Flavor mixing and the eigenstates of the weak interactions

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The quark mixing matrix for two and three families is diagonalized. Weak eigenstates are defined in an unambiguous way based on the eigenvalues of the Cabibbo-Kobayashi-Maskawa matrix. For the real 3×3 mixing matrix a geometrical interpretation of flavor mixing is given, and it is argued that in the complex case such an interpretation is not possible. Furthermore, if the third generation decouples, we get democratic mass matrices in terms of the weak eigenstates for the first two families. There are indications that the Cabibbo mixing is different in character from the mixings of the third generation with the other ones.

I. INTRODUCTION

In addition to the lepton-quark mass spectrum, the phenomenon of weak-interaction mixing remains an important and yet unresolved feature of the present model of electroweak forces. It implies that the interaction of a W boson with a quark state of definite mass leads to a mixture of mass eigenstates. The charged weak current can be written as $\bar{Q}_u \gamma_u [(1 + \gamma_5)/2] Q'_d$, where Q_u stands for

$$Q_u = \begin{bmatrix} u \\ c \\ t \end{bmatrix} \quad (1.1)$$

(u, c, t are eigenstates of the quark mass matrix) and Q'_d is related to the mass eigenstates by a unitary transformation, given by the Cabibbo-Kobayashi-Maskawa (CKM) matrix^{1,2} V :

$$Q'_d = \begin{bmatrix} d' \\ s' \\ b' \end{bmatrix} = V \begin{bmatrix} d \\ s \\ b \end{bmatrix}. \quad (1.2)$$

For the parametrization of V , we shall use the three angles θ_{12} , θ_{23} , and θ_{13} as well as a phase parameter δ :³

$$V = \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{bmatrix} \quad (1.3)$$

($c_{12} = \cos\theta_{12}$, $s_{12} = \sin\theta_{12}$, etc).

Sometimes the states d', s', b' , which are mixtures of mass eigenstates, are referred to as the eigenstates of the weak interactions. This is a misleading concept, since one could just as well act with the unitary matrix V^{-1} on the mass eigenstates u, c, t in order to define analogous mixtures u', c', t' as weak-interaction eigenstates.

Here we should like to point out that there is an unambiguous way to define the eigenstates of weak interactions. It is well known that unitary matrices are quite analogous to Hermitian matrices, as far as their properties with respect to a diagonalization are concerned. For any unitary matrix V , one can find another unitary matrix U such that $U^{-1}VU$ is diagonal. The eigenvalues λ_j of this matrix are complex numbers, whose absolute values are equal to 1; i.e., we can write $\lambda_j = e^{i\phi_j}$ (ϕ_j is the phase parameter).⁴

Using this features of unitary matrices, we diagonalize in the following the CKM matrix (see also Ref. 5). Recently, such an approach to the mass-mixing problem of the quark was also used by Barnhill⁶ in a more formal way. In contrast with this author, we are working here with the explicit parametrization (1.3) of the CKM matrix.

Let us first discuss the general diagonalization of the mixing matrix and the definition of the weak eigenstates. Afterwards, we consider two special cases before turning to the complex CKM matrix.

II. DIAGONALIZATION OF THE CKM MATRIX

The general 3×3 CKM matrix V can always be diagonalized by another 3×3 unitary matrix U :

$$U^{-1} V U = \begin{bmatrix} e^{i\phi_1} & & \\ & e^{i\phi_2} & \\ & & e^{i\phi_3} \end{bmatrix} = D(\phi_j). \quad (2.1)$$

We can write $V = U D U^{-1}$. The weak current has the form

$$(\bar{u}, \bar{c}, \bar{t})_L U D U^{-1} \begin{bmatrix} d \\ s \\ b \end{bmatrix}_L. \quad (2.2)$$

Let us define mixtures of mass eigenstates by

$$\begin{bmatrix} d' \\ s' \\ b' \end{bmatrix} = U^{-1} \begin{bmatrix} d \\ s \\ b \end{bmatrix}, \quad \begin{bmatrix} u' \\ c' \\ t' \end{bmatrix} = U^{-1} \begin{bmatrix} u \\ c \\ t \end{bmatrix}. \quad (2.3)$$

These transformations are meant both for the left-handed (L) quark fields and for the right-handed (R) fields. Of course, the weak currents are defined only in terms of the L fields. For the charged weak current, we obtain

$$(\bar{u}', \bar{c}', \bar{t}')_L \begin{bmatrix} e^{i\phi_1} & & \\ & e^{i\phi_2} & \\ & & e^{i\phi_3} \end{bmatrix} \begin{bmatrix} d' \\ s' \\ b' \end{bmatrix}_L. \quad (2.4)$$

Thus, in the new basis, the weak interactions are diagonal. The interaction of, e.g., the state d' with a W boson leads uniquely to the state u' , whose wave function is multiplied by the phase factor $e^{i\phi_1}$. Thus it is justified to define the new states $d' \dots$ and $u' \dots$ as the *weak-interaction eigenstates*.

We emphasize that these states as well as the corresponding eigenvalues are uniquely defined once the CKM matrix V is given. However, there exists a large ambiguity in defining V because of the fact that the quark fields can be multiplied by phase parameters without changing the underlying physics. This freedom can be used in order to arrive, e.g., at the parametrization given above [Eq. (1.3)]. Because of this freedom, one cannot say that weak-interaction physics singles out uniquely the weak-interaction eigenstates defined above, but rather certain equivalence classes of states. Of course, this ambiguity vanishes, once we consider only CKM matrices of the type given in Eq. (1.3). We shall do so below.

We emphasize that the matrix U connecting the mass eigenstates and the weak eigenstates is, of course, the same for up- and down-type quarks. A weak eigenstate is a certain superposition of mass eigenstates; i.e., it consists of a certain probability of a u quark, a certain probability of a c quark, etc. The charged-current interaction leads to a transformation of this state to an analogous state with the same coefficients in terms of mass eigenstates. Since the interaction amounts to a multiplication with a phase $e^{i\phi_j}$, a mixing is generated between the various mass eigenstates. Only if all phases ϕ_j are zero, no mixing takes place, and mass eigenstates are connected only to mass eigenstates of the same kind.

III. CASE OF TWO-LEPTON-QUARK GENERATIONS

If we disregard the t and b quarks, the weak-interaction mixing can be described by one angle, the Cabibbo angle $\theta_C = \theta_{12}$. After absorbing irrelevant phase parameters, the mixing matrix V takes the form

$$V = \begin{bmatrix} c_{12} & s_{12} \\ -s_{12} & c_{12} \end{bmatrix} = \begin{bmatrix} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{bmatrix}. \quad (3.1)$$

This matrix is, of course, not only unitary, but orthogonal. It can be diagonalized by noting

$$\begin{bmatrix} c_{12} & s_{12} \\ -s_{12} & c_{12} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ i & i \end{bmatrix} \begin{bmatrix} e^{-i\theta_{12}} & \\ & e^{i\theta_{12}} \end{bmatrix} \times \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & -i \\ 1 & -i \end{bmatrix}. \quad (3.2)$$

Thus the matrix U introduced in Eq. (2.1) is given by

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ i & i \end{bmatrix}, \quad U^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & -i \\ 1 & -i \end{bmatrix}. \quad (3.3)$$

The unitary transformation on the mass eigenstates (both for left- and right-handed quarks),

$$\frac{1}{\sqrt{2}} \begin{bmatrix} -1 & -i \\ 1 & -i \end{bmatrix} \begin{bmatrix} d \\ s \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}, \quad \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & -i \\ 1 & -i \end{bmatrix} \begin{bmatrix} u \\ c \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad (3.4)$$

leads to a diagonal mixing matrix, whose eigenvalues are $e^{\mp i\theta_{12}}$, i.e., their phases are nothing but the Cabibbo angle. The states $d_1, d_2; u_1, u_2$, defined above in terms of the mass eigenstates, are independent of the Cabibbo angle:

$$d_{1,2} = \frac{1}{\sqrt{2}} (\mp d - is), \quad u_{1,2} = \frac{1}{\sqrt{2}} (\mp u - ic). \quad (3.5)$$

The weak current can be rewritten as

$$\cos\theta_C (\bar{u}d + \bar{c}s) + \sin\theta_C (\bar{u}s - \bar{c}d) = (\bar{u}_1 d_1) e^{-i\theta_C} + (\bar{u}_2 d_2) e^{i\theta_C}. \quad (3.6)$$

Thus the states $d_{1,2}$ and $u_{1,2}$, defined above, are states which are transformed into each other, i.e., $u_1 \leftrightarrow d_1$, $u_2 \leftrightarrow d_2$, in a weak-interaction vertex; the interaction of such a state with a W boson produces a phase $e^{\mp i\theta_C}$. We emphasize that the definition of the states u_i, d_i ($i=1,2$) does not involve the Cabibbo angle. The latter appears solely as a phase rotation in the weak interaction of a W boson and quark state.

We define the states u_i, d_i ($i=1,2$) as the *eigenstates* of the weak interaction, short as the *weak eigenstates*, to be distinguished from the mass eigenstates.

It is interesting to note that the weak eigenstates are

mixtures of mass eigenstates with equal weight; e.g., the state d_1 consists with a probability of 50% of a d quark, and with the same probability of an s quark.

The question arises whether the definition of the weak eigenstates, given above, is merely a useful mathematical exercise, or these states are of special physical interest. In the standard electroweak model, our definition is, of course, nothing but a reformulation of the weak currents in a new basis, which could be interesting in special circumstances, but does not lead to new physical insight.

However, if one speculates about the physics beyond the standard model, it is likely that the weak eigenstates play an interesting role. Since the weak interaction is

“pure,” i.e., diagonal in terms of these states, one expects that they are the states which have a simple structure and play a role as the basic units of such an extended dynamical theory. For example, in a composite model of the quarks and leptons, the weak eigenstates would be the states which have a simple internal structure, quite analogous to the situation in QCD. There the meson states $\pi_0 = (\bar{u}u - \bar{d}d)/\sqrt{2}$ and $\eta = (\bar{u}u + \bar{d}d)/\sqrt{2}$ are mass eigenstates (we neglect the $\bar{s}s$ component), but they are mixtures of the states $(\bar{u}u)$ and $(\bar{d}d)$, which are the quark-model eigenstates.

Let us consider the mass matrices of the quarks in terms of the weak eigenstates:

$$\begin{aligned} (\bar{d}, \bar{s}) \begin{bmatrix} m_d & \\ & m_s \end{bmatrix} \begin{bmatrix} d \\ s \end{bmatrix} &= (\bar{d}_1, \bar{d}_2) \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & -i \\ 1 & -i \end{bmatrix} \begin{bmatrix} m_d & \\ & m_s \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ i & i \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \\ &= (\bar{d}_1, \bar{d}_2) \frac{1}{2} \begin{bmatrix} m_d + m_s & -m_d + m_s \\ -m_d + m_s & m_d + m_s \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \end{aligned} \quad (3.7)$$

(analogous relations are valid for the up-type quarks).

Using as an example the quark mass values, normalized at 1 GeV, $m_u = 5$ MeV, $m_d = 8$ MeV, $m_s = 150$ MeV, and $m_c = 1350$ MeV, we find the following mass matrices (in MeV):

Q_u quarks :

$$\begin{bmatrix} 678 & 673 \\ 673 & 678 \end{bmatrix} = 678 \begin{bmatrix} 1 & 0.993 \\ 0.993 & 1 \end{bmatrix}. \quad (3.8)$$

Q_d quarks :

$$\begin{bmatrix} 79 & 71 \\ 71 & 79 \end{bmatrix} = 79 \begin{bmatrix} 1 & 0.899 \\ 0.899 & 1 \end{bmatrix}. \quad (3.9)$$

Thus the quark mass matrices written in terms of weak eigenstates turn out to be remarkably simple. In the limit $m_u \rightarrow 0$, $m_d \rightarrow 0$, they are proportional to the matrix

$$M = \text{const} \times \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \quad (3.10)$$

Such matrices have been studied previously in connection to the mass hierarchy problem.⁷ It is interesting that we are led to such “democratic” mass matrices for a reason, which is not directly related to the mass hierarchy.

IV. THREE GENERATIONS: REAL CKM MATRIX

The CKM matrix in the case of three generations is described by the three angles $\theta_{12}, \theta_{23}, \theta_{13}$ and the phase parameter δ . For the angles θ_{12}, θ_{23} , we shall assume the central values as given by the Particle Data Group,⁸ $V_{us} = s_{12}c_{13} = 0.220$, $\theta_{12} = 12.7^\circ$; $V_{cb} = s_{23}c_{13} = 0.046$, $\theta_{23} = 2.6^\circ$. The experimental errors, especially the fairly large one for θ_{23} , are not considered for the subsequent calculations, but, if needed, can easily be taken into account by the reader.

As far as the third angle θ_{13} and the phase parameter δ are concerned, the experimental situation is still unclear. A conservative upper limit on $|V_{ub}| = s_{13}$ seems to be about 0.01, corresponding to $\theta_{13} = 0.6^\circ$. There are indications^{9,10} that $|V_{ub}|$ is of the order of 0.005, which corresponds to an angle of about 0.29° . In any case it is evident that θ_{13} is significantly smaller than θ_{23} and much smaller than the Cabibbo angle θ_{12} . For this reason we consider first the case $\theta_{13} = 0$, i.e., a limit, in which CP violation cannot be described within the standard electroweak model. Nevertheless, this limit is a very good approximation to the real world. The CKM matrix takes the form of the product of two rotation matrices:

$$\begin{aligned} V &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{bmatrix} \begin{bmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} c_{12} & s_{12} & 0 \\ -s_{12}c_{23} & c_{12}c_{23} & s_{23} \\ s_{12}s_{23} & -c_{12}s_{23} & c_{23} \end{bmatrix}. \end{aligned} \quad (4.1)$$

We proceed to diagonalize this matrix. The matrix V describes an orthogonal transformation in the space of the three generations, which (in the limit $V_{ub} = 0$) is isomorphic to the ordinary space of three dimensions. It is well known that such a transformation can always be described by a simple rotation (angle α) around a specific rotation axis. The three-vector which defines this axis remains invariant under the transformation. This fact is a special feature of three dimensions.

In the case of two dimensions, all vectors are rotated by the same angle under an orthogonal transformation. No vector, except the null vector, remains invariant. As a result, the diagonalization of the CKM matrix in the case of three generations leads to new features not

present in the case of two generations. In particular, one vector, i.e., a certain superposition of the mass eigenstates, will remain invariant under the CKM transformation; i.e., this state is reproduced in a charged-current interaction with a W boson.

The eigenvalues of the CKM matrix denoted above must be $e^{-i\alpha}, e^{i\alpha}, 1$, since they must have the absolute values 1 and must be pairwise complex conjugated with respect to each other. The eigenvector belonging to the eigenvalue 1 is the vector remaining invariant under the transformation; it defines the rotation axis. The phase angle α is nothing but the angle of rotation around this axis.

The simplest way to determine the angle is to use the invariance of the trace. We have

$$V = U \begin{bmatrix} e^{-i\alpha} & & \\ & e^{i\alpha} & \\ & & 1 \end{bmatrix} U^{-1} \quad (4.2)$$

(U = unitary 3×3 matrix),

$$\text{tr} V = 1 + 2 \cos \alpha = c_{12} + c_{12}c_{23} + c_{23}. \quad (4.3)$$

If the (small) angle θ_2 is set to zero, we find, of course, $\alpha = \theta_{12}$; i.e., the rotation corresponds exactly to the Cabibbo rotation among the first two generations. The vector describing the third generation defines the rotation axis. Let us take

$$\alpha = \theta_{12} + \Delta\alpha. \quad (4.4)$$

One finds, setting $c_{23} = 1 - \theta_{23}^2/2$ and neglecting terms of order θ_{23}^3 and higher,

$$\Delta\alpha \simeq \frac{\theta_{23}^2(1 + c_{12})}{4s_{12}}. \quad (4.5)$$

Using the numbers given above, we obtain

$$\begin{aligned} \Delta\alpha &\simeq 2.246 \theta_{23}^2 \simeq 0.0046 = 0.265^\circ, \\ \alpha &= \theta_{12} + 0.265^\circ \simeq 12.96^\circ. \end{aligned} \quad (4.6)$$

Thus the actual rotation angle α is only slightly larger than the Cabibbo angle.

This result for the value of the phase in the diagonal mixing matrix is similar to that found by Barnhill.⁶

The rotation axis $\mathbf{a} = (a_1, a_2, a_3)$ is obtained as follows. We have $V \cdot \mathbf{a} = \mathbf{a}$ or $V^{-1} \mathbf{a} = \mathbf{a}$. Since V is orthogonal, we can also write $V^\dagger \mathbf{a} = \mathbf{a}$ or $(V - V^\dagger) \mathbf{a} = 0$. This matrix system leads to the relations

$$\begin{aligned} (v_{12} - v_{21})a_2 + (v_{13} - v_{31})a_3 &= 0, \\ (v_{21} - v_{12})a_1 + (v_{23} - v_{32})a_3 &= 0, \\ (v_{31} - v_{13})a_1 + (v_{32} - v_{23})a_2 &= 0. \end{aligned} \quad (4.7)$$

The direction of the rotation axis is then given by

$$a_1 : a_2 : a_3 = (v_{23} - v_{32}) : (v_{31} - v_{13}) : (v_{12} - v_{21}). \quad (4.8)$$

Inserting the particular values of the matrix elements, we find

$$a_1 : a_2 : a_3 = s_{23}(1 + c_{12}) : s_{12}s_{23} : s_{12}(1 + c_{23}). \quad (4.9)$$

Numerically, we obtain

$$a_1 : a_2 : a_3 = 0.090 : 0.010 : 0.440. \quad (4.10)$$

The unit vector in the $\mathbf{a}$ direction has the components

$$\mathbf{a} = \frac{1}{\sqrt{N}} \begin{bmatrix} (1 + c_{12})s_{23} \\ s_{12}s_{23} \\ (1 + c_{23})s_{12} \end{bmatrix}, \quad (4.11)$$

where the normalization factor $\sqrt{N}$ is given by $2 \sin \alpha$:

$$\begin{aligned} N &= (1 + c_{12})^2 s_{23}^2 + s_{12}^2 s_{23}^2 + (1 + c_{23})^2 s_{12}^2 \\ &= 4 \sin^2 \alpha. \end{aligned} \quad (4.12)$$

[This relation is derived by using the definition of α in terms of the $\text{tr}(V) = c_{12} + c_{12}c_{23} + c_{23} = 1 + 2 \cos \alpha$.]

Neglecting terms of order s_{23}^2 , one finds, for the unit vector in the $\mathbf{a}$ direction,

$$\mathbf{a} \simeq \frac{1}{[4s_{12}^2 + s_{23}^2(1 + c_{12})^2]^{1/2}} \begin{bmatrix} s_{23}(1 + c_{12}) \\ s_{12}s_{23} \\ 2s_{12} \end{bmatrix}, \quad (4.13)$$

$$\mathbf{a} \approx \begin{bmatrix} s_{23}/s_{12} \\ s_{23}/2 \\ 1 \end{bmatrix}. \quad (4.14)$$

Specifically, we obtain, using $\theta_{12} = 12.7^\circ$, but keeping θ_{23} free,

$$\mathbf{a} \simeq \frac{1}{(0.193 + 3.903s_{23}^2)^{1/2}} \begin{bmatrix} s_{23}1.976 \\ s_{23}0.220 \\ 0.440 \end{bmatrix}. \quad (4.15)$$

For example, setting $\theta_{23} = 2.6^\circ$, one finds

$$\mathbf{a} \simeq \begin{bmatrix} 0.200 \\ 0.022 \\ 0.980 \end{bmatrix}. \quad (4.16)$$

The mass eigenstates of the quarks determine the x axis (first generation), the y axis (second generation), and the z axis (third generation) of our three-dimensional coordinate system. We observe that the rotation axis is fairly aligned with the direction of the third generation, but tilted by an angle $\arcsin(0.200) \approx 11.5^\circ$ in the direction of the first-generation axis and by a (small) angle $\arcsin(0.022) \approx 1.26^\circ$ into the direction of the second generation. The ratio of these two angles is given by the Cabibbo angle θ_{12} : $(1 + c_{12})/s_{12} \approx 9$ (see Fig. 1).

Since the CKM transformation leaves the vector $\mathbf{a}$ invariant, the charged-current weak interaction is such that the state $u_a = a_1 u + a_2 c + a_3 t$ is transformed into the state $d_a = a_1 d + a_2 s + a_3 b$. In other words, the doublet $\begin{bmatrix} u_a \\ d_a \end{bmatrix}$ acts as a doublet of weak eigenstates. Numerically, we have

FIG. 1. Flavor mixing (in the case $V_{ub}=0$) is just a rotation by the angle α around the axis which is determined by the eigenvector $\mathbf{a}$ belonging to the real eigenvalue of V . The mass eigenstates of the quarks determine the x axis (first generation), the y axis (second generation), and the z axis (third generation) of the coordinate system.

$$\begin{aligned} u_a &= 0.200u + 0.022c + 0.980t, \\ d_a &= 0.200d + 0.022s + 0.980b; \end{aligned} \quad (4.17)$$

i.e., the third generation mixes fairly strongly with the first generation, much less so with the second one. We note that the corresponding mixing angles are proportional to θ_{23} . If this angle turns to zero, u_a, d_a become mass eigenstates: $u_a \rightarrow t, d_a \rightarrow b$.

We note that the angle θ_{23} , which is observed to be very small, leads to a fairly large bending of the $\mathbf{a}$ axis away from the axis of the third generation. This is due to the fact that the associated angle is determined by the ratio $s_{23}/s_{12} \approx 0.2$, which happens to be of the same order as s_{12} itself.

We proceed to calculate the eigenvectors belonging to the eigenvalues $e^{\mp i\alpha}$. We find

$$\mathbf{v}_1 = \frac{1}{\sqrt{K}} \begin{bmatrix} -1 \\ (c_{12} - e^{-i\alpha})/s_{12} \\ \frac{(c_{12}e^{-i\alpha} - 1)s_{23}}{(e^{-i\alpha} - c_{23})s_{12}} \end{bmatrix}, \quad (4.18)$$

$$\mathbf{v}_2 = \frac{1}{\sqrt{K}} \begin{bmatrix} 1 \\ (e^{i\alpha} - c_{12})/s_{12} \\ \frac{(1 - c_{12}e^{i\alpha})s_{23}}{(e^{i\alpha} - c_{23})s_{12}} \end{bmatrix}. \quad (4.19)$$

The normalization coefficient K is given by

$$K = 1 + 2 \frac{(1 - c_{23}\cos\alpha)(1 + c_{12}^2 - 2c_{12}\cos\alpha)}{s_{12}^2(1 + c_{23}^2 - 2c_{23}\cos\alpha)}. \quad (4.20)$$

The components of the three eigenvectors $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{a}$ can be combined to a matrix U , which diagonalizes the CKM matrix V :

$$U = \begin{bmatrix} (v_1)_1 & (v_2)_1 & a_1 \\ (v_1)_2 & (v_2)_2 & a_2 \\ (v_1)_3 & (v_2)_3 & a_3 \end{bmatrix}, \quad (4.21)$$

$$U^{-1} V U = \begin{bmatrix} e^{-i\alpha} & & \\ & e^{i\alpha} & \\ & & 1 \end{bmatrix}.$$

It is useful to consider the limit $\theta_{23} \rightarrow 0, \theta_{12} \neq 0$, in which the rotation axis $\mathbf{a}$ coincides with the axis of the third generation. In this limit we find ($K \rightarrow 2$)

$$\mathbf{v}_1 = \begin{bmatrix} -1/\sqrt{2} \\ i/\sqrt{2} \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \\ 0 \end{bmatrix}, \quad (4.22)$$

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 & 0 \\ i & i & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}. \quad (4.23)$$

The corresponding quark fields are $d_{1,2} = (1/\sqrt{2}) \times (\mp d - is)$, $u_{1,2} = (1/\sqrt{2}) (\mp u - ic)$, $d_3 = b$, and $u_3 = t$. Of course, this limit is analogous to the Cabibbo rotation discussed in the two-generation case.

In this limit the mass matrices (in the new basis) are proportional to the matrix (neglecting corrections of order $m_d/m_s, m_u/m_c$, respectively)

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & R \end{bmatrix}, \quad (4.24)$$

where R is $2m_b/m_s, 2m_t/m_c$, respectively.

In general, we can introduce the matrix

$$U_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i & 0 \\ 1 & -i & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}, \quad (4.25)$$

and transform the complex diagonal matrix into a rotation matrix:

$$U_0^{-1} \begin{bmatrix} e^{-i\alpha} & & \\ & e^{i\alpha} & \\ & & 1 \end{bmatrix} U_0 = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (4.26)$$

This shows explicitly that the effect of the flavor mixing is just a rotation by the angle α around the eigenvector belonging to the real eigenvalue.

Four our choice of θ_{12} and θ_{23} , we obtain

$$\mathbf{v}_1 \simeq \frac{1}{\sqrt{K}} \begin{bmatrix} -1 \\ 0.0045 + i1.0195 \\ 0.2065 - i0.0235 \end{bmatrix}, \quad (4.27)$$

$$\mathbf{v}_2 \simeq \frac{1}{\sqrt{K}} \begin{bmatrix} 1 \\ -0.0045 + i1.0195 \\ -0.2065 - i0.0235 \end{bmatrix}, \quad (4.28)$$

where $K \simeq 2.081$,

$$U \simeq \begin{bmatrix} -0.693 & 0.693 & 0.200 \\ 0.003+i0.707 & -0.003+i0.707 & 0.022 \\ 0.143-i0.016 & -0.143-i0.016 & 0.980 \end{bmatrix}. \quad (4.29)$$

In the new basis we find for the mass matrices in MeV, using as an example $m_t = 130$ GeV, $m_b = 5.7$ GeV (both masses are defined at an energy scale of 1 GeV),

$$M(u_1, u_2, u_3) = 3368.40 \begin{bmatrix} 1 & -0.58 & 5.41 \\ -0.58 & 1 & -5.41 \\ 5.41 & -5.41 & 37.07 \end{bmatrix} + i592.33 \begin{bmatrix} 0 & -1 & 3.41 \\ 1 & 0 & 3.41 \\ -3.41 & -3.41 & 0 \end{bmatrix}, \quad (4.30)$$

$$M(d_1, d_2, d_3) = 196.83 \begin{bmatrix} 1 & -0.22 & 4.05 \\ -0.22 & 1 & -4.05 \\ 4.05 & -4.05 & 27.81 \end{bmatrix} + i25.59 \begin{bmatrix} 0 & -1 & 3.40 \\ 1 & 0 & 3.40 \\ -3.40 & -3.40 & 0 \end{bmatrix}. \quad (4.31)$$

Note that the matrices of the imaginary part of the up- and down-type quark mass matrices turn out to be proportional. Thus the phenomenon that in the case of two generations the mass matrix is close to the “democratic” mass matrix does not repeat itself in the case of three generations. The third “eigenvector” coincides almost with the mass eigenvector of the third generation. This is, of course, not dictated *a priori*, but is merely an observed fact. We interpret this as an indication that the Cabibbo mixing among the two “light” flavors and the mixing of the third “heavy” family with the light ones look like two different phenomena, i.e., of a somewhat different dynamical origin.

V. THREE GENERATIONS: COMPLEX CKM MATRIX

Now we allow the angle θ_{13} as well as the phase parameter δ to be different from zero. We use the parametrization given in Eq. (1.3). The diagonalization leads in general to three complex eigenvalues with modulus 1:

$$U^{-1} V U = \begin{bmatrix} e^{i\alpha} & & \\ & e^{i\beta} & \\ & & e^{i\gamma} \end{bmatrix}. \quad (5.1)$$

Now we are dealing with a unitary matrix, which is not orthogonal. The three phase angles α , β , and γ cannot be interpreted, as in the previous case, as rotation angles in a three-dimensional Euclidean space. Thus a geometrical interpretation of the flavor mixing is not possible, only in the sense that the flavor mixing, as observed in reality, is very close to the limiting case $V_{ub} = 0$ discussed previously, in which a geometrical interpretation can be given.

The invariant coefficients of the characteristic polynomial

$$P(\lambda) = \det(\lambda I - V) = \lambda^3 - \text{tr}(V)\lambda^2 + p(V)\lambda - \det V \quad (5.2)$$

are given by

$$\det V = 1 = e^{i(\alpha+\beta+\gamma)}, \quad (5.3)$$

$$\begin{aligned} \text{tr} V &= c_{12}c_{23} + c_{23}c_{13} + c_{12}c_{13} - s_{12}s_{23}s_{13}^{i\delta} \\ &= e^{i\alpha} + e^{i\beta} + e^{i\gamma}, \end{aligned} \quad (5.4)$$

$$p(V) = [\text{tr}(V)]^* = e^{i(\alpha+\beta)} + e^{i(\beta+\gamma)} + e^{i(\alpha+\gamma)}. \quad (5.5)$$

In accordance with these constraints, we choose the following parametrization of the eigenvalues:

$$U^{-1} V U = \text{diag}(e^{-i(\alpha+\epsilon/2)}, e^{i(\alpha\epsilon/2)}, e^{i\epsilon}). \quad (5.6)$$

In the limit $V_{ub} \rightarrow 0$ ($s_{13} \rightarrow 0$), the eigenvalues will be the ones given in Eq. (4.2). Thus ϵ will be zero in that limit, and in reality ϵ will be small compared to α , while α is expected to be close to the Cabibbo angle.

We shall not compute α and ϵ exactly in terms of the three angles and the phase δ , but rather compute the departures from the limiting case $V_{ub} = 0$, discussed above. According to the trace condition, we have

$$c_{12}c_{23} + c_{23}c_{13} + c_{12}c_{13} - s_{12}s_{23}s_{13}e^{i\delta} = (e^{i\alpha} + e^{-i\alpha})e^{-i\epsilon/2} + e^{i\epsilon}. \quad (5.7)$$

The real part of this expression is of the order of 3, while the imaginary part proportional to $\sin\delta$, is multiplied by $s_{12}s_{23}s_{13}$, which is of the order of 10^{-4} or less. Thus we can expand the right-hand side in terms of ϵ and neglect all but the terms linear in ϵ . We find

$$c_{12}c_{23} + c_{23}c_{13} + c_{12}c_{13} - s_{12}s_{23}s_{13}e^{i\delta} \simeq 2\cos\alpha + 1 + i\epsilon(1 - \cos\alpha), \quad (5.8)$$

$$\text{Re: } c_{12}c_{23} + c_{23}c_{13} + c_{12}c_{13} - s_{12}s_{23}s_{13}\cos\delta \simeq 2\cos\alpha + 1, \quad (5.9a)$$

$$\text{Im: } -s_{12}s_{23}s_{13}\sin\delta \simeq \epsilon(1 - \cos\alpha). \quad (5.9b)$$

Neglecting terms of order 10^{-4} , we find that the angle α coincides with the one calculated in the limit $V_{ub} = 0$, i.e., $\alpha = \theta_{12} + 0.265^\circ$. Furthermore, we find a relation between ϵ and $\sin\delta$:

$$\epsilon \simeq \frac{-s_{12}s_{23}s_{13}\sin\delta}{1-\cos\alpha}, \quad (5.10)$$

$$\epsilon \approx -0.397s_{13}\sin\delta. \quad (5.11)$$

Thus ϵ is of the order of 4×10^{-3} or less, depending on δ . Let us consider the three eigenvectors, which we denote by $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{a}$, in analogy to the vectors introduced in Eqs. (4.18), (4.19), and (4.11). We find

$$\mathbf{v}_1 = \frac{1}{\sqrt{K_1}} \begin{bmatrix} -1 \\ [(c_1 - c_{13}e^{-i(\alpha+\epsilon/2)})s_{23} + c_{23}s_{12}s_{13}e^{-i\delta}]/A_1 \\ [(1 - c_{12}c_{13}e^{-i(\alpha+\epsilon/2)})s_{23}^2 - s_{13}C_1]/B_1 \end{bmatrix}, \quad (5.12)$$

$$\mathbf{v}_2 = \frac{1}{\sqrt{K_2}} \begin{bmatrix} 1 \\ [(c_{13}e^{i(\alpha-\epsilon/2)} - c_{12})s_{23} - c_{23}s_{12}s_{13}e^{-i\delta}]/A_2 \\ [(c_{12}c_{13}e^{i(\alpha-\epsilon/2)} - 1)s_{23}^2 + s_{13}C_2]/B_2 \end{bmatrix}, \quad (5.13)$$

$$\mathbf{a} = \frac{1}{\sqrt{K_3}} \begin{bmatrix} [s_{23}(1 - c_{12}^2) + s_{13}e^{-i\delta}s_{12}(e^{i\epsilon} - c_{12}c_{23})]/D \\ s_{12}s_{23} - s_{13}e^{-i\delta}s_{12}^2c_{23}/D \\ s_{12}[e^{i\epsilon}(e^{i\epsilon} - c_{12}c_{13}) + c_{23}(c_{13} - c_{12}e^{i\epsilon}) + s_{13}e^{i\delta}s_{12}^2s_{23}e^{i\epsilon}]/D \end{bmatrix}, \quad (5.14)$$

with the abbreviations

$$A_{1,2} = s_{12}s_{23} + s_{13}e^{-i\delta}(e^{\mp i(\alpha \pm \epsilon/2)} - c_{12}c_{23}),$$

$$B_{1,2} = (c_{23}c_{13} - e^{\mp i(\alpha \pm \epsilon/2)})A_{1,2},$$

$$C_{1,2} = s_{12}s_{23}e^{\mp i(\alpha \pm \epsilon/2)}(c_{23}c_{13}e^{i\delta} - e^{-i\delta}) + c_{23}s_{13}(c_{12}e^{\mp i(\alpha \pm \epsilon/2)} - c_{23}),$$

$$D = c_{13}e^{i\epsilon} - c_{12},$$

$$K_3 = 2s_{12}^2[2 + c_{12}^2(c_{23}^2 + c_{13}^2) - \frac{1}{2}(c_{12}^2 + c_{23}^2 + c_{13}^2 + c_{12}^2c_{23}^2c_{13}^2) + c_{12}^2c_{23}c_{13} + c_{23}c_{13}\cos(2\epsilon) - c_{12}(c_{23} + c_{13})(2\cos\epsilon + s_{12}s_{23}s_{13}\cos\delta) + 2s_{12}s_{23}s_{13}\cos(\delta - \epsilon)]/(c_{12}^2 + c_{13}^2 - 2c_{12}c_{13}\cos\epsilon) \approx 0.2097.$$

The normalization coefficients K_1 and K_2 are lengthy expressions in terms of the mixing angles and δ . They are not given here.

Numerically, we find, for $\mathbf{a}$ as a function of θ_{13} and δ ,

$$\mathbf{a} \approx \begin{bmatrix} 0.198 + 0.500s_{13}e^{-i\delta} \\ 0.022 - 4.308s_{13}e^{-i\delta} \\ 0.960 + 0.198s_{13}e^{i\delta} \end{bmatrix}, \quad (5.15)$$

with $\theta_{13} \simeq 0.6^\circ$, and we get

$$\mathbf{a} \approx \begin{bmatrix} 0.198 \\ 0.022 \\ 0.960 \end{bmatrix} + \begin{bmatrix} 0.005 \\ -0.043 \\ 0.002 \end{bmatrix} \cos\delta + i \begin{bmatrix} -0.005 \\ 0.043 \\ 0.002 \end{bmatrix} \sin\delta. \quad (5.16)$$

VI. CONCLUSIONS

Because of the freedom of rotating the phases of the quark fields, the eigenvalues of the quark mixing matrix do not have *a priori* a physical meaning. However, once a special representation of the mixing matrix is chosen,

there is an unambiguous way in defining the weak-interaction eigenstates via the eigenvalues of the CKM matrix.

In the case of two generations, we arrive at weak eigenstates which are mixtures of quark mass eigenstates of equal weight; their definition is independent of the Cabibbo angle. In this weak eigenbasis, we get mass matrices of democratic structure.

Also, in the case of a decoupled third generation, we are led to democratic mass matrices for the first two generations. But if the mixing between the second and third family is turned on, the quark mass matrices depart substantially from the previous ones.

Furthermore, in the three-generation case the diagonalization of the mixing matrix results in the existence of an invariant eigenvector, showing a qualitatively new feature of the mixing phenomenon. For a decoupled third family, this invariant eigenvector is oriented in the direction of the coordinate axis of the third generation, while for $\theta_{23} \neq 0$ the vector $\mathbf{a}$ lies almost in the plane, spanned by the axis of the first and third generation, but has a sizeable inclination away from the third coordinate axis. A quark state belonging to $\mathbf{a}$ mixes fairly strongly the third and first generations.

Considering the complex mixing matrix, two phase angle parameters α, ϵ are needed in order to describe the flavor-mixing phenomenon. In the general case, neither

does an invariant eigenvector exist nor is a geometrical interpretation of flavor mixing possible. The second component of the third eigenvector is strongly affected by the CP phase δ .

Note added. In Ref. 11 a two-angle parametrization of the CKM matrix was considered, based on the eigenvalue idea, which, however, departs both in intention and results from our approach.

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¹N. Cabibbo, Phys. Rev. Lett. **10**, 531 (1963).

²M. Kobayashi and T. Maskawa, Prog. Theor. Phys. **49**, 652 (1973).

³L. Maiani, in *Proceedings of the International Symposium on Lepton and Photon Interactions at High Energies*, Hamburg, West Germany, 1977, edited by F. Gutbrod (DESY, Hamburg, 1977); L. L. Chau and W. Y. Keung, Phys. Rev. Lett. **53**, 1802 (1984); H. Fritzsch, Phys. Rev. D **32**, 3058 (1985); H. Harari and M. Leurer, Phys. Lett. B **181**, 123 (1986); H. Fritzsch and J. Plankl, Phys. Rev. D **35**, 1732 (1987).

⁴E. Wigner, *Group Theory and its Application to the Quantum Mechanics of Atomic Spectra, Pure and Applied Physics*

(Academic, New York, 1959), Vol. 5.

⁵J. Plankl, in *Proceedings of the XII Warsaw Symposium on High Energy Physics*, Kazimierz, Poland, 1989 (unpublished).

⁶M. V. Barnhill III, Phys. Rev. D **36**, 192 (1987).

⁷H. Fritzsch and J. Plankl, Phys. Lett. B **237**, 451 (1990).

⁸Particle Data Group, Phys. Lett. B **204**, 107 (1988).

⁹M. V. Danilov, in *Proceedings of the XIVth International Symposium on Lepton and Photon Interactions*, Stanford, 1989, edited by M. Riordan (World Scientific, Singapore, 1990).

¹⁰D. L. Kreinick, in *Proceedings of the XIVth International Symposium on Lepton and Photon Interactions* (Ref. 9).

¹¹P. Kielanowski, Phys. Rev. Lett. **63**, 2189 (1989).