)}. The $m \rightarrow \infty$ limit furnishes a representation of the second factor in the coset pairs (1)-(3) (i.e. of $(A_1^{(1)})_1, (A_1^{(1)})_2, (A_n^{(1)})_1$, respectively). Instead of thinking of $(A_1^{(1)})_2$ as being realized by one free boson and one free fermion, it is more fruitful to think in terms of three free fermions transforming under the adjoint representation of A_1 . This is the reason why a level-2 representation of $(A_1^{(1)})$ is necessary. We now ask how this is to be generalized in order to obtain super W -algebras.

General cosets

$$(A_n^{(1)})_k \otimes (A_n^{(1)})_1 / (A_n^{(1)})_{k+l},$$

$$c_{k,l}^n = n \left(1 - \frac{l(n+1)(n+2)}{(k+n+1)(k+l+n+1)} \right) + \frac{n(n+1)(l-1)}{n+1+l} \quad (4)$$

have been considered in ref. 5. They can be represented in terms of n free bosons

with a background charge and generalized parafermions. In order to be able to construct generators of a superconformal subalgebra, the $(A_n^{(1)})_l$ factor should again be realized by $\dim A_n$ free fermion fields in the adjoint representation. In fact, this construction was already carried out in the classical GKO paper¹⁾, although there the main interest was on $(A_1^{(1)})$. Hence we choose l to be equal to the Coxeter number: $l = n+1$. Next, one should explicitly construct higher integer and half-integer spin currents, as was done for the W_3 -algebra in ref. 3. This, however, will not be done here. Instead, we will give some evidence that the coset (4) with $l = n+1$ has a good chance to realize (eventually after a projection) super W -algebras.

Let us find out which of the general coset theories (4), with $n > 1$, are also contained in the superconformal series (2). The only case where the central charges match is $c = \frac{10}{7}$ which corresponds to $m = 12$ of the superconformal series (2) and to $n = 2, k = 1, l = 3$ in (4). Note that $l = n+1$ as expected. We also remark, that it has already been found previously⁶⁾ by studying associativity constraints that a super W_3 -algebra should have a unitary representation for $c = \frac{10}{7}$. We should stress, however, that our above argument does *not* mean that a super W -algebra can exist only for $c = \frac{10}{7}$. It only shows that the $c = \frac{10}{7}$ theory is the only one which is *finitely* reducible with respect to the superconformal subalgebra, since all others have $c > \frac{3}{2}$.

We now take a closer look at the $m = 12$ superconformal theory ($c = \frac{10}{7}$) and argue that it realizes a super W_3 -algebra. The highest weights of this $m = 12$ theory are given by

$$h_{rs} = \frac{(14r-12s)^2 - 4}{8 \times 12 \times 14} + \frac{\epsilon}{8} \quad (5)$$

with $\epsilon = 0$ and $r-s$ even in the NS sector, and $\epsilon = \frac{1}{2}$ and $r-s$ odd in the R sector. It is easy to see that there is no h_{rs} equal to 3, but in the NS sector there exists a spin- $\frac{5}{2}$ primary field $\phi_{5/2}$: $h_{51} = \frac{5}{2}$. Since the superconformal generator G has half-integer modes in the NS sector, $G_{-1/2}\phi_{5/2}$ has spin 3. Hence, in the $m = 12$ superconformal theory it is the spin- $\frac{5}{2}$ current that appears as a primary field and

the spin-3 current as a descendent. Working out the values of the different $h_{r,s}$ of eq. (5) one sees that most of them can be grouped into multiplets, the values in each multiplet differing by integers or half-integers. Clearly, this can be interpreted as being due to a bigger symmetry: acting with the integer or half-integer modes of the spin- $\frac{5}{2}$ field on a weight h primary state gives another primary* state of weight $h + \frac{1}{2}, h + 1$, etc.

It is most interesting to study the modular invariants of the $m = 12$ superconformal theory. There exist two exceptional invariants of the E_6 type⁷⁾: (E_6, A_{13}) and (E_6, D_8) . The latter is particularly interesting since it is "maximally non-diagonal":

$$\frac{1}{4} \sum_{\substack{i=1 \\ \text{odd}}}^{13} (X_{1i} + X_{5i} + X_{7i} + X_{11i})^2 + |\tilde{X}_{1i} + \tilde{X}_{5i} + \tilde{X}_{7i} + \tilde{X}_{11i}|^2 + 2|\tilde{X}_{4i} + \tilde{X}_{8i}|^2 \quad (6)$$

Here $\hat{X}$ refer to characters in the R sector, and χ and $\tilde{X}$ to characters in the NS sector (without and with $(-)^F$ insertion in the trace). It is quite generally believed that such non-diagonal combinations of characters reflect a bigger symmetry algebra, and that in terms of characters of the bigger algebra the modular invariant is diagonal. Of course, we suspect that this bigger algebra is just the super W -algebra.[†] Let us consider which highest weights are grouped together in the modular invariant (6):

$$(h_{1s}, h_{5s}, h_{7s}, h_{11s}) = \left(0, \frac{5}{2}, \frac{11}{2}, 15\right), \quad s = 1$$

$$\left(\frac{5}{14}, \frac{6}{7}, \frac{20}{7}, \frac{145}{14}\right), \quad s = 3$$

$$\left(\frac{11}{7}, \frac{1}{14}, \frac{15}{14}, \frac{46}{7}\right), \quad s = 5$$

$$\left(\frac{51}{14}, \frac{1}{7}, \frac{1}{7}, \frac{51}{14}\right), \quad s = 7$$
(7)

* Of course, here we mean primary with respect to the superconformal algebra, not with respect to the (bigger) super W -algebra.

† A similar reasoning, interpreting the $c = \frac{4}{5}$ ($m = 5$) exceptional modular invariant of the D_4 type (the 3-state Potts model) as the diagonal invariant of the W_3 -algebra, was successfully carried out in ref. 3.

The weights for $s = 9, 11, 13$ are the same as for $s = 5, 3, 1$ (but occur in a reversed order). The pairs of Ramond weights occurring in (6) are

$$(h_{4s}, h_{8s}) = \left(\frac{3}{2}, \frac{15}{2}\right), \quad s = 1$$

$$\left(\frac{5}{14}, \frac{61}{14}\right), \quad s = 3$$

$$\left(\frac{1}{14}, \frac{29}{14}\right), \quad s = 5$$

$$\left(\frac{9}{14}, \frac{9}{14}\right), \quad s = 7$$
(8)

and again the weights for $s = 9, 11, 13$ are the same as for $s = 5, 3, 1$ (in a reversed order). One sees in eqs (7), (8) that the weights of a given multiplet differ by integers in the R sector and by integers or half-integers in the NS sector. This is consistent with the existence of an integer-moded (R) or half-integer-moded (NS) extra symmetry generator, and the reduction of a diagonal invariant of the bigger algebra to the non-diagonal invariant (6) of the (smaller) superconformal algebra. Since the theory contains a spin- $\frac{5}{2}$ primary field in the same multiplet as the identity, we would like to interpret the bigger algebra as a super W_3 -algebra.

As mentioned above, the only coset (4) with $n > 1$ that has $c = \frac{10}{7}$ is the one with $n = 2, k = 1$ and $l = 3$. But since k and l play a symmetric role, this is also the $m = 6$ representation of the W_3 -algebra. Hence we know the highest weights²⁾ (with respect to the W_3 -algebra). A priori, however, they are not necessarily all primary with respect to a super W_3 -algebra. Hence we should only consider the W_3 highest weights modulo half-integers. If the above interpretation of the multiplets (7) and (8) is correct, the lowest highest weight of each multiplet, i.e. $0, \frac{5}{14}, \frac{1}{14}, \frac{1}{7}, \frac{3}{2}, \frac{9}{14}$ should be a highest weight of the super W_3 -algebra. Indeed, the W_3 highest weights are, for $m = 6$

$$h_{r,s}^{m=6} = \frac{3(7(r+r') - 6(s+s'))^2 + (7(r-r') - 6(s-s'))^2 - 12}{12 \times 6 \times 7}$$

$$1 < r + r' < 6, \quad 1 < s + s' < 7$$
(9)

Evaluating this we find the following list of highest weights:

$$\begin{aligned}
 h_{W_3}^{m=6} = & 0, \frac{3}{2}, 4, \\
 & 1 \ 4 \ 6 \ 11 \ 15 \ 20 \ 22 \\
 & \overline{7}, \overline{7}, \overline{7}, \overline{7}, \overline{7}, \overline{7}, \overline{7}, \\
 & 1 \ 5 \ 9 \\
 & \overline{14}, \overline{14}, \overline{14}, \\
 & 5 \ 17 \ 29 \\
 & \overline{9}, \overline{9}, \overline{9}, \\
 & 2 \ 5 \ 8 \ 23 \ 29 \ 44 \ 47 \ 50 \ 86 \ 89 \ 107 \ 128 \\
 & \overline{63}, \overline{63}, \overline{63}, \overline{63}, \overline{63}, \overline{63}, \overline{63}, \overline{63}, \overline{63}, \overline{63}, \overline{63}, \overline{63}.
 \end{aligned} \tag{10}$$

We see that, as expected, the values $0, \frac{5}{14}, \frac{1}{4}, \frac{1}{7}, \frac{3}{2}, \frac{9}{14}$ are present. But we also see that the $c = \frac{10}{7}$ W_3 -theory contains many highest weights that are not contained in the multiplets (7) and (8). Hence it is only after a consistent projection that we expect the $c = \frac{10}{7}$ W_3 -theory to become the super W_3 -theory.

In conclusion, we have made a rather detailed study of the $m = 12$ superconformal theory with the (E_6, D_8) modular invariant and shown that it exhibits a bigger symmetry algebra. It was argued that this is a super W_3 -algebra, and that this latter can be realized (after a projection) from the coset $\hat{su}(3)_1 \otimes \hat{su}(3)_3 / \hat{su}(3)_4$. We also argued that one has some chance to realize general super W -algebras from cosets $\hat{g}_k \otimes \hat{g}_l / \hat{g}_{k+l}$ with $l = h_c$, the dual Coxeter number of the (simply-laced) Lie algebra g .

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