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Barycentre Motion of Beams due to Beam-Beam Interaction in Asymmetric Ring Colliders

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Abstract

The effects of the beam-beam interaction on the coherent motion of the barycentres of the colliding bunches are considered for asymmetric (two-ring) e^+e^- colliders, where circumference, energy, tune, etc. may be different between two rings. As well as integer and half-integer resonances, sum resonances occur. These resonance may be quite dense in the tune diagram, for the cases with different circumferences. This effect poses a severe restriction on the choice of the circumference ratio. On these resonances, the beams separate spontaneously at the interaction point. In particular, on the sum resonance, the barycentres follow a limit cycle in four-dimensional phase space, rather than reaching a fixed point. This causes a severe drop in the luminosity. Results of analytical calculations and computer simulations for the motion of the barycentres and the excitation of the resonances are presented.

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1 Introduction

The beam-beam interaction drives the coherent barycentre motion of bunches. The motion is a superposition of several characteristic modes referred to as the beam-beam modes. In order to keep the beams stable, all of the modes should be stable. This effect has been studied well [1,2,3,4,5,6,7,8,9] for symmetric colliders, where two beams circulate in the same ring and collide at some interaction points: the energies, tunes, etc. are almost identical to both beams. The stability of the modes can be studied by linear analysis. This approach has been integrated into the program BBMODE[10].

Recently, however, less symmetric colliders were considered: two-ring colliders with different energies as a possible type of B-meson (and other meson) factory[11]. Some projects proposed a pair of rings even with different circumferences[12]. The beam-beam effects in such machines have not been studied seriously.

In a previous paper[13], we studied the asymmetric colliders with different circumferences by a linear approximation of the beam-beam force. We found that the beam-beam modes can become unstable quite easily due to the beam-beam interaction in some cases, quite contrary to the conventional symmetric colliders, where they are more useful as a diagnostic tool[7] than a cause of dangerous instabilities. In this paper, we extend the study

by considering the nonlinear nature of the beam-beam force including the multiparticle tracking studies.

In the next section, some basic formulae are reviewed for later convenience. In section 3, linear instability is studied. Section 4 is devoted to an analysis on what occurs beyond the linear instability, using a simple model which incorporates the nonlinear nature of the force. In section 5, we confirm results obtained thus far by multiparticle tracking. The case of unequal circumferences is studied in section 6. Section 7 is devoted to discussions.

2 Beam-Beam Force for Barycentre Motion

We divide our complicated problem into two parts. In the first part, we study the effects on the bunch motion of differences in parameters (energy, tune and the beta value, for example) between the two rings with the same circumference. It is still a complicated problem and we will study it in detail first. In the second part, we study the additional effects arising from the difference in the circumferences. The structures studied in the first part will appear in more complicated fashion.

Thus, until section 6 we confine ourselves to the case where the both ring have the same circumferences. Beams collide at one interaction point (IP). We assume only one bunch in each beam: it is general since this bunch does not collide with any other bunches, even if they exist. The relativistic Lorentz factor ($\gamma_{\pm}$), the value of the beta functions at the IP ($\beta_{x,y}^{\pm}$), the beam sizes at the IP ($\sigma_{x,y}^{\pm}$) and the tunes ($\nu_{x,y}^{\pm}$) may be different for e^{+} and e^{-} beams. The usual symmetric collider can be considered as a special case.

The coordinate and slope of each particle in the $e^{\pm}$ bunch are denoted by $z_{\pm}$ and $z'_{\pm}$, where z refers to either horizontal (x) or vertical (y) coordinates. The barycentre and its slope of the each bunch are denoted by $\bar{z}_{\pm}$ and $\bar{z}'_{\pm}$, respectively:

$$\bar{z}_{\pm} = \langle z_{\pm} \rangle, \quad \text{and} \quad \bar{z}'_{\pm} = \langle z'_{\pm} \rangle.$$

We assume that the density distributions of both bunches are Gaussian. According to Bassetti and Erskine[14], the kick which acts on each particle is

$$\delta y'_{\pm} + i \delta x'_{\pm} = -\frac{N_{\mp} r_e}{\gamma_{\pm}} f(x_{\pm} - \bar{x}_{\mp}, y_{\pm} - \bar{y}_{\mp}; \sigma_x^{\mp}, \sigma_y^{\mp}). \quad (2.1)$$

Here

$$f(x, y; \sigma_x, \sigma_y) = \sqrt{\frac{2\pi}{(\sigma_x^{\mp})^2 - (\sigma_y^{\mp})^2}} \left\{ w \left(\frac{x + iy}{\sqrt{2((\sigma_x^{\mp})^2 - (\sigma_y^{\mp})^2)}} \right) - \exp \left(-\frac{x^2}{2(\sigma_x^{\mp})^2} - \frac{y^2}{2(\sigma_y^{\mp})^2} \right) w \left(\frac{\frac{(\sigma_y^{\mp})}{(\sigma_x^{\mp})}x - i\frac{(\sigma_x^{\mp})}{(\sigma_y^{\mp})}y}{\sqrt{2((\sigma_x^{\mp})^2 - (\sigma_y^{\mp})^2)}} \right) \right\},$$

where w is the complex error function.

Assuming that the beams always remain Gaussian with fixed variances (rigid Gaussian model), it was shown[6] that the kick for the barycentre coordinate can be expressed by

the same expression as Eq.(2.1) but with the replacement

$$(\sigma_x^\mp) \longrightarrow \Sigma_x \equiv \sqrt{(\sigma_x^+)^2 + (\sigma_x^-)^2},$$

and

$$(\sigma_y^\mp) \longrightarrow \Sigma_y \equiv \sqrt{(\sigma_y^+)^2 + (\sigma_y^-)^2}.$$

That is,

$$\delta \bar{y}'_\pm + i \delta \bar{x}'_\pm = -\frac{N_\mp r_e}{\gamma_\pm} f(\bar{x}_\pm - \bar{x}_\mp, \bar{y}_\pm - \bar{y}_\mp; \Sigma_x, \Sigma_y). \quad (2.2)$$

Note that the difference in the beam sizes is incorporated automatically by using Σ 's. When two bunches have identical beam sizes, it is implied that σ 's should be multiplied by $\sqrt{2}$. Therefore, the force represented by Eq.(2.2) extends over a longer range than that of Eq.(2.1). When one bunch is point-like, as in the case of the a weak bunch colliding with a strong bunch (where one bunch is just a particle), Σ is equivalent to σ of the strong bunch.

3 Linear Analysis

When the separations

$$\bar{x}_\pm - \bar{x}_\mp, \quad \text{and} \quad \bar{y}_\pm - \bar{y}_\mp$$

are small compared to Σ_x and Σ_y , respectively, we can linearize the force f , Eq.(2.2), and get

$$\delta \bar{z}'_\pm \simeq -\frac{4\pi}{\beta_z^\pm} \Xi_z^\pm (\bar{z}_\pm - \bar{z}_\mp), \quad (3.1)$$

where $\Xi_z^\pm$ is the effective beam-beam strength parameter

$$\Xi_z^\pm = \frac{r_e N_\mp \beta_z^\pm}{2\pi \gamma_\pm \Sigma_z (\Sigma_x + \Sigma_y)}. \quad (3.2)$$

Here r_e is the classical electron radius. Note the unconventional definition of the beam-beam parameter: when the beam sizes of both the bunches are the same, we get

$$\Xi = \xi/2,$$

where ξ is the usual beam-beam strength parameter. On the other hand, in the weak-strong case, we get

$$\Xi = \xi.$$

We employ the following set of canonical variables for the barycentres:

$$Z_\pm = \sqrt{N_\pm \gamma_\pm} \bar{z}_\pm, \quad P_\pm = \sqrt{N_\pm \gamma_\pm} \frac{\beta_z^\pm \bar{z}'_\pm}{\Sigma_z}, \quad (3.3)$$

Here $\sqrt{N\gamma}$ plays the role of the bunch mass. A bunch with more particles and/or more energy is heavier in the sense that it is less kicked by the same force: one usually uses momentum (mass $\times$ velocity) rather than the velocity when one studies interaction between particles with different masses.

The beam-beam kick and the successive map for the betatron oscillation can be represented by a 4×4 matrix M , acting on the 4-vector $(Z_+, P_+, Z_-, P_-)^t$:

$$M = VR,$$

with

$$V = \begin{pmatrix} V_+ & 0 \\ 0 & V_- \end{pmatrix}, \quad \text{and} \quad V_{\pm} = \begin{pmatrix} \cos \mu_{\pm} & \sin \mu_{\pm} \\ -\sin \mu_{\pm} & \cos \mu_{\pm} \end{pmatrix}, \quad (3.4)$$

where $\mu = 2\pi\nu$, and

$$R = I - 4\pi \begin{pmatrix} 0 & 0 & 0 & 0 \\ \Xi_+ & 0 & -\sqrt{\Xi_+ \Xi_-} & 0 \\ 0 & 0 & 0 & 0 \\ -\sqrt{\Xi_+ \Xi_-} & 0 & \Xi_- & 0 \end{pmatrix}, \quad (3.5)$$

I being the 4×4 unit matrix. Differences in γ , N , β_{IP} and σ_{IP} are integrated into Ξ 's. Only $\Xi_{\pm}$ and the tunes $\nu_{\pm}$ are the independent parameters (and possibly the damping decrement[15], $\delta_{\pm}$, which we do not discuss seriously, however, except in the Sect.4.3).

The mapping M is stable if and only if all eigenvalues of it are unity in absolute value. Let ρ be an eigenvalue of M , we have

$$\rho + \frac{1}{\rho} = \cos \mu_+ + \cos \mu_- - 2\pi\Xi_+ \sin \mu_+ - 2\pi\Xi_- \sin \mu_- \pm \sqrt{Q}, \quad (3.6)$$

$$Q = [\cos \mu_+ - \cos \mu_- - 2\pi\Xi_+ \sin \mu_+ + 2\pi\Xi_- \sin \mu_-]^2 + 16\pi^2 \Xi_+ \Xi_- \sin \mu_+ \sin \mu_-.$$

The stability condition is equivalent to that the r.h.s. of Eq.(3.6) is real and lies between -2 and 2 . When the motion is stable, the l.h.s. can be written as

$$2 \cos \bar{\mu}_{\omega_{\pm}},$$

and

$$\bar{\nu}_{\omega_{\pm}} = \bar{\mu}_{\omega_{\pm}}/2\pi,$$

is called the eigentune of $\omega_{\pm}$ mode. In the particular case of $\nu_+ = \nu_- = \nu$, it becomes

$$\cos \bar{\mu}_{\omega_{\pm}} = \cos \mu - \pi(\Xi_+ + \Xi_-)(\sin \mu \pm |\sin \mu|).$$

These two modes are often referred to as σ and π modes.

The cases where $|\rho| \neq 1$ can be classified as follows:

'2/2' resonance ($\cos \bar{\mu} \geq 1$) It occurs when

$$\nu_+ \lesssim \text{integer} \quad \text{or} \quad \nu_- \lesssim \text{integer}.$$

The symbol $\lesssim$ indicates that the instability occurs just below the resonant tune. The condition is

$$1 < -2\pi(\Xi_+ \cot \frac{\mu_+}{2} + \Xi_- \cot \frac{\mu_-}{2}). \quad (3.7)$$

'1/2' resonance ($\cos \bar{\mu} \leq -1$) It occurs when

$$\nu_+ \lesssim 1/2 + \text{integer} \quad \text{or} \quad \nu_- \lesssim 1/2 + \text{integer},$$

The condition is

$$1 < 2\pi(\Xi_+ \tan \frac{\mu_+}{2} + \Xi_- \tan \frac{\mu_-}{2}). \quad (3.8)$$

Sum resonance It occurs when

$$\nu_+ + \nu_- \lesssim \text{integer}. \quad (3.9)$$

The condition is

$$Q < 0. \quad (3.10)$$

These give all possibilities of linear instability: more precisely, onset of instability. With large Ξ , some instabilities can co-exist. Usually, however, we are not interested in such cases. Note that the sum resonance is the new feature for asymmetric colliders.

4 Spontaneous Separation of Beams

In this section, we will study what happens when a linear instability occurs. The linear consideration in the previous section would predict infinite divergence of the barycentre variables with constant growth rate. Actually, however, once they begin to grow, the nonlinear nature of the force becomes important and the linear model is no longer reliable. Since the nonlinear part of the beam-beam force contains terms which couple horizontal and vertical motion, the full analysis is complicated. Instead, we will study it in terms of a simpler model.

4.1 The Model

Let us assume that the beams are flat

$$\Sigma_x \gg \Sigma_y,$$

and consider the vertical motion only. That is, we assume the horizontal beam-beam force weak enough. If we assume that $\bar{x}_\pm = 0$ and that $\bar{y} \ll \Sigma_x$, we can replace the first w contained in Eq.(2.2) by unity. We thus get

$$\delta \bar{y}'_\pm = -\frac{N_\mp r_e}{\gamma_\pm} \frac{\sqrt{2\pi}}{\Sigma_x} \text{erf} \left(\frac{\bar{y}_\pm - \bar{y}_\mp}{\sqrt{2\Sigma_y}} \right).$$

We call this model the rigid (flat) Gaussian model. When the separation $\bar{y}_+ - \bar{y}_-$ is small compared to Σ_y , we recover Eq.(3.1) for y . If the separation becomes so large that it is comparable to the horizontal beam size, the force should begin to decrease. Such a case is not considered here.

In this section, we employ the following set of variables:

$$\hat{Y}_{\pm} = \frac{\bar{y}_{\pm}}{\Sigma_y}, \quad \hat{P}_{\pm} = \frac{\beta_y^{\pm} \bar{y}'_{\pm}}{\Sigma_y}.$$

(This choice of variables is not explicitly symplectic — the beam-beam kick is not expressible by a Hamiltonian — but can easily be made so by rescaling.) In these variables, the dynamics is as follows:

At the IP, (Beam-Beam Kick)

$$\begin{aligned} \hat{Y}_{\pm} &\longrightarrow \hat{Y}_{\pm}, \\ \hat{P}_{\pm} &\longrightarrow \hat{P}_{\pm} - \Xi_y^{\pm} F\left(\frac{\hat{Y}_{\pm} - \hat{Y}_{\mp}}{\sqrt{2}}\right), \end{aligned} \quad (4.1)$$

where

$$F(x) = (2\pi)^{3/2} \text{erf}(x).$$

During the arc, (Betatron Oscillation)

$$\begin{pmatrix} \hat{Y}_{\pm} \\ \hat{P}_{\pm} \end{pmatrix} \longrightarrow V_{\pm} \begin{pmatrix} \hat{Y}_{\pm} \\ \hat{P}_{\pm} \end{pmatrix}, \quad (4.2)$$

where $V_{\pm}$ is defined by Eq.(3.4). These two mappings are applied in alternation. We also consider the damping by multiplying $V_{\pm}$ with the damping rate $\lambda_{\pm} = \exp(-\delta)$.

Clearly, the origin

$$(\hat{Y}_+, \hat{P}_+, \hat{Y}_-, \hat{P}_-) = (0, 0, 0, 0)$$

is the fixed point of the mapping. When the origin is stable, the vector

$$\mathcal{W} \equiv (\hat{Y}_+, \hat{P}_+, \hat{Y}_-, \hat{P}_-)^t$$

falls into it eventually due to the radiation damping, provided the vector was already not far from it. Its stability can be seen by studying the linearized mapping around the origin: this is what we have done in the previous section.

4.2 Period One and Two Fixed Points

It seems appropriate first to study the period-one fixed point of the nonlinear mapping.

The period-one fixed point is the solution of

$$\begin{aligned} \mathcal{W} &= V(\mathcal{W} + \mathcal{F}), \\ \mathcal{F} &= \begin{pmatrix} 0 \\ -\Xi_+ F((\hat{Y}_+ - \hat{Y}_-)/\sqrt{2}) \\ 0 \\ +\Xi_- F((\hat{Y}_+ - \hat{Y}_-)/\sqrt{2}) \end{pmatrix}. \end{aligned}$$

Thus the solution is expressed as

$$\mathcal{W} = \frac{V}{1-V} \mathcal{F}.$$

Figure 1: The period-one fixed points of the nonlinear mapping. The cases with and without exceeding the threshold correspond to dashed ($- \cdot -$) and dotted ($\cdots$) lines, respectively. The solution is an intersection with the solid line.

For $\dot{Y}_{\pm}$ at the IP, it gives either $\dot{Y}_{\pm} = 0$ or

$$\dot{Y}_{\pm} = \mp \frac{1}{2} \Xi_{\pm} \cot \frac{\mu_{\pm}}{2} F((\dot{Y}_{+} - \dot{Y}_{-})/\sqrt{2}).$$

Thus,

$$\dot{Y}_{+} - \dot{Y}_{-} = -(2\pi)^{3/2} \left\{ \Xi_{+} \cot \frac{\mu_{+}}{2} + \Xi_{-} \cot \frac{\mu_{-}}{2} \right\} \text{erf}((\dot{Y}_{+} - \dot{Y}_{-})/\sqrt{2}).$$

The solution is shown in Fig. 1. For small value of Ξ 's, $\dot{Y}_{+} - \dot{Y}_{-} = 0$ is the only solution. When

$$1 < -2\pi \left\{ \Xi_{+} \cot \frac{\mu_{+}}{2} + \Xi_{-} \cot \frac{\mu_{-}}{2} \right\},$$

there appears a pair of new fixed points. Note that the condition is the same as Eq.(3.7). Thus the origin becomes unstable at the same time. The system chooses one or the other fixed point according to the initial conditions. The barycentres separate from each other spontaneously: each bunch has a different closed orbit at the IP. In general, the amount of the shift of the barycentre from the origin is also different for the e^{+} and e^{-} bunches. We call this "the spontaneous separation of the first kind".

Thus, we state

Lemma 1 (Instability of the First Kind) The condition of the 2/2 instability (and the absence of the other two instabilities) is the necessary and sufficient condition of the spontaneous separation of the first kind.

The next simplest possibility is that the system reaches a period-two fixed point. The analysis in this case seems quite complicated. However, we may find an approximate solution by requiring that the period-two fixed point appear in the phase space as if they change sign from one turn to the next. That is, a vector $\mathcal{W}$ becomes $-\mathcal{W}$ after one turn. Thus we get

$$-\mathcal{W} = V(\mathcal{W} + \mathcal{F}).$$

The solution is

$$\mathcal{W} = \frac{-V}{1+V} \mathcal{F}.$$

Thus,

$$\hat{Y}_{\pm} = \pm(2\pi)^{3/2} \Xi_{\pm} \tan \frac{\mu_{\pm}}{2} \operatorname{erf}((\hat{Y}_{+} - \hat{Y}_{-})/\sqrt{2}).$$

As before, we get either $\hat{Y}_{\pm} = 0$ or the two nonzero roots of

$$\hat{Y}_{+} - \hat{Y}_{-} = (2\pi)^{3/2} \left\{ \Xi_{+} \tan \frac{\mu_{+}}{2} + \Xi_{-} \tan \frac{\mu_{-}}{2} \right\} \operatorname{erf}((\hat{Y}_{+} - \hat{Y}_{-})/\sqrt{2}). \quad (4.3)$$

When

$$1 < 2\pi \left\{ \Xi_{+} \tan \frac{\mu_{+}}{2} + \Xi_{-} \tan \frac{\mu_{-}}{2} \right\},$$

there appear period-two fixed points: $\hat{Y}_{+} - \hat{Y}_{-}$ changes from one to the other of the roots of Eq.(4.3) every turn. Each bunch has two closed orbits at the IP and passes them alternately. We call it “the spontaneous separation of the second kind”. Note that the condition is the same as Eq.(3.8).

Thus, we state

Lemma 2 (Instability of the Second Kind) The condition of the 1/2 instability (and the absence of the other two instabilities) is the necessary and sufficient condition of the spontaneous separation of the second kind.

In the instability of the first (second) kind, the centre of the two beams, $(\hat{Y}_{+} + \hat{Y}_{-})/2$, is also different from zero and has a period one (two) fixed point(s) at the IP.

For these two kinds of instabilities, the damping decrement δ is not an important parameter. It affects how fast the system falls into the fixed points but the resulting fixed points are not so sensible to δ .

4.3 Sum Resonance Case

Also at the sum resonance, the origin becomes unstable. Neither period-one nor period-two fixed points can appear, because they have one-to-one correspondence between 2/2 and 1/2 resonances, respectively. What possibility is left?

Since an analytical study seems a little too complicated, let us study the case by a numerical tracking of the mapping system of Eqs.(4.1) and (4.2), with a damping rate λ close to one. Some of the results are shown in Fig. 2. It appears that the system is either in a limit cycle or in multi period state in the space $(\hat{Y}_{+} - \hat{Y}_{-}, \hat{P}_{+} - \hat{P}_{-})$. We call it “the spontaneous separation of the third kind”.

The multi-period behaviour is rather common in nonlinear systems: the system is caught by a nonlinear resonance. If the system were two-dimensional, the limit cycle would be a well-known consequence[16] when the origin becomes unstable. Since it is four-dimensional, it is remarkable that this limiting set is a one-dimensional object (as long as surveyed) in the four-dimensional phase space, as shown in Fig. 3. This behaviour is quite different from that of the first and second kind instabilities, where the equilibrium value of the separation can be calculated and is not so sensitive to parameters. In the third kind instability, the behaviour is almost unpredictable in the sense that the motion depends strongly on the

Figure 2: The phase space $(\dot{Y}_+ - \dot{Y}_-, \dot{P}_+ - \dot{P}_-)$, for the sum-resonance case. (a) An example of multi-period fixed point: parameters are $\nu = (0.35, 0.6)$, $\Xi = (0.025, 0.025)$ and $\delta = (7 \times 10^{-3}, 7 \times 10^{-3})$. (b) An example of limit cycle: parameters are $\nu = (0.268, 0.73)$, $\Xi = (0.03, 0.03)$ and $\delta = (7 \times 10^{-3}, 7 \times 10^{-3})$.

parameters, including λ and tunes. In case of the multi-period fixed point, the fixed points depend also on the initial conditions.

We summarize the results based on the nonlinear-rigid-Gaussian model in Table 1.

Mode	Cases	Behaviour	Remark
1	2/2 resonance	period-one fixed point	Pitchfork bifurcation
2	1/2 resonance	period-two fixed point	Period doubling bifurcation
3	Sum resonance	multi period fixed point or limit cycle	unpredictable Hopf bifurcation

Table 1: Classification of all kind of spontaneous separations and linear instabilities.

In all cases, the separation becomes larger and larger as Ξ 's increase. Since the luminosity is the sum of the instantaneous luminosity

$$\frac{1}{2\pi} \frac{N_+ N_-}{\Sigma_x \Sigma_y} \exp\left[-\frac{1}{2}(\dot{Y}_+ - \dot{Y}_-)^2\right],$$

over the unit time, the spontaneous separation reduces the luminosity. The reduction increases rapidly once one of the instabilities occurs.

Figure 3: The same limit cycle as that in Fig. 2 (b) in (a) $(\hat{Y}_+ + \hat{Y}_-, \hat{P}_+ + \hat{P}_-)$ plane, (b) $(\hat{Y}_+, \hat{P}_+)$ plane.

The spontaneous separation of the first and second kinds can exist also in the weak-strong case[17] and in the strong-strong but symmetric case[18]. The third kind separation, related to the sum resonance, is the new feature of the asymmetric collider.

There are regions in tune diagram where two kinds of instability conditions are satisfied at the same time. A numerical study shows that the resulting motion is very sensitive on initial W . Since this region is very tiny and not important, we do not discuss it.

5 Multiparticle tracking

The results presented so far depend on the linear- and/or nonlinear-rigid-Gaussian model, where we ignore the possible change of the beam sizes and nonlinear horizontal-vertical coupling. We remove these restrictions by computer simulation, using a multiparticle tracking program.

The computer program AB is an adaptation of the similar program BB[3]. It tracks many particles in asymmetric colliders. Its main features are:

- The K_+ bunches of e^+ and the K_- bunches of e^- in the simulation collide in the same sequence as they would do in a real collider. (This is necessary for the cases discussed in Sect.6).
- Each bunch is represented by a few hundred superparticles.
- The means $\langle x^\pm \rangle$ and $\langle y^\pm \rangle$ and the standard deviations $\sigma_x^\pm$ and $\sigma_y^\pm$ of the colliding $e^\pm$ bunch are used to calculate the forces on the opposite $e^\mp$ bunch, assuming that the density distribution in the bunches is a Gaussian.

Figure 4: Results of the multiparticle tracking program AB. (a) The barycentre motion in $(\dot{Y}_+ - \dot{Y}_-, \dot{P}_+ - \dot{P}_-)$ plane, (b) The same plot in $(\dot{Y}_+, \dot{P}_+)$ plane. Parameters are $\nu_y = (0.268, 0.73)$, $\nu_x = (0.362, 0.362)$, $\Xi_y = (0.03, 0.03)$, $\delta_y = (0.002, 0.002)$.

- The transformations around the circumference include radiation damping with decrements per turn (δ_x and δ_y) and quantum excitation.
- The complex error function $w(z)$ appearing in the beam-beam force is obtained in the first quadrant by six-point-two-dimensional interpolation in a table for arguments $z = x + iy$ in the interval $0 \leq x \leq 3.6$ and $0 \leq y \leq 2.5$ while for arguments outside this range $w(z)$ is approximated by a rational function. Symmetry relations are used to obtain $w(z)$ in the other quadrants.
- As our analytical work, the simulation includes only transverse betatron motion.
- The beam energies, the damping rates, and the tunes of the two rings are independent.
- The beams collide in a single interaction point.

In Fig. 4, we show multiparticle tracking results for the case of one e^+ bunch colliding with one e^- bunch ($K_+ = K_- = 1$), as discussed in sections 3 and 4. Fig. 4 (a) shows the vertical separation in the vertical phase space. The motion converges towards a straight line, crossing the origin and reaching maximum separations of about $\pm 40\sigma_y$. This convergence implies that the motion of the two bunches is locked onto the sum resonance such that ν_y^- is exactly equal to $1 - \nu_y^+$. Fig. 4 (b) shows the vertical phase space motion of the barycentre of the bunch in the second beam. The motion converges towards a limit cycle.

The luminosity estimate as a function of ν_y^+ is shown in Fig. 5. All the dips in luminosity can be identified with resonances from left to right:

Figure 5: Results of the multiparticle tracking program AB. The luminosity (normalized by the nominal value) as a function of ν_y^+ . Parameters are $\nu_x = (0.412, 0.412)$, $\nu_y^- = 0.318$, $\Xi = (0.03, 0.03)$, $\delta_y = (0.005, 0.005)$.

1. Coupling resonance $2\nu_x^+ + 2\nu_y^+ = 1$ below $\nu_y^+ \simeq 0.088$.
2. Nonlinear resonance $4\nu_y^+ = 1$ below $\nu_y^+ \simeq 0.25$.
3. Coupling resonance $2\nu_x^+ - 2\nu_y^+ \simeq 0$ above $\nu_y^+ = 0.412$.
4. Half integer resonance $2\nu_y^+ = 1$ below $\nu_y^+ \simeq 0.5$.
5. Coupling resonance $2\nu_x^+ + 2\nu_y^+ = 2$ below $\nu_y^+ \simeq 0.588$.
6. A large dip at the sum resonance $\nu_y^+ + \nu_y^- = 1$ below $\nu_y^+ \simeq 0.682$.
7. Nonlinear resonance $4\nu_y^+ = 1$ below $\nu_y^+ \simeq 0.75$.
8. Coupling resonance $2\nu_x^+ - 2\nu_y^+ = 1$ below $\nu_y^+ = 0.912$.
9. Integer resonance below $\nu_y^+ = 1$.

Three of the resonances observed by multiparticle tracking were predicted in sections 3 and 4: the integral, half-integral and sum resonances. Among all resonances, the sum resonance causes the largest reduction of the luminosity. The remaining resonances were already found in the earlier beam-beam simulations[3].

The qualitative results obtained in sections 3 and 4 are confirmed by multiparticle tracking with the simulation program AB. In particular, it is found that:

1. The behaviour listed in Table 1 is valid also in the multiparticle tracking.
2. In the cases with spontaneous separation, the change of the beam sizes are rather minor so that the assumption used in the rigid-Gaussian model is permissible.

6 Different Circumferences

Let us consider rings with different circumferences. If the energies are different, it appears more natural to use smaller circumference for the beam with the lower energy, not only for economical reasons but also for rapid damping which may help some instabilities. However, we will show that there is a severe restriction on the choice of the circumference ratio.

6.1 The Dynamics

We will consider a pair of asymmetric rings with different circumferences. The ratio of circumferences of both rings is K_+/K_- . Without loss of generality, we can assume that $K_+ \leq K_-$ and that K_+ and K_- have no common factor except for unity. The values of $\gamma_{\pm}$, $\beta_{x,y}^{\pm}$, $\sigma_{x,y}^{\pm}$ and $\nu_{x,y}^{\pm}$ may be different for the two beams, as before. The ring for the $e^{\pm}$ beam can have $K_{\pm}K_0$ bunches, which are spaced equally and contain $N_{\pm}$ particles each, where K_0 is an arbitrary integer. While an e^+ bunch completes K_- turns in its ring, the e^- bunch completes K_+ turns in its ring. This period is called ‘super period’. At the IP, the e^+ and e^- bunches are coupled by the beam-beam force. A common factor $K_0 > 1$ is irrelevant for the dynamics of the beam-beam interaction, because an $e^{\pm}$ bunch is coupled to only $K_{\mp}e^{\mp}$ bunches among a total of $K_{\mp}K_0$ bunches. We will assume $K_0 = 1$ in this section.

We label the bunches as $n = 1, 2, \dots, K_+$ for e^+ beam and $m = 1, 2, \dots, K_-$ for e^- beam. The collision of one particular set of bunches (n, m) occurs once in every super period. In this connection, it is useful to define the ‘super tunes’,

$$\tilde{\nu}_{\pm} \equiv K_{\mp}\nu_{\pm},$$

which correspond to the number of betatron oscillations of each bunch per one super period.

We employ, as canonical variables, Eq.(3.3) and form a $2(K_+ + K_-)$ vector

$$\mathcal{W} = (\vec{W}_1^+, \dots, \vec{W}_n^+, \dots, \vec{W}_{K_+}^+, \vec{W}_1^-, \dots, \vec{W}_m^-, \dots, \vec{W}_{K_-}^-)^t, \quad (6.1)$$

where each $\vec{W}_i^{\pm}$ is (Z, P) , defined by Eq.(3.3) for the i -th bunch. The evolution of $\mathcal{W}$ in a super period is described by multiplying it from the left with a matrix C :

$$C = OR_{(K_+, K_-)} \cdots OR_{(n, m)} \cdots OR_{(1, 1)}$$

Here the 2×2 block-diagonal matrix O represents the betatron oscillation in the arc:

$$O = \text{diag} \left[\overbrace{U(\mu_+/K_+), \dots, U(\mu_+/K_+)}^{K_+}, \overbrace{U(\mu_-/K_-), \dots, U(\mu_-/K_-)}^{K_-} \right]$$

where

$$U(\alpha) = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}$$

and $\mu_{\pm} = 2\pi\nu_{\pm}$. The matrices $R_{(n,m)}$ represent the kicks due to the collision between the n -th and the m -th bunches: In the subspace of $(\vec{W}_n^+, \vec{W}_m^-)^t$, $R_{(n,m)}$ is identical with R , Eq.(3.5). In the rest of the space, $R_{(n,m)}$ is the unit matrix.

The perturbed tunes of the system can be obtained from the eigenvalues of C . There are $K_+ + K_-$ independent pairs of eigenvalues. The members of each pair are either complex conjugate or real and reciprocal. The system is unstable if at least one of the eigenvalues becomes greater than unity in absolute value. The eigenvectors are modes of coherent dipole oscillations.

6.2 Eigenvectors

Contrary to the case of the same circumference, discussed in section 3, it is difficult to obtain exact eigenvalues analytically. Instead, in order to get physical insight, let us consider approximate eigenvectors and eigenvalues. When the currents, i.e. Ξ 's, are small enough, we can regard the beam-beam kicks as if they act continuously. For convenience, we rescale the canonical variables as

$$P_{\pm} \rightarrow P_{\pm}/\sqrt{\tilde{\nu}_{\pm}}, \quad \text{and} \quad Z_{\pm} \rightarrow Z_{\pm}\sqrt{\tilde{\nu}_{\pm}}.$$

The Hamiltonian of the system is then $H = T_+ + T_- + H_i$, where

$$\begin{aligned} T_{\pm} &= \frac{1}{2} \sum_{k=1}^{K_{\pm}} (P_k^{\pm})^2 \\ H_i &\simeq \frac{1}{2} \sum_{\pm} \sum_{k=1}^{K_{\pm}} (\tilde{\nu}_{\pm} Z_k^{\pm})^2 + \sum \left(\sqrt{\tilde{\nu}_+ \Xi_+} Z_n^+ - \sqrt{\tilde{\nu}_- \Xi_-} Z_m^- \right)^2 \\ &\equiv \frac{1}{2} \sum_{(i,j)=(1,1)}^{K_++K_-} V_{ij} Z_i Z_j \end{aligned}$$

Here, the Z_i form a $(K_+ + K_-)$ vector, the order being half of that of W . Since V defined here is in symmetric quadratic form and the kinetic term $(T_+ + T_-)$ has $(K_+ + K_-)$ -fold degeneracy, we can perform an orthogonal transformation to V without changing the kinetic term.

It is easy to find the eigenvectors and eigenvalues of V , which give approximate eigenvectors and 'eigen tunes' squared of C . The vectors

$$(a_1, a_2, \dots, a_{K_+}, 0, 0, \dots, 0)', \quad \text{with} \quad \sum a_n = 0$$

and

$$(0, 0, \dots, 0, b_1, b_2, \dots, b_{K_-})^t, \quad \text{with} \quad \sum b_m = 0$$

form $K_+ - 1$ and $K_- - 1$ independent vectors with super tunes $\tilde{\nu}_+ + K_- \Xi_+$ and $\tilde{\nu}_- + K_+ \Xi_-$, and correspond to relative motions within e^+ and e^- bunches, respectively. Two vectors of the form

$$\left(\overbrace{A, A, \dots, A}^{K_+}, \overbrace{B, B, \dots, B}^{K_-} \right)^t \quad (6.2)$$

are also eigenvectors, where A and B are the nonzero solutions of

$$\begin{pmatrix} \tilde{\nu}_+^2 + 2\tilde{\nu}_+ K_- \Xi_+ & -2K_- \sqrt{\tilde{\nu}_+ \tilde{\nu}_- \Xi_+ \Xi_-} \\ -2K_+ \sqrt{\tilde{\nu}_+ \tilde{\nu}_- \Xi_+ \Xi_-} & \tilde{\nu}_-^2 + 2\tilde{\nu}_- K_+ \Xi_- \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \lambda \begin{pmatrix} A \\ B \end{pmatrix},$$

and λ is the eigenvalue of V corresponding to the vector Eq.(6.2). The components A and B of the eigenvectors are the common barycentre positions of the e^+ and e^- bunches, respectively. The super tunes of the eigenmodes are given by

$$\tilde{\nu}_\omega^2 = \frac{1}{2}(\tilde{\nu}_+^2 + \tilde{\nu}_-^2) + K_- \tilde{\nu}_+ \Xi_+ + K_+ \tilde{\nu}_- \Xi_- \pm \frac{1}{2} \sqrt{Q'}, \quad (6.3)$$

$$Q' = (\tilde{\nu}_+^2 - \tilde{\nu}_-^2)^2 + 4(\tilde{\nu}_+^2 - \tilde{\nu}_-^2)(K_- \tilde{\nu}_+ \Xi_+ - K_+ \tilde{\nu}_- \Xi_-) + 4(K_- \tilde{\nu}_+ \Xi_+ + K_+ \tilde{\nu}_- \Xi_-)^2$$

Let us call modes with $\pm$ in Eq.(6.3) $\omega_\pm$ modes as before. When Q' becomes negative, the $\omega_\pm$ mode becomes unstable. This occurs at the sum resonance:

$$\tilde{\nu}_+ + \tilde{\nu}_- = K_- \nu_+ + K_+ \nu_- \lesssim \text{integer} \quad (6.4)$$

In general, when we fix $\nu_\mp$ and increase $\nu_\pm$ starting from some integer, we meet the sum resonance $K_\mp$ times while we go from that integer to the next. When $K_\pm$ is large, the resonant region can cover almost all range of $\nu_\mp$. In the particular case where $\tilde{\nu}_+ = \tilde{\nu}_-$, Eq.(6.3) implies that one of the super tunes is not shifted by the beam-beam interaction and the other becomes

$$\tilde{\nu} \longrightarrow \tilde{\nu} + K_- \Xi_+ + K_+ \Xi_-.$$

This is the largest shift of the super tunes. To obtain the tunes of the $e^\pm$ rings, we should divide $\tilde{\nu}$'s by $K_\mp$. Thus, when $K_+ \ll K_-$, the tune shift of the e^- ring can be quite large. Also the half-integer resonances $2\nu_+ \lesssim \text{integer}$ or $2\nu_- \lesssim \text{integer}$ lead to an instability.

6.3 Numerical Results

The precise eigenvalues of C are calculated with a computer code ABMODE (Asymmetric Beam-beam Modes), derived from the BBMODE code[10]. In Fig. 6, we show the maximum stable value of Ξ as a function of $\nu_\pm$, for the case $(K_+, K_-) = (2, 3)$. We assume that $\Xi_+ = \Xi_- \equiv \Xi$, which seems reasonable for maximizing the luminosity. The dips in Ξ due to the sum and half-integer resonances are clearly visible. From symmetry considerations, it is sufficient to show only $0 \leq \nu_+ \leq 1$ and $0 \leq \nu_- \leq 0.5$.

In Fig. 7, we show the growth rate of the instabilities, defined as the largest absolute eigenvalue Λ of C , for $(K_+, K_-) = (4, 5)$. The area in the tune diagram where the growth

Figure 6: The largest Ξ as a function of (ν_+, ν_-) for $(K_+, K_-) = (2, 3)$. The right axis is $1 - \nu_+$ while the left one is ν_- . The plateau corresponds to $\Xi \geq 0.03$.

rate is larger than unity is called instability region. The rest is the safe region. The instability region increases as $K_{\pm}$ increases, as shown in Table 2. It is roughly proportional to the density of sum resonances, i.e. $\sqrt{K_+^2 + K_-^2}$, apart from the contribution of the half-integer resonance lines. At the same time, the diameter of the largest circle between neighbouring beam-beam resonances is just $1/\sqrt{K_+^2 + K_-^2}$. The area of the instability region gives a rough measure of the difficulties associated with such rings.

Figure 7: The growth rate Λ as a function of (ν_+, ν_-) for $(K_+, K_-) = (4, 5)$ and $\Xi = 0.03$. The right and left axes are ν_+ and ν_- , respectively. The base is at $\Lambda = 1$, the plateau corresponds to $\Lambda \geq 2$.

6.4 Computer Simulation

An example of the results of computer simulation with the multiparticle tracking program AB for the luminosity in the case $(K_+, K_-) = (2, 3)$ is shown in Fig. 8. The sum resonances at $\nu_y^+ = 0.155, 0.488, 0.821$ are the strongest sources of the luminosity reduction, as in the case of $(K_+, K_-) = (1, 1)$. In addition, the resonances identified in section 5 are also observed in the simulation: the half-integral and integral resonances in ν_y^+ , and the fourth-

	1	2	3	4	5	6	7
1	32.5	34.9	41.9	43.2	44.0	48.6	51.9
2	-	-	48.4	-	55.3	-	61.6
3	-	-	-	57.1	61.7	-	67.9
4	-	-	-	-	66.6	-	73.6
5	-	-	-	-	-	74.8	78.2
6	-	-	-	-	-	-	82.6

Table 2: Instability region (%) in the tune diagram for different set of (K_+, K_-) in case of $\Xi_{\pm} = 0.03$.

order coupling resonances between the horizontal and vertical tunes of the e^+ beam.

6.5 Unequal Population

Until now, we assumed that $N_{\pm}$ is common to all the $e^{\pm}$ bunches in the same ring. In practice, there is always a variation of N from bunch to bunch within the same beam.

Let us first consider the extreme case that particles are only in the first bunches in both beams ($n = 1$ and $m = 1$), while the other bunches are empty. In this case, we can apply the same argument as we did for the $(K_+, K_-) = (1, 1)$ case. The only difference is that the collision occurs only once in a super period. Thus all we have to do is to replace the tunes by the super tunes,

$$\nu_{\pm} \longrightarrow \hat{\nu}_{\pm},$$

in expressions obtained for the $(K_+, K_-) = (1, 1)$ case. It follows that resonance lines becomes denser than the equally populated case. That is,

- the sum resonance lines remain as they were for the equally populated case.

$$\hat{\nu}_+ \lesssim \text{integer} \text{ or } \hat{\nu}_- \lesssim \text{integer},$$

- the $1/2$ resonances become denser as

$$\left\{ \begin{array}{c} \nu_+ \lesssim 1/2 + \text{integer} \\ \text{or} \\ \nu_- \lesssim 1/2 + \text{integer} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \hat{\nu}_+ \lesssim 1/2 + \text{integer} \\ \text{or} \\ \hat{\nu}_- \lesssim 1/2 + \text{integer} \end{array} \right. :$$

the $1/2$ resonances are $K_+ K_-$ times denser than equally populated case.

- the $2/2$ resonances become $K_+ K_-$ times denser in the same manner as $1/2$ case.

In more general cases with unequal population these new resonances, called (half-) integral super resonances, will also occur. They were absent in the equally populated cases.

Figure 8: Luminosity as a function of ν_y^+ . Parameters: $\nu_x = (0.312, 0.312)$, $\nu_y^- = 0.268$, $\Xi = (0.03, 0.03)$.

The precise eigenvalues were again computed with the ABMODE program. In Fig. 9, we show the variation of the instability region and the growth rate due to the inequality between bunches for the simplest nontrivial case of $(K_+, K_-) = (1, 2)$: we fix N of the e^+ bunch and change the population between two e^- bunches with the sum kept fixed. We conclude that

1. The sum resonances and the resonances at integral and half-integral values of the **tune** are driven by the average bunch population. They do not change considerably when the population varies as long as the sum is kept constant.
2. The resonances at integral and half-integral values of the **super tune**, i.e. the (half-) integral super resonances, are driven by differences in the bunch populations. They disappear when all bunches have the same population.

Figure 9: The growth rate in (ν_+, ν_-) space for $(K_+, K_-) = (1, 2)$ for $\Xi_- = 0.03$ and a) $\Xi_+ = (0.06, 0.00)$ b) $\Xi_+ = (0.05, 0.01)$ c) $\Xi_+ = (0.04, 0.02)$ d) $\Xi_+ = (0.03, 0.03)$.

7 Discussion

7.1 Summary

Asymmetric colliders have the following resonance regions in tune space, where the beams are separated spontaneously.

Resonance	Definition	Remark
Sum	$\tilde{\nu}_+ + \tilde{\nu}_- \lesssim \text{integer}$	Dangerous (dense)
Integral tune	$\nu_+ \lesssim \text{integer}$ or $\nu_- \lesssim \text{integer}$	Dangerous but avoidable (not dense)
Half-integral tune	$\nu_+ \lesssim 1/2 + \text{integer}$ or $\nu_- \lesssim 1/2 + \text{integer}$	
Integral super tune	$\tilde{\nu}_+ \lesssim \text{integer}$ or $\tilde{\nu}_- \lesssim \text{integer}$	Dense but weak
Half-integral super tune	$\tilde{\nu}_+ \lesssim 1/2 + \text{integer}$ or $\tilde{\nu}_- \lesssim 1/2 + \text{integer}$	

Among these resonances, the sum resonances are the most dangerous: as seen from the multiparticle tracking results, they are the main source of the luminosity reduction. In addition, the sum resonances are more and more closely spaced in the tune diagram for colliders with larger and larger $\sqrt{K_+^2 + K_-^2}$. The resonances at integral and half-integral tunes are dangerous but not dense in tune space. Hence they can be avoided. The resonances at integral and half-integral super tunes are weak in the sense that they disappear when all bunches in the same beam have the same population.

These resonances, in particular the sum resonances, pose a severe restriction on the choice of K_+ and K_- , because enough space in the tune diagram is needed for the operation of the rings. Two criteria may be applied, which both yield an upper limit on $K_{\pm}$.

- If we require that the instability region, i.e. the fraction of tune space where the largest absolute eigenvalue Λ exceeds unity is not larger than 0.5, we find that for $\Xi = 0.03$

$$(K_+, K_-) = (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6) \text{ and } (2, 3)$$

are the only acceptable choices of $K_{\pm}$.

- In many colliders, the tunes have to be controlled to better than the synchrotron tune ν_s in order to avoid difficulties associated with synchro-betatron resonances. If we require that the sum resonances do not cause a further reduction of the tolerance in the tunes, we find

$$K_+^2 + K_-^2 \leq \nu_s^{-2}$$

The former criterion is usually tighter than the latter.

7.2 Possible Cures

Feed-Back

Possibility of feed-back is restricted in two ways:

1. The growth rate is relatively large so that the gain in the feed-back system needed is also large. If the gain of the feed-back system is known, the maximum tolerable

growth rate, and therefore the maximum tolerable value of Ξ , can be calculated. Since in most B-factory proposals the circulating current is already rather high, a drop in luminosity in proportion to the drop in Ξ seems inevitable.

2. The sum resonance leads to unpredictable motion. It will make the design of the feed-back systems more difficult.

More Bunches

Once the maximum tolerable value of Ξ is given, we can achieve the required luminosity only by increasing K_0 with the circumference kept fixed. Since, however, it reduces the bunch spacing and increases the total current, there arise many other difficulties.

Equal Circumferences

The difficulties described in this paper can be best avoided by choosing $(K_+, K_-) = (1, 1)$, i.e. by making the circumferences of the two rings identical. In this case, the beam-beam effect couples each bunch in one ring to only one bunch in the other ring, even if there is a large – but equal – number of bunches in both rings, and there is only one sum resonance, at $\nu_+ + \nu_- \lesssim \text{integer}$, which is easily avoided by an appropriate choice of the tunes $\nu_{\pm}$.

For the colliders with different energies, however, this choice leads to the difficulty that the ring with lower energy needs an unnaturally large circumference: wigglers are needed to obtain a high enough damping rate.

7.3 Deviation from the Rigid-Gaussian Model

In our analytical studies in sections 3 and 4, we employed the rigid Gaussian model to analyze the beam-beam kick. We assumed that the rms radii of the beams were not changed by the beam-beam interaction. This approximation has two shortcomings.

- In the vicinity of the threshold of the linear instability, the beam sizes will grow as observed in multiparticle tracking. This effect reduces Ξ so that the motion can become linearly stable. Thus the threshold value of Ξ for the spontaneous separation is a little larger than that estimated by the rigid model. For Ξ just below the threshold, only a decrease of the luminosity but no spontaneous separation are observed.

In contrast to multiparticle tracking, the nonlinear-rigid-Gaussian model showed, in some rare cases, multi-period fixed points for the barycentre motion. (See Fig. 2). This seems to be due to the fact we ignored the changes in the beam sizes. Once a bunch is trapped in a nonlinear resonance, the beam will adapt itself to the new potential well so that the beam sizes can no longer be considered as fixed. Usually, the nonlinear resonances are not large enough to hold all particles within them. Thus the multi-period fixed points should be considered as one of the shortcomings of the rigid-Gaussian model.

This effect can, in principle, be included in the solvable model[19] for the beam-beam interaction by extending it to the barycentre motions. Because of some technical

problems only partial results were obtained[20] so far.

- Even for small currents, the deviation from the rigid Gaussian model can manifest itself as indicated by the correction factors[8,9] for the tune shift. For the sake of self-consistency, we did not take them into account. For the time being, the following questions remain unanswered: 1) whether the factor should be applied to the kick or to the tune shift, 2) how the factor depends on the separation between beams, 3) how to calculate the factor for fully asymmetric cases. If these factors are to be applied to the dipole kick, as asserted in Ref.[9], they would enhance the beam-beam kick and the strength of the resonances, and make the problem more serious.

Our computer simulation by multiparticle tracking, presented in section 5, includes the variation of the beam radii due to the beam-beam interaction, and implies the correction factors[4,8,9]. Since the results of our analytical studies and of multiparticle tracking are essentially in agreement, the shortcomings of our approximations in the analytical studies are not serious.

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