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The QCD Effective Action at Long Distances

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ABSTRACT

We describe the derivation, under a well defined set of approximations, of a low energy effective chiral lagrangian of the strong interactions from QCD. The effective lagrangian includes only the Nambu-Goldstone modes (i.e. the octet of pseudoscalar particles) in the presence of external sources, with inclusion of the explicit chiral symmetry breaking due to the light quark masses $\mathcal{M}$. The coupling constants of the $\mathcal{O}(p^4)$ terms of the chiral lagrangian are calculated first to leading order in the $1/N_c$ expansion but with $\mathcal{O}(\alpha)$ terms and higher orders neglected. Next, the corrections $\mathcal{O}(\alpha)$ to the couplings which remain in the chiral limit $\mathcal{M} \rightarrow 0$ are also taken into account. In view of our results, an improved chiral power counting for higher order terms in the effective chiral lagrangian is proposed.

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1. Introduction

The importance of the flavour $SU(3)_L \otimes SU(3)_R$ global symmetry in Particle Physics has been recognized for many years starting in the "old" days of current algebra. The existence of light pseudoscalar mesons, with a mass much smaller than those of the rest of the hadrons, suggests that in nature this symmetry is spontaneously broken down to vectorial $SU(3)_V$ (the Eightfold Way symmetry). The light pseudoscalars are the Nambu-Goldstone bosons (massless in the limit of exact flavour symmetry) associated with the broken continuous axial symmetry, and the unbroken $SU(3)_V$ manifests itself in the approximate classification of hadrons in multiplets of given spin and parity, degenerate in mass. According to the Goldstone theorem, the Nambu-Goldstone bosons (π^a) couple to the axial currents A_μ^a

$$\langle 0 | A_\mu^a(x) | \pi^b(q) \rangle = i q_\mu f_\pi \delta^{ab} e^{-iqx} \quad (1.1)$$

with $q^2 = m_\pi^2$, where a, b label flavour indices, q_μ is the momentum of the meson and f_π is a constant finite in the chiral limit with the dynamical content of the theory. Experimentally, from $\pi \rightarrow \mu\nu$ decay, $f_\pi = 93.3$ MeV.

The approximate $SU(3)_L \otimes SU(3)_R$ symmetry — exact in the chiral limit of massless quarks — is incorporated naturally in QCD. Regrettably, many important observables at low energies which are governed by chiral symmetry breaking and, in general, the mechanism of hadronization have not been yet understood on the basis of QCD.

In a series of papers, Gasser and Leutwyler [1] have shown how to systematically extract from experiments the low energy structure of strong interactions among pseudoscalar mesons in powers of the external momenta and quark masses; an idea that goes back to early work by Weinberg [2] (see also [3]). One uses for this purpose an effective chiral lagrangian involving Nambu-Goldstone degrees of freedom only. The form of the chiral lagrangian is determined by symmetry requirements. The number of allowed terms is infinite, so the low energy effective lagrangian contains an infinite number of coupling constants. In practice, however, to a given order in the chiral expansion, the couplings can be fitted with the data available and then one proceeds to make further predictions. Of course all these low energy constants should be obtainable from QCD.

The need for an infinite number of terms in the four dimensional chiral lagrangian can be understood in several ways. The simplest chiral lagrangian one can build is the

nonlinear sigma model which contains two derivatives and one free parameter (f_π); this is a non renormalizable model in four dimensions. More and more counterterms (with an increasing number of derivatives, by power counting) are needed to remove the divergences generated by loops induced by the lower terms. Not all the structures that can possibly appear need have divergent coefficients, though. It is well known that, in the chiral limit, at the one loop level, only a particular combination involving two out of all the possible operators with four derivatives does actually require a divergent coefficient to cancel the one loop divergences [1, 3]. Another way of looking at the same question is the following. Interpreting the chiral lagrangian as the result of integrating out all the QCD degrees of freedom, but the pseudoscalars (the lowest lying states), it is natural to expect the result to be a non local functional of the local Nambu-Goldstone fields. When suitably expanded this would lead to an infinite number of local terms with an increasing number of derivatives.

The whole idea of obtaining a low energy effective lagrangian makes sense as far as the degrees which are left after the integration are indeed the lightest ones. This is guaranteed to be the case for the pseudoscalars in QCD. Not only because they are Goldstone bosons of a spontaneously broken symmetry, as it is commonly stated, but also because even if chiral symmetry were not spontaneously broken, the pseudoscalar channel contains the lightest excitations in any vector-like theory [4]. Of course, if chiral symmetry is not spontaneously broken there is nothing special about pions. Their mass would not be much lower than those of excitations in other channels, and in that case a chiral lagrangian would only be useful in a rather restricted range of energies. But the point is that pseudoscalars are necessarily part of a low energy effective lagrangian irrespectively of the way the symmetries of the theory are realized.

The four dimensional chiral lagrangian contains terms of increasing dimensionality. They are classified according to the number of derivatives and the power of the current quark mass when explicit chiral breaking terms are taken into account (counting a current quark mass as two chiral powers as is required by current algebra). One has to ask which is the scale that gives the right dimensionality to the operators. The natural answer in the framework of chiral perturbation theory is a scale $\Lambda_{CSB} \simeq 4\pi f_\pi$, as discussed in [5]. Can one prove this? Section 5 is dedicated to this question. As we will see the scale that normalizes the coefficients of higher dimensional operators is not $4\pi f_\pi$. It is quite easy to see, however, that the scale that suppresses higher order terms in the chiral lagrangian

must indeed be related to f_π . We will give a detailed derivation in the following sections, but for the sake of this discussion, let us imagine that we have integrated all the degrees of freedom of QCD above some energy Λ , such that only the pseudoscalars are explicitly left. On dimensional and symmetry grounds, the leading term in the resulting effective action will be of the form

$$\frac{c\Lambda^2}{16\pi^2} \text{tr} \partial_\mu U \partial^\mu U^\dagger \quad (1.2)$$

where U is a $SU(3)$ matrix which contains the pseudoscalar fields and c is a pure number to be computed. Making contact with the real world we find for the cut off the value

$$\Lambda = \frac{1}{\sqrt{c}} 2\pi f_\pi \quad (1.3)$$

All higher dimensional operators will then be suppressed by additional powers of Λ . The problem of course is to compute the constant factors like c and similar ones in higher dimensional operators. Since the coefficients of the chiral lagrangian are measurable quantities, certainly one wants to obtain results which are cut-off and regularization independent. Eventually the cut-off has to be traded for some physical quantity to yield regularization invariant results. We will discuss in the following sections how chiral symmetry breaking plays a very important role in settling the scale Λ and how this affects the long distance effective action of QCD.

Finally, we would like to make a short digression into two dimensions. It is well known that in two dimensions there is an established set of bosonization rules. That is, we can express the partition function of fermionic theories as well as their current algebra in terms of Bose fields and vice versa. There are many ways of seeing this equivalence, but a rather simple one is the power counting argument just sketched. Let us consider a two dimensional fermionic theory coupled to a gauge field ('t Hooft's QCD₂, in fact [6a]) and let us integrate all the degrees of freedom, but the local chiral phase — i.e., the pseudoscalar degrees of freedom. The result will be an expression similar to (1.2), but now, of course, the coefficient will not be quadratically divergent in the cut-off, but a calculable constant which would be, apart from "gluonic" corrections, the exact analogue of f_π [6b]. Therefore in two dimensions the physics does not constrain the cut-off Λ and nothing prevents us from removing it, hence making the higher order operators, which are suppressed by powers of the cut-off, negligible and bosonization exact.

In this article we would like to study the four dimensional counterpart of this bosonization. In other words, we would like to construct from QCD an equivalent bosonic theory

involving the fields which are generators of the global symmetries of the theory. Unfortunately, as we just stated, in four dimensions this can only be done approximately, order by order in the chiral expansion. We shall be mostly concerned with the next to leading terms of the chiral lagrangian (the L_i 's in the notation of reference [1b]). These are local $SU(3)_L \otimes SU(3)_R$ invariant terms of the effective action involving an even number of pseudoscalar fields. (The form of the Wess-Zumino terms which involve an odd number of fields is well established and will not be considered here at all, except for a small point which is made in section 4.) There are already several approaches in the literature to the computation of these even parity terms from QCD and, despite the fact that they share the feature that none of them take into account the effect of the gluon fields, the agreement of some of the coupling constants with the available experimental information is surprisingly good. The question is obviously why such a drastic approximation leads to rather accurate results. To answer this question we compute the leading gluonic corrections and discuss whether the agreement is physically relevant or not. As we shall see the outcome turns out to be quite satisfactory.

We have organized the article as follows. The second section deals with the $O(p^4)$ effective terms in the chiral limit. Terms with masses are studied in the third and fourth sections. For those the calculation is rather more complicated due to the subtleties of spontaneous symmetry breaking, as we will see. The question of chiral power counting is discussed in section 5. The conclusions of this work and an outlook are given in section 6. Some technical points and useful formulae have been relegated to two appendices.

2. The Chiral Limit

The chiral lagrangian describes the strong interactions when only the pseudoscalar meson fields are present. There is a range of energies, a typical scale, where the chiral lagrangian is predictive. Beyond that range new particles can be created, meaning that the expansion in derivatives ceases to be useful. With this idea in mind we begin by introducing an ultraviolet cut off Λ in QCD. The meaning of this cut off is the following: only the modes of action lower than Λ contribute to physics at low energies and the rest of the modes are taken into account through the effective couplings, which will, in principle, depend on Λ . The value of Λ is arbitrary a priori, but in a world with only pions we will see in a short while that it is related to f_π in a form that we will derive. It is perhaps of use to think of Λ as some sort of inverse lattice cut off or regulator à la Wilson. The above identification is thus equivalent to renormalizing the theory. The outcome of the calculation will be an effective lagrangian valid at energies much lower than Λ . Eventually, once everything is expressed in terms of physical quantities, we will want to remove the cut-off Λ .

Let us now proceed to compute the effective chiral lagrangian up to terms of $\mathcal{O}(p^4)$ included, taking into account the first gluonic corrections in the chiral limit. Massless QCD with three flavours is symmetric under $SU(3)_L \otimes SU(3)_R$ transformations and the corresponding Noether currents are (vector and axial-vector, respectively)

$$\begin{aligned} V_\mu^a &= \bar{q} \gamma_\mu \frac{\lambda^a}{2} q \\ A_\mu^a &= \bar{q} \gamma_\mu \gamma_5 \frac{\lambda^a}{2} q \end{aligned} \quad (2.1)$$

The generating functional for the Green functions of such currents is denoted by $\Gamma[v, a]$. In euclidean space

$$\exp \Gamma[v, a] = \frac{1}{Z} \int \mathcal{D}(G_\mu^a \mathcal{D}q \mathcal{D}q) \exp - \int d^4x \left(\frac{1}{4} G_{\mu\nu}^a G_{\mu\nu}^a + \bar{q} \gamma_\mu (D_\mu + v_\mu + \gamma_5 a_\mu) q \right) \quad (2.2)$$

where $D_\mu q = (\partial_\mu + ig \frac{\lambda^a}{2} G_\mu^a) q$ acts only on color indices, G_μ^a stands for the gluon fields, $\mathcal{D}G_\mu^a$ includes the Faddeev-Popov determinant, $G_{\mu\nu}^a = \partial_\mu G_\nu^a - \partial_\nu G_\mu^a - gf^{abc} G_\mu^b G_\nu^c$, and v_μ and a_μ are the vector and the axial-vector sources for the currents. They are traceless flavour matrices. The denominator in (2.2) is just a constant that makes $\Gamma[0, 0] = 0$. Our euclidean gamma matrices are such that $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$, $\{\gamma_\mu, \gamma_5\} = 0$ and both γ_μ and γ_5 are hermitean.

The quadratic integral over the quark fields is a one loop integral and its result is, formally, a determinant.

$$\exp \Gamma[v, a] = \frac{1}{Z} \int \mathcal{D}G_\mu^a \det \Delta \exp - \int d^4x \frac{1}{4} G_{\mu\nu}^a G_{\mu\nu}^a \quad (2.3)$$

where $\Delta = \gamma_\mu \Delta_\mu = \gamma_\mu (D_\mu + v_\mu + \gamma_5 a_\mu)$.

Under the infinitesimal local chiral transformation

$$\begin{aligned} \delta q &= -\lambda \gamma_5 q \\ \delta \bar{q} &= -\bar{q} \lambda \gamma_5 \\ \delta v_\mu &= [a_\mu, \lambda] \\ \delta a_\mu &= \partial_\mu \lambda + [v_\mu, \lambda] \equiv d_\mu \lambda \end{aligned} \quad (2.4)$$

where $\lambda(x)$ is a traceless, antihermitean ($\lambda^\dagger = -\lambda$) flavour matrix, one finds $\delta(\bar{q} \Delta q) = 0$. This symmetry is, however, anomalous at the quantum level. As noted by Fujikawa[7] the fermionic measure $\mathcal{D}q \mathcal{D}q$ is not invariant under (2.4). It is known to transform as

$$\delta(\mathcal{D}q \mathcal{D}q) = \mathcal{D}q \mathcal{D}q (2\text{Tr} \lambda \gamma_5) \quad (2.5)$$

which implies for the determinant of (2.3)

$$\delta X \equiv \delta \ln \det \Delta = 2\text{Tr} \lambda \gamma_5 \quad (2.6)$$

(Tr stands for the trace over space, Dirac, flavour and colour indices.) The right hand side is an ill-defined quantity and, following Fujikawa, we choose to regularize it with a damping factor

$$\delta X = 2\text{Tr} \lambda \gamma_5 \exp \Delta^2 / \Lambda^2 \quad (2.7)$$

Equation (2.7) is the precise mathematical formulation of our cut-off Λ . Expanding (2.7) in inverse powers of Λ , up to next to leading chiral order, and keeping the first gluonic corrections one gets for the real part of the effective action (see Appendix A for details)

$$\begin{aligned} \delta X = \text{tr} \int d^4x \lambda \left\{ \frac{N_c \Lambda^2}{2\pi^2} \left(1 + \frac{\pi^2}{6N_c} \frac{G_{\mu\nu}^a G_{\mu\nu}^a}{\Lambda^4} \right) d_\mu a_\mu \right. \\ \left. + \frac{N_c}{12\pi^2} \left(2[d_\mu V_{\mu\nu}, a_\nu] + \frac{1}{2}[V_{\mu\nu}, \Lambda_{\mu\nu}] + \frac{1}{2}[[a_\mu, a_\nu], \Lambda_{\mu\nu}] \right) \right. \\ \left. + \{d_\mu a_\mu, a_\nu a_\nu\} + 2a_\nu (d_\mu a_\mu) a_\nu + 2\{\{d_\mu a_\nu, a_\nu\}, a_\mu\} \right. \\ \left. + d_\mu d_\mu d_\nu a_\nu \right\} \\ + \frac{1}{120} \frac{G_{\mu\nu}^a G_{\mu\nu}^a}{\Lambda^4} \left(-\frac{1}{2}[V_{\mu\nu}, \Lambda_{\mu\nu}] + d_\mu d_\mu d_\nu a_\nu - 4\{\{d_\mu a_\nu, a_\nu\}, a_\mu\} \right. \\ \left. - 5\{d_\mu a_\mu, a_\nu a_\nu\} + 2a_\nu (d_\nu a_\nu) a_\mu - \frac{1}{2}[[a_\mu, a_\nu], \Lambda_{\mu\nu}] \right) \} \end{aligned} \quad (2.8)$$

where $d_\mu a_\nu = \partial_\mu a_\nu + [v_\mu, a_\nu]$, (r refers only to flavour indices and $V_{\mu\nu}$ and $A_{\mu\nu}$ are given by

$$[\Delta_\mu, \Delta_\nu] = V_{\mu\nu} + [a_\mu, a_\nu] + \gamma_5 A_{\mu\nu} \quad (2.9)$$

In (2.8) we have made the identification

$$G_{\mu\nu}^a G_{\rho\sigma}^b = \frac{1}{12} (\delta_{\mu\rho} \delta_{\nu\sigma} - \delta_{\mu\sigma} \delta_{\nu\rho}) G_{\rho\sigma}^a G_{\mu\nu}^b \quad (2.10)$$

since this will be true for the terms we are interested in once the gluonic fields are integrated over.

After integrating eq. (2.8) with respect to the local chiral transformation one gets three kinds of terms: those containing only $G_{\mu\nu}$, which will be denoted by X_G ; those containing only the external sources v_μ and a_μ , denoted by X_S ; and the rest, which we will represent by X_R . Then we have

$$e^{\Gamma[v,a]} = e^{X_S} e^{X_R} \quad (2.11)$$

Since X_G is nothing but the effective action of the gauge field induced by the fermion loop, the brackets denote the expectation value of the operator in question in QCD (without sources)

$$\langle A \rangle = \frac{1}{Z} \int \mathcal{D}G_\mu^a \exp - \int d^4x \left(\frac{1}{4} G_{\mu\nu}^a G_{\mu\nu}^a + X_G \right) \quad (2.12)$$

Now we can use the cumulant expansion

$$\langle e^{X_R} \rangle = \exp \left[\langle X_R \rangle + \frac{1}{2} (\langle X_R^2 \rangle - \langle X_R \rangle^2) + \dots \right] \quad (2.13)$$

to extract the leading dependence in the $1/N_c$ expansion. The leading piece is given by the first term in the expansion, which is $\mathcal{O}(N_c)$. This term also contains gluonic corrections which are leading in the large N_c limit; that is to say $\mathcal{O}(N_c)$ in this case. In particular the $(\frac{g}{\pi} GG)$ terms we are computing. The higher terms in the cumulant expansion are $\mathcal{O}(\alpha^2)$ at least (like $(\frac{g^2}{\pi} GGG)$ or $(\frac{g}{\pi} GG)^2$, for instance) and non-leading in the large N_c limit. We shall neglect them. At the approximation we are working we get from the first term in (2.13)

$$\begin{aligned} \Gamma[v,a] = & -\frac{N_c \Lambda^2}{4\pi^2} \left(1 + \frac{\pi^2}{6N_c} \frac{\langle \frac{g}{\pi} G_{\mu\nu}^a G_{\mu\nu}^a \rangle}{\Lambda^4} \right) \int d^4x \text{tr } a_\mu a_\nu \\ & + \frac{N_c}{12\pi^2} \int d^4x \text{tr} \left(2a_\mu a_\nu (V_{\mu\nu} + [a_\mu, a_\nu]) - a_\mu a_\nu a_\mu a_\nu \right. \\ & \left. + \frac{1}{4} A_{\mu\nu} A_{\mu\nu} + \frac{1}{2} (d_\mu a_\nu)^2 \right) \\ & + \frac{1}{120} \frac{\langle \frac{g}{\pi} G_{\mu\nu}^a G_{\mu\nu}^a \rangle}{\Lambda^4} \int d^4x \text{tr} \left(\frac{1}{4} A_{\mu\nu} A_{\mu\nu} + \frac{1}{2} (d_\mu a_\nu)^2 + 2a_\mu a_\nu a_\mu a_\nu \right) \end{aligned} \quad (2.14)$$

There are several remarks to point out. The first one is that (2.14) is not unambiguously determined from the anomalous variation (2.8). Indeed, we can add to (2.14) any local polynomial invariant by itself under a local chiral transformation. We will return to this issue shortly. As we will see, this arbitrariness is of no consequence to us. Setting $\alpha = 0$ (2.14) coincides with the results given in [8]. To the order we are considering $\delta\Gamma[v,a]$ coincides with (2.8) by just replacing $\frac{g}{\pi} GG$ by its expectation value. To be able to compute this expectation value one should also regulate the functional integral over the gluon fields. A higher covariant derivative with the normalizing factor Λ seems a good choice. However, we will not attempt to do this as we know that in QCD in addition to the computable, perturbative piece there is an uncalculable (in a weak field expansion), genuinely non perturbative contribution to $\langle \frac{g}{\pi} GG \rangle$ [9]. We rather keep $\langle \frac{g}{\pi} GG \rangle$ as a quantity that we can either fit phenomenologically or get from Monte Carlo simulations after proper identification of the corresponding cut-offs.

In [8] a sharp cut-off was used to evaluate the terms that do not depend on the cut-off (the second and third lines in (2.14)). The fact that, setting $\alpha = 0$, our results and those of [8] coincide as well as with a heat kernel calculation, as we will show in the next section, is good evidence of regularization independence for the terms that we are discussing (see also [7]). The arbitrariness lies, in any case, in the locally invariant polynomial that can be added to (2.14). Terms that contain Λ explicitly are non-universal. We shall return to them later.

Let us now make contact with the chiral lagrangian. Chiral symmetry implies that $\Gamma[v,a]$ admits a low energy representation of the form

$$\exp \Gamma[v,a] = \int \mathcal{D}U \exp W[U,v,a] \quad (2.15)$$

where the pseudoscalar fields are collected in a unitary matrix $U(x)$ with $\det U = 1$. Local chiral invariance implies that in the chiral limit U necessarily appears through covariant derivatives $D_\mu U$ with respect to the external fields v, a or commutators of covariant derivatives. Under $SU(3)_L \otimes SU(3)_R$ local chiral transformations (g_L, g_R), the matrix $U(x)$ transforms linearly

$$U(x) \rightarrow g_R U(x) g_L^\dagger \quad (2.16)$$

The functional $W[U,v,a]$ consists of two pieces: the Wess-Zumino term containing the imaginary part of the effective action and the term we are concerned with here, which is real and involves interactions among an even number of pseudoscalar fields.

We have seen that $\Gamma[v, a]$ is not invariant under a local chiral transformation due to the existence of the axial anomaly. Since bosonic path integrals have no anomalies, this restricts $W[U, v, a]$ to be a solution of the equation

$$\delta(W[U, v, a] - \Gamma[v, a]) = 0 \quad (2.17)$$

where U is a classical field, solution of the equations of motion of the bosonic representation (2.15). Chiral loops in (2.15) are down by powers of $1/N_c$ (see section 5), hence in the large N_c limit it is correct to replace the functional integral over the U fields by the value of the integrand at the classical solution. Equation (2.17) then follows. We take the boundary condition $W[U, v, a] = 0$, for any external fields v, a , which amounts to saying that the effective action vanishes when the Nambu-Goldstone fields are set equal to zero. Invariance under local $SU(3)_L \otimes SU(3)_R$ transformations and eq. (2.17) imply then

$$W[g_R^\dagger g_L, v, a] = \Gamma[v, a] - \Gamma[v', a'] \quad (2.18)$$

where

$$v' \pm a' = g_{R,L}(v \pm a)g_{R,L}^\dagger + g_{R,L}\partial g_{R,L}^\dagger \quad (2.19)$$

Without loss of generality we can choose

$$g_R^\dagger = g_L = \xi(x) = \exp\left(-\frac{i}{f_\pi} \frac{\lambda^a}{2} \pi^a\right) \quad (2.20)$$

i.e. $U(x) = \xi(x)\xi(x)$, from which an explicit representation for the effective action follows. As an example let us show how the lowest term in the chiral lagrangian, i.e. the non-linear sigma model term appears.

From (2.11) we have for the leading term

$$\Gamma^{(2)}[v, a] = -\frac{N_c \Lambda^2}{4\pi^2} \int d^4x \text{tr} a_\mu a_\mu \quad (2.21)$$

A straightforward substitution and some algebra gives

$$W^{(2)}[U, v, a] = -\frac{N_c \Lambda^2}{16\pi^2} \int d^4x \text{tr} D_\mu U D_\mu U^\dagger - \frac{N_c \Lambda^2}{4\pi^2} \int d^4x \text{tr} a_\mu a_\mu \quad (2.22)$$

with $D_\mu U = \partial_\mu U + (v_\mu + a_\mu)U - U(v_\mu - a_\mu)$ and $D_\mu U^\dagger = \partial_\mu U^\dagger - U^\dagger(v_\mu + a_\mu) + (v_\mu - a_\mu)U^\dagger$. The second term is a polynomial in the external field only.

With this approach it is clear that we cannot get the coefficients of terms which are invariant under (2.4) which is reflected in the freedom to add to W (or to Γ) any invariant piece. However, $R_{\mu\nu} \equiv [\Delta_\mu, \Delta_\nu] = V_{\mu\nu} + [a_\mu, a_\nu] + \gamma_5 A_{\mu\nu}$ is the only gauge covariant tensor in the sense of local $SU(3)_L \otimes SU(3)_R$. Therefore, we are missing combinations which are local polynomials in $R_{\mu\nu}$. Since the dependence of W on U is obtained from Γ by an (anomalous) pure gauge transformation, it is also clear that these terms can never give rise to any dependence on U in the effective action. Some of the local invariants are not even well defined in the infrared, at least in perturbation theory. The fact that they only depend on the external sources makes them irrelevant for our purposes.

It is not difficult to get the $\mathcal{O}(p^4)$ terms. One can use the lowest order equations of motion in the next to leading chiral terms. In (2.14) $\text{tr}(d_\mu a_\mu)^2$ gives rise to $\text{tr}[(D^2 U)(D^2 U^\dagger) - (D_\mu U D_\mu U^\dagger)^2]$ which vanishes by the equations of motion. One then finds (upon Wick rotating to Minkowski space)

$$\begin{aligned} W^{(4)} = & \int d^4x \left(L_1 \text{tr}(D_\mu U D_\mu U^\dagger)^2 + L_2 \text{tr} D^\mu U D^\mu U^\dagger D_\mu U D_\mu U^\dagger \right. \\ & + L_3 \text{tr} D^\mu U D_\mu U^\dagger D^\nu U D_\nu U^\dagger \\ & - i L_9 \text{tr}(F_R^{\mu\nu} D_\mu U D_\nu U^\dagger + F_L^{\mu\nu} D_\mu U D_\nu U^\dagger) \\ & \left. + L_{10} \text{tr} U^\dagger F_R^{\mu\nu} U F_L^{\mu\nu} \right) \end{aligned} \quad (2.23)$$

where $-i F_{\mu\nu}^{R,L} = \partial_\mu(v_\nu \pm a_\nu) - \partial_\nu(v_\mu \pm a_\mu) + [v_\mu \pm a_\mu, v_\nu \pm a_\nu]$ and

$$8L_1 = 4L_2 = L_9 = \frac{N_c}{48\pi^2} (1 + \mathcal{O}(\frac{1}{\Lambda^6})) \quad (2.24)$$

$$L_3 = L_{10} = -\frac{N_c}{96\pi^2} \left(1 + \frac{\pi^2}{10N_c} \left(\frac{GG}{\Lambda^4} + \mathcal{O}(\frac{1}{\Lambda^6})\right)\right) \quad (2.25)$$

Recall that $(\frac{GG}{\Lambda^4})$ is $\mathcal{O}(N_c)$ so the gluonic corrections are also $\mathcal{O}(N_c)$. One may ask whether any gluonic correction can contribute to the Wess-Zumino term. The answer is that it does not as can be understood after Witten's remark [10] that the single-valuedness of the action together with topological reasons make the coefficient of the Wess-Zumino term an integer.

If for the time being we neglect the leading gluonic corrections, we obtain the following set of values for the L_i

Constant	Theory	Experiment
L_1	0.79×10^{-3}	$(0.7 \pm 0.3) \times 10^{-3}$
L_2	1.58×10^{-3}	$(1.3 \pm 0.7) \times 10^{-3}$
L_3	-3.17×10^{-3}	$(-1.1 \pm 2.5) \times 10^{-3}$
L_9	6.33×10^{-3}	$(6.9 \pm 0.7) \times 10^{-3}$
L_{10}	-3.17×10^{-3}	$(-5.2 \pm 0.3) \times 10^{-3}$

The experimental values are taken, as is now customary, at the M_ρ scale. As is apparent in (2.21) and (2.25) the L_i we have obtained are independent of the ultraviolet regulator Λ , while to make the comparison with the phenomenological values we have taken a scale (the M_ρ in this case). The reason is, of course, that the chiral loops that are responsible for the running of the L_i are always $\mathcal{O}(1)$ (see section 5), while we are only retaining the $\mathcal{O}(N_c)$ contribution to the $\mathcal{O}(p^4)$ coupling constants. Therefore, for the comparison, any scale is really as good as any other. In fact, L_3 and $L_9 + L_{10}$ are scale independent.

As we said, the gluonic corrections are not included in the first column, so the values for the L_i merely reproduce those obtained previously in [8]. It is worth noting, however, that the regulators are different, so it is certainly reassuring to obtain the same values (as we said, in the next section the same values will be found again using a heat kernel regularization of the determinant). As announced in the introduction, the agreement between the two columns is quite good. The approximation provides not only the order of magnitude, but also the correct sign of the coefficients; even the values are right within a factor of 2 in the worst case.

Let us now turn to the leading term (2.22). Rotating back to Minkowski space we see that, to this order, if we want to reproduce the chiral lagrangian, $\frac{N_c \Lambda^2}{16\pi^2}$ is to be identified with $f_\pi^2/4$ (using the conventional normalization of [1]). This gives $\Lambda \simeq 320\text{MeV}$. If we interpret Λ as a cut-off à la Wilson; that is, a typical scale built in the theory such that only lower modes contribute to the physics of the problem and that we are not going to remove, we see that the value we get is indeed a very reasonable one for a low energy cut-off.

How is this picture changed by the gluonic corrections? To begin with, at the order

computed, L_1 , L_2 and L_9 do not get any corrections at all! We find this fact quite remarkable and probably it accounts for the rather good accuracy of the results. Both L_3 and L_{10} are negative, but their absolute values are smaller than the experimental values: the gluonic corrections shift them in the right direction making L_3 and L_{10} more negative. We can also predict relations between the L_i that could not really be seen without the gluonic corrections: $L_3 = L_{10}$ and $L_9 = 4L_2$, apart from the well known large N relation $L_2 = 2L_1$. It is interesting that these relations are also satisfied by the coefficients found in [11], where a QCD effective chiral lagrangian with vector and axial-vector fields has been considered.

The relation between f_π and the cut-off also gets modified by gluonic corrections

$$\frac{f_\pi^2}{4} = \frac{N_c \Lambda^2}{16\pi^2} \left(1 + \frac{\pi^2}{6N_c} \frac{\langle \frac{g}{\pi} GG \rangle}{\Lambda^4} + \mathcal{O}\left(\frac{1}{\Lambda^6}\right) \right) \quad (2.26)$$

The gluonic corrections are of course essential to get a regularization independent result. If we just ignore them and adopt the “wilsonian” point of view discussed above we cannot remove the cut-off. Doing so would lead us in the $\Lambda \rightarrow \infty$ limit to a theory of free bosons (after rescaling the π^a fields). However, $\langle \frac{g}{\pi} GG \rangle$ and the rest of the gluonic corrections have a Λ dependence as well (on dimensional grounds $\langle \frac{g}{\pi} GG \rangle = f(\alpha)\Lambda^4$), so, although we have not analyzed this point, one should be able to tune α so as to have the physical value of f_π in the limit when the cut-off is removed. Then everything else is fixed.

The $\mathcal{O}(\alpha')$ contribution to the L_i is independent of the value of Λ , but as we see from (2.25) the leading gluonic corrections are important in some cases. If we simply forget about next to leading gluonic corrections and use the experimental value of L_{10} we find for the ratio $\langle \frac{g}{\pi} GG \rangle/\Lambda^4$ a value of order 1. The gluonic corrections to L_3 , L_{10} are large, at the 50% level. For some coupling constants it is also possible to determine the contribution proportional to the cubic condensate $\langle g^3 GG \rangle$, or even higher dimensional ones, but we will postpone this result to section 4.

As we said before, we know that in QCD $\langle \frac{g}{\pi} GG \rangle$ and similar condensates have a genuinely non perturbative part. Even if we cannot compute this, we get some non trivial information in the form of relations between the $\mathcal{O}(p^4)$ coupling constants. Another possibility would be to borrow $\langle \frac{g}{\pi} GG \rangle$, etc. from some independent determination. However, one then faces the issue of regularization and scale dependence of the condensates. If however we just go ahead and use the values most commonly stated in QCD sum rules and

the value $\Lambda \simeq 320$ MeV that we have obtained from the lowest order (in α) calculation, we again obtain that the gluonic corrections to L_3 , L_{10} are about right to give quite good agreement with the experimental data. The argument, as it stands, is however no more (than) indicative. A proper job would require determination of the condensates within the same regularization method. In section 4 we will rederive some of the above results (and others) in a way which is much more reminiscent of the techniques used in sum rules and where we feel much more confident to carry over known results for the condensates derived from QCD duality arguments.

So far we have ignored the quark masses. Let us briefly discuss how we can incorporate them in the above formalism. We modify the operator Δ in (2.3) to incorporate scalar and pseudoscalar sources s and p , respectively

$$\Delta = \gamma_\mu (D_\mu + v_\mu + \gamma_5 a_\mu) + s + \gamma_5 p \quad (2.27)$$

The effective action Γ is now a function of v, a, s and p . The infinitesimal local chiral transformations under which the action in (2.3) is invariant are the same as written in (2.4) plus

$$\delta s = \{\lambda, p\}$$

$$\delta p = \{\lambda, s\} \quad (2.28)$$

The calculation proceeds in much the same way and eventually we get the following additional anomalous variation (to be added to (2.8))

$$\delta X_{\mathcal{M}} = \frac{N_c \Lambda^2}{2\pi^2} \int d^4 x \text{tr} \lambda \{s, p\} \quad (2.29)$$

This variation can be integrated to get the following addition to the effective action (modulo the usual ambiguities that are of no consequence for the chiral lagrangian, as discussed above)

$$\frac{N_c \Lambda^2}{4\pi^2} \int d^4 x \text{tr} s^2 \quad (2.30)$$

Neglecting terms that only depend on the external sources we get a new piece in the effective chiral lagrangian ($\mathcal{M} = \text{diag}(m_u, m_d, m_s)$)

$$-\frac{f_\pi^2}{4} \int d^4 x \text{tr} (\mathcal{M} U \mathcal{M} U + \mathcal{M} U^\dagger \mathcal{M} U^\dagger) \quad (2.31)$$

This is a contribution to L_8 . (See eqs. (4.16) and (4.17) for the conventional normalization of L_8 .) The numerical value for L_8 rather sensitive to the current quark masses (or,

alternatively, to $\langle \bar{q}q \rangle$). If we use $m_u = m_d = 10$ MeV we get $L_8 = -0.58 \times 10^{-3}$. In absolute value this is not too different from the experimental value $(0.9 \pm 0.3) \times 10^{-3}$, but it has the wrong sign. There are no contributions to L_8 or to the term linear in the current quark masses in the $\mathcal{O}(p^2)$ effective chiral lagrangian (i.e. we have no term proportional to $\mathcal{M} U + \mathcal{M} U^\dagger$). The approach to the calculation of the effective action, as it stands, can never give rise to terms which are linear in the current quark mass matrix $\mathcal{M}$. This can be easily seen on dimensional grounds, since a factor Λ^3 would be needed to make the dimensions match, but by construction Λ always turns up in powers of Λ^2 (see (2.7)). There is another, more physical, reason for this to happen and this is that nowhere have we assumed spontaneous chiral symmetry breaking! Of course we are never going to obtain such a breaking in a weak field expansion such as the one we are carrying out.

How can we solve this problem? One possibility is to repeat the above calculation with a new mass-like parameter that breaks the chiral symmetry (some sort of "effective" spectral asymmetry of the Dirac operator Δ). This has been advocated in [12a]. However, the approach we followed in this section gets then rather cumbersome so this is not the way we choose to take. Another approach to incorporate spontaneous symmetry breaking has been proposed in [12b] by arguing that in strong coupling QCD can be approximated by a Nambu-Jona-Lasinio type lagrangian. In the leading large N_c limit this model can be solved yielding chiral symmetry breaking. In the next section we will derive an effective chiral theory by perturbing around a chirally non-symmetric vacuum, but still keeping local chiral invariance. In this way, we believe that we will provide a much better description of the long distance properties of QCD than the one given in this section. The assumption that we are implicitly making is that the coefficients of the chiral lagrangian are, to a large extent, independent of the actual mechanism responsible for chiral symmetry breaking. Like [12a] or [12b] our procedure calls for the introduction of a new scale related to $\langle \bar{q}q \rangle$. This new scale will also play an important role in setting the size of the higher dimensional coefficients. In fact, it will allow us to obtain non-zero results for higher dimensional operators even when we remove the ultraviolet cut-off. Unlike [12a] or [12b] our approach allows an easy incorporation of gluonic corrections.

Finally, note that in an effective theory of strong interactions without spontaneous symmetry breaking such as the one derived in this section, the leading "mass" term for the pion must come from the L_8 piece. However, as we have already mentioned, eq. (2.31) has the wrong sign. A negative value of L_8 means in fact a tachyonic mass for the pion.

Eq. (2.31) implies that for euclidean momenta squared, p^2 , smaller than $4m^2$ (assuming all quark masses equal) the effective action is unstable. That is precisely what we expect. Since we have not included gluonic corrections in L_R yet, we are dealing with a theory of free quarks. For minkowskian momenta squared greater than $4m^2$ a cut develops in the Green functions and the above instability in euclidean space is just a reflection of this fact.

3. The massive terms

We would like to compute the effective couplings of the chiral lagrangian when the current quark masses are taken into account. As noted before, the global $SU(3)_L \otimes SU(3)_R$ is explicitly broken by the mass terms of the u , d and s quarks. These mass terms can thus be regarded as a perturbation of the spontaneously broken chiral symmetry limit. The quarks being spin 1/2 particles, their mass term $\bar{q}Mq$ is linear in the mass flavour matrix $M = \text{diag}(m_u, m_d, m_s)$, and we therefore expect the lowest correction in the chiral lagrangian to be of the form $\text{tr}(M\mathcal{U}\mathcal{U}^\dagger + M\mathcal{U}^\dagger)$, also linear in M , with $\mathcal{U}$ defined in (2.15).

We shall add to the QCD lagrangian the following $SU(3)_L \otimes SU(3)_R$ invariant term

$$M_Q(\bar{q}_R \mathcal{U} q_L + \bar{q}_L \mathcal{U}^\dagger q_R) \quad (3.1)$$

This serves us to introduce the $\mathcal{U}$ field in a way non invariant under the transformation $\mathcal{U} \rightarrow -\mathcal{U}$, and a parameter M_Q with dimensions of mass that will regulate the infrared behaviour of the effective action. With the inclusion of this term the QCD Dirac operator in the presence of external vector and axial-vector sources as well as quark masses reads as follows (in euclidean space)

$$L = \bar{q}_L \gamma_\mu (D_\mu - i L_\mu) q_L + \bar{q}_R \gamma_\mu (D_\mu - i R_\mu) q_R - \bar{q}_R M q_L - \bar{q}_L M^\dagger q_R - M_Q(\bar{q}_R \mathcal{U} q_L + \bar{q}_L \mathcal{U}^\dagger q_R) \quad (3.2)$$

The sign of M_Q is a priori arbitrary but we shall see that for consistency M_Q must have the same sign as M . It is convenient to introduce new quark fields, which we shall call the "rotated basis" and are defined as follows

$$\begin{aligned} Q_L &= \xi q_L, & \bar{Q}_L &= \bar{q}_L \xi^\dagger \\ Q_R &= \xi^\dagger q_R, & \bar{Q}_R &= \bar{q}_R \xi \end{aligned} \quad (3.3)$$

with ξ chosen such that $\xi^2 = \mathcal{U}$. The lagrangian (3.2) then becomes

$$L = \bar{Q} \mathcal{D} Q \quad (3.4)$$

where

$$D = \gamma_\mu (D_\mu + i L_\mu - \frac{1}{2} \gamma_5 \xi_\mu) - \frac{1}{2} (\gamma_5 - \gamma_5 \Delta) - M_Q \quad (3.5)$$

and

$$\begin{aligned} \Gamma_\mu &= \frac{1}{2} [\xi^\dagger (\partial_\mu - i R_\mu) \xi + \xi (\partial_\mu - i L_\mu) \xi^\dagger], & \Gamma_\mu^\dagger &= -\Gamma_\mu \\ \xi_\mu &= i [\xi^\dagger (\partial_\mu - i R_\mu) \xi - \xi (\partial_\mu - i L_\mu) \xi^\dagger], & \xi_\mu^\dagger &= \xi_\mu \\ \Sigma &= \xi^\dagger M \xi^\dagger + \xi M^\dagger \xi, & \Sigma^\dagger &= \Sigma \\ \Delta &= \xi^\dagger M \xi^\dagger - \xi M^\dagger \xi, & \Delta^\dagger &= -\Delta \end{aligned} \quad (3.6)$$

The chiral rotation (3.3) induces a non trivial jacobian in the path integral. However, it only depends on the external sources L_μ and R_μ and it is irrelevant for the obtention of the chiral lagrangian. The M_Q term in the "rotated basis" is an ordinary mass term and M_Q can, perhaps, be interpreted as a "constituent" quark mass. Under $SU(3)_L \otimes SU(3)_R$ transformations (g_L, g_R)

$$\xi(x) \rightarrow g_R \xi(x) h^\dagger(x) = h(x) \xi^\dagger(x) g_L^\dagger \quad (3.7)$$

which defines the non-linear realization $h(x)$ of $SU(3)_L \otimes SU(3)_R$. Under chiral transformations the quark fields of the "rotated basis" $Q_{L,R}$ transform like "constituent chiral quarks" [5,13], i.e.

$$Q_L \rightarrow h(x) Q_L, \quad Q_R \rightarrow h(x) Q_R \quad (3.8)$$

In fact, the lagrangian in eqs. (3.4) and (3.5), in the absence of external sources and explicit quark masses (i.e. $M \rightarrow 0$) is rather similar to the constituent chiral quark model of Georgi and Manohar [13].

In the presence of the term in eq. (3.1) the operator $\bar{q}q$ acquires a vacuum expectation value. This is the whole point of this extra term, as we will never get the terms in the effective action which are sensitive to chiral symmetry breaking via a weak field expansion (such as the one performed in section 2). We are interested in the "physical" vacuum expectation value that we define to be

$$\langle \bar{q}q \rangle = \langle \bar{q}q \rangle_{\sigma_\mu = 0} - \langle \bar{q}q \rangle_{\sigma_\mu = 0} \quad (3.9)$$

where the subtraction term means the spurious vacuum condensate generated when the gluon fields are switched off. The role of M_Q is thus not really to provide an explicit value of $\langle \bar{q}q \rangle$ but to trigger the breaking of chiral symmetry.

For values of M_Q sufficiently large $\langle \bar{q}q \rangle$ can be obtained as an expansion in inverse powers of M_Q using operator product expansion techniques. The calculation of the first terms have been made by Novikov, Shifman, Vainshtein and Zakharov [1], with the result

$$\langle \bar{q}q \rangle = -\frac{1}{12} \frac{\alpha \langle GG \rangle}{\pi M_Q} - \frac{1}{360} \frac{\alpha}{\pi} g f_{abc} \frac{\langle G^a G^b G^c \rangle}{M_Q^3} + \dots \quad (3.10)$$

We shall consider this equation as the definition of M_Q . The term in eq. (3.1) is then an effective way to generate the order parameter due to spontaneous chiral symmetry breaking. Clearly, from eq. (3.10), M_Q must be positive, but the precise value is harder to pin down. The value of $\frac{\alpha}{\pi} \langle GG \rangle$ extracted from phenomenology is rather uncertain. We favor the latest determination from $e^+e^- \rightarrow$ hadrons data made in [15], i.e.

$$\frac{\alpha}{\pi} \langle GG \rangle = [(410 \pm 80) \text{ MeV}]^4 \quad (3.11)$$

but smaller values have also been claimed before [9]. The error in the value of $\langle \bar{q}q \rangle$ extracted from QCD sum rules is smaller. The renormalization scale invariant determination of $\langle \bar{q}q \rangle$ found in ref. [16] gives

$$\langle \bar{q}q \rangle = -[(194 \pm 8) \text{ MeV}]^3 \quad (3.12)$$

With the central values of (3.11) and (3.12) and from the first term only of (3.10) one gets $M_Q \simeq 320$ MeV. Remarkably this value coincides with the "low energy" cut-off Λ introduced in section 2. The corrections from the $\langle g^3 GGG \rangle$ piece in (3.10) are estimated to be at the few percent level [9, 14].

4. Calculation of the Effective Action

We are now in a position to compute the effective couplings of the chiral lagrangian up to $\mathcal{O}(p^4)$, with explicit mass breaking terms included. For this purpose we shall use the heat kernel technique [17, 18]. The relevant Dirac operator has been defined in eqs. (3.5) and (3.6). Notice that

$$\mathcal{D}^\dagger = -\gamma_\mu (D_\mu + \Gamma_\mu + \frac{i}{2} \gamma_5 \xi_\mu) - \frac{1}{2} (\Sigma + \gamma_5 \Delta) - M_Q \quad (4.1)$$

We introduce a covariant derivative ∇_μ which on the quarks Q acts as

$$\nabla_\mu Q = (D_\mu + \Gamma_\mu - \frac{i}{2} \gamma_5 \xi_\mu) Q \quad (4.2)$$

The commutator of two such derivatives defines the tensor $\mathcal{R}_{\mu\nu}$

$$[\nabla_\mu, \nabla_\nu] = \mathcal{R}_{\mu\nu} = iG_{\mu\nu} + \Gamma_{\mu\nu} - \frac{1}{4} [\xi_\mu, \xi_\nu] - \frac{i}{2} \gamma_5 \xi_{\mu\nu} \quad (4.3)$$

with $G_{\mu\nu} = \partial_\mu G_\nu - \partial_\nu G_\mu + i[G_\mu, G_\nu]$, $G_\mu = i\frac{\Delta}{2} G_\mu^a$, $\Gamma_{\mu\nu} = \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu + [\Gamma_\mu, \Gamma_\nu]$, $\xi_{\mu\nu} = d_\mu \xi_\nu - d_\nu \xi_\mu + [\Gamma_\mu, \xi_\nu]$; i.e. d_μ is also a covariant derivative with respect to the Γ_μ connection. We then have

$$\mathcal{D}^\dagger \mathcal{D} - M_Q^2 = -\nabla_\mu \nabla_\mu + E \quad (4.4)$$

with

$$\begin{aligned} E = & iM_Q \gamma_\mu \gamma_5 \xi_\mu - \frac{i}{2} \sigma_{\mu\nu} \mathcal{R}_{\mu\nu} \\ & + \frac{i}{4} \gamma_\mu \{ \gamma_5 \xi_\mu, \Sigma - \gamma_5 \Delta \} + \frac{1}{2} \gamma_\mu d_\mu (\Sigma - \gamma_5 \Delta) \\ & + M_Q \Sigma + W + \gamma_5 W_5 \end{aligned} \quad (4.5)$$

where $\sigma_{\mu\nu} = -\frac{i}{2} [\gamma_\mu, \gamma_\nu]$ and

$$\begin{aligned} W &= \frac{1}{2} (\xi^\dagger M M^\dagger \xi + \xi M^\dagger M \xi^\dagger) \\ W_5 &= \frac{1}{2} \gamma_5 (\xi^\dagger M M^\dagger \xi - \xi M^\dagger M \xi^\dagger) \end{aligned} \quad (4.6)$$

In the chiral limit where $M \rightarrow 0$ only the terms in the first line in eq. (4.5) will appear.

We are interested in the real part of the effective action, therefore we only keep the modulus of the determinant

$$\Gamma = \frac{1}{2} \log \det \mathcal{D}^\dagger \mathcal{D} \quad (4.7)$$

and since $\mathcal{P}^\dagger \mathcal{P}$ is a second order elliptic operator, $\det \mathcal{P}^\dagger \mathcal{P}$ can be defined using the ζ -function regularization method [18]

$$\det \mathcal{P}^\dagger \mathcal{P} = \exp \left(-\frac{d}{ds} \zeta_{\mathcal{P}^\dagger \mathcal{P}}(s) \Big|_{s=0} \right) \quad (4.8)$$

where

$$\zeta_{\mathcal{P}^\dagger \mathcal{P}}(s) = \frac{\mu^2}{\Gamma(s)} \int_0^\infty d\tau \tau^{s-1} \text{Tr} \exp(-\tau \mathcal{P}^\dagger \mathcal{P}) \quad (4.9)$$

Tr stands for trace over Dirac, colour, flavour and space indices; μ is an arbitrary mass scale introduced so that $\zeta(s)$ is dimensionless, and will eventually serve to regulate ultraviolet divergences. Inserting (4.1) in (4.9) we see that $\exp(-\tau M_Q^2)$ can be factored out and the infrared behaviour (large τ) is then regulated by this factor. The integral gets dominant contributions from $\tau \leq 1/M_Q^2$.

We shall now recall the Seeley-DeWitt expansion of the exponential of an elliptic operator in order to evaluate (4.9) perturbatively

$$\langle x | \exp(-\tau(E - \nabla_\mu \nabla_\mu)) | x \rangle = \frac{1}{16\pi^2 \tau^2} \sum_{n=0}^\infty H_n(x, x) \tau^n \quad (4.10)$$

which is exact when $\tau \rightarrow 0^+$. The first few coefficients $H_n(x, x)$ are known [17] (see also [18]) It is not difficult to reproduce them up to $n = 3$. We find

$$\begin{aligned} H_0(x, x) &= 1 \\ H_1(x, x) &= -E \\ H_2(x, x) &= \frac{1}{2} E^2 - \frac{1}{16} \nabla_\mu \nabla_\mu E + \frac{1}{12} \mathcal{R}_{\mu\nu} \mathcal{R}_{\mu\nu} \\ H_3(x, x) &= -\frac{1}{6} E^3 + \frac{1}{12} [E \nabla_\mu \nabla_\mu E + (\nabla_\mu \nabla_\mu E) E + (\nabla_\mu E)(\nabla_\mu E)] \\ &\quad - \frac{1}{60} \nabla_\mu \nabla_\mu \nabla_\nu \nabla_\nu E \\ &\quad - \frac{1}{30} (E \mathcal{R}_{\mu\nu} \mathcal{R}_{\mu\nu} + \frac{1}{2} \mathcal{R}_{\mu\nu} E \mathcal{R}_{\mu\nu} + \mathcal{R}_{\mu\nu} \mathcal{R}_{\mu\nu} E) \\ &\quad + \frac{1}{60} [(\nabla_\mu E) \nabla_\mu \mathcal{R}_{\nu\rho} - (\nabla_\nu \mathcal{R}_{\nu\rho}) \nabla_\mu E] \\ &\quad + \frac{1}{15} [\frac{1}{3} (\nabla_\mu \mathcal{R}_{\nu\rho}) \nabla_\mu \mathcal{R}_{\nu\rho} + \frac{1}{12} (\nabla_\mu \mathcal{R}_{\nu\rho}) \nabla_\rho \mathcal{R}_{\nu\rho} \\ &\quad + \frac{1}{4} (\nabla_\mu \nabla_\mu \mathcal{R}_{\nu\rho}) \mathcal{R}_{\nu\rho} + \frac{1}{4} \mathcal{R}_{\nu\rho} (\nabla_\mu \nabla_\mu \mathcal{R}_{\nu\rho}) \\ &\quad - \frac{1}{2} \mathcal{R}_{\mu\nu} \mathcal{R}_{\nu\rho} \mathcal{R}_{\rho\mu}] \\ H_4(x, x) &= \frac{1}{2} E^4 + \dots \end{aligned} \quad (4.11)$$

The effective action (4.7) in terms of these $H_n(x, x)$ coefficients reads then

$$\Gamma = -\frac{1}{32\pi^2} \int d^4x \text{Tr} \left[M_Q^4 \left(\frac{3}{4} + \frac{1}{2} \ln \frac{\mu^2}{M_Q^2} \right) H_0(x, x) - M_Q^2 \left(1 + \ln \frac{\mu^2}{M_Q^2} \right) H_1(x, x) \right. \\ \left. + \ln \left(\frac{\mu^2}{M_Q^2} \right) H_2(x, x) + \sum_{n \geq 2} \frac{(n-3)!}{(M_Q^2)^{n-2}} H_n(x, x) \right] \quad (4.12)$$

where Tr stands now for trace over Dirac, colour and flavour indices only. The result is an expansion in inverse powers of M_Q . Notice, however, that the H_n coefficients do not correspond to a fixed power in the chiral expansion. This is due to the fact that the terms which compose E in eq. (4.5) have different chiral power behaviour. For instance, $iM_Q \gamma_\mu \gamma_5 \xi_\mu$ is $\mathcal{O}(p)$; $-\frac{1}{2} \sigma_{\mu\nu} \mathcal{R}_{\mu\nu}$ is a mixture of $\mathcal{O}(1)$ and $\mathcal{O}(p^2)$; $M_Q \Sigma$ is $\mathcal{O}(m)$, i.e. $\mathcal{O}(p^2)$. On the other hand, $\frac{1}{4} \gamma_\mu (\gamma_5 \xi_\mu, \Sigma - \gamma_5 \Delta)$ and $\frac{1}{2} \gamma_\mu d_\mu (\Sigma - \gamma_5 \Delta)$ are $\mathcal{O}(mp)$, that is $\mathcal{O}(p^3)$. Finally $W + \gamma_5 W_5$ is $\mathcal{O}(m^2)$, i.e. $\mathcal{O}(p^4)$. Also, since H_k always has an E^k term, it is clear that $H_k(x, x)$ contains M_Q powers up to k ; this mismatch between the order of the Seeley coefficients and the order of powers of M_Q makes the computation of the explicit expansion in M_Q^{-1} rather cumbersome since a given power $0 \leq p \leq 2$ of M_Q^{-1} can get contributions from H_3 up to H_{p+4} , and from H_{p+1} to H_{p+4} if $p \geq 3$.

Inserting the Seeley coefficients (4.11) in (4.12) with the expression for $\mathcal{R}_{\mu\nu}$ and E given in (4.3) and (4.5), we get after some tedious algebra and to zeroth order in the gluon field strength tensor

$$\begin{aligned} \Gamma = \frac{N_c}{16\pi^2} \int d^4x \text{Tr} & \left[-\frac{1}{4} M_Q^4 \left(\frac{3}{2} + \ln \frac{\mu^2}{M_Q^2} \right) - 2M_Q^2 \left(1 + \ln \frac{\mu^2}{M_Q^2} \right) (M_Q \Sigma + W) \right. \\ & - M_Q^2 \ln \frac{\mu^2}{M_Q^2} \xi_\mu \xi_\mu \\ & - M_Q \ln \frac{\mu^2}{M_Q^2} (i \xi_\mu d_\mu \Delta + \xi_\mu \xi_\mu \Sigma + M_Q \Sigma^2) \\ & + \frac{1}{12} \ln \frac{\mu^2}{M_Q^2} (f_{\mu\nu}^{(+)} f_{\mu\nu}^{(+)} + f_{\mu\nu}^{(-)} f_{\mu\nu}^{(-)}) \\ & + \frac{1}{2} \xi_\mu \xi_\nu f_{\mu\nu}^{(+)} \\ & + \frac{1}{6} (d_\mu \xi_\nu)^2 + \frac{1}{12} (2 \xi_\mu \xi_\nu \xi_\mu \xi_\nu - \xi_\mu \xi_\mu \xi_\nu \xi_\nu) \\ & + M_Q \xi_\mu \xi_\mu \Sigma \\ & \left. + \mathcal{O}(p^6) \right] \end{aligned} \quad (4.13)$$

where $f_{\mu\nu}^{(\pm)}$ are defined by

$$f_{\mu\nu}^{(\pm)} = \xi F_{L,\mu\nu} \xi^\dagger \pm \xi^\dagger F_{R,\mu\nu} \xi \quad (4.14)$$

and F_L , F_R are the external L , R field strength tensors defined in (B.6). Here tr stands for a trace over the flavour indices only.

In the course of the calculation there are also terms containing both the tensor $\epsilon_{\mu\nu\alpha\beta}$ and an odd number of ξ_μ 's in an even parity combination. These terms turn out to be total derivatives and have thus been neglected. For the two such terms that show up we have

$$\begin{aligned} \epsilon_{\alpha\beta\mu\nu} \text{tr} \xi_\alpha \xi_\beta d_\mu \xi_\nu &= \frac{1}{3} \text{tr} d_\alpha (\epsilon_{\alpha\beta\mu\nu} \xi_\beta \xi_\mu \xi_\nu) \\ \epsilon_{\alpha\beta\mu\nu} \text{tr} \xi_\alpha \xi_\beta \xi_\mu &= \text{tr} d_\alpha (\epsilon_{\alpha\beta\mu\nu} \xi_\beta \xi_\mu \xi_\nu) \end{aligned} \quad (4.15)$$

(The last relation follows from the Bianchi identity for the d_μ covariant derivative, which can be cast in the form $\epsilon_{\alpha\beta\mu\nu} [d_\alpha, F_{\mu\nu}] = 0$.)

The first term in the first line of eq. (4.13), proportional to M_Q^4 is a remnant of the regularized UV quartic divergence. It is a constant term which we will subtract to define the physical effective action. The second term in the first line stems from the regularized UV quadratic divergence, which we shall discuss later. The terms in the second, third and fourth line are modulated by the same regularized UV logarithmic divergence. The other terms are all UV finite and, as we shall see, regularization independent. Higher order terms in inverse powers of M_Q , with the gluonic strength tensor $G_{\mu\nu}$ absent, lead to contributions in the chiral effective lagrangian which are $\mathcal{O}(p^6)$ or higher. However, in the presence of $G_{\mu\nu} \neq 0$, there appear new contributions to the various coefficients of the terms in eq. (4.13) which we shall discuss later.

Using now the formulae collected in Appendix B as well as the constraints from the lowest order equations of motion in the next-to-leading terms of the chiral lagrangian, we finally get in the Minkowski space the functional structure of the Gasser and Leutwyler

lagrangian[1b]

$$\begin{aligned} \Gamma = \int d^4x & \left(\frac{f_\pi^2}{4} \text{tr} (D_\mu U^\dagger D^\mu U + (X^\dagger U + U^\dagger X)) \right. \\ & + (L_1 - \frac{1}{2} L_2) \text{tr} L_\mu L^\mu \text{tr} L_\nu L^\nu \\ & + L_2 \text{tr} \left(\frac{1}{2} [L_\mu, L_\nu] [L^\mu, L^\nu] + 3 L_\mu L^\mu L_\nu L^\nu \right) \\ & + L_3 \text{tr} L_\mu L^\mu L_\nu L^\nu + L_4 \text{tr} D_\mu U^\dagger D^\mu U \text{tr} (X^\dagger U + U^\dagger X) \\ & + L_5 \text{tr} D_\mu U^\dagger D^\mu U (X^\dagger U + U^\dagger X) \\ & + L_6 (\text{tr} (X^\dagger U + U^\dagger X))^2 + L_7 (\text{tr} (X^\dagger U - U^\dagger X))^2 \\ & + L_8 \text{tr} (U X^\dagger U X^\dagger + U^\dagger X U^\dagger X) \\ & - i L_9 \text{tr} (F_R^{\mu\nu} D_\mu U D_\nu U^\dagger + F_L^{\mu\nu} D_\mu U^\dagger D_\nu U) \\ & + L_{10} \text{tr} U^\dagger F_R^{\mu\nu} U F_{L,\mu\nu} \\ & + H_1 \text{tr} (F_R^{\mu\nu} F_{R,\mu\nu} + F_L^{\mu\nu} F_{L,\mu\nu}) \\ & \left. + H_2 \text{tr} X^\dagger X + \dots \right) \end{aligned} \quad (4.16)$$

where $L_\mu = i U^\dagger D_\mu U$. The matrix X is such that, with $v = \frac{1}{2} |\langle \bar{q} q \rangle|$,

$$\frac{1}{4} f_\pi^2 X = v M \quad (4.17)$$

The identification of the various constants f_π , v , L_i and H_1 , H_2 with the results of our calculation leads in the absence of gluonic corrections to the following expressions

$$L_1 - \frac{1}{2} L_2 = L_4 = L_6 = L_7 = 0 \quad (4.18)$$

and with x and y parameters defined by

$$x = M_Q f_\pi^2 / 4v \quad y = \ln \frac{\mu^2}{M_Q^2} \quad (4.19)$$

we find

$$f_\pi^2 = \frac{N_c M_Q^2 y}{16\pi^2}$$

$$v = -\frac{N_c M_Q^3 (1+y)}{8\pi^2}$$

$$l_2 = \frac{N_c}{16\pi^2} \frac{1}{12}$$

$$l_3 = -\frac{N_c}{16\pi^2} \frac{1}{6}$$

$$l_5 = \frac{N_c}{16\pi^2} x(y-1)$$

$$l_8 = \frac{N_c}{16\pi^2} \left[\left(\frac{1}{2} x - x^2 \right) y - \frac{1}{24} \right]$$

$$l_9 = \frac{N_c}{16\pi^2} \frac{1}{3}$$

$$l_{10} = -\frac{N_c}{16\pi^2} \frac{1}{6}$$

$$H_1 = -\frac{N_c}{16\pi^2} \frac{1}{6} \left(y - \frac{1}{2} \right)$$

$$H_2 = -\frac{N_c}{16\pi^2} \left[2x^2(1+y) + (x+2x^2)y - \frac{1}{12} \right]$$

Several remarks are in order:

i) In the chiral limit, only the $\mathcal{O}(p^4)$ couplings l_1, l_2, l_3, l_8 and l_{10} are of concern. We find for these constants the same results as the ones obtained in the first part. Given that the derivation in section 2 did not include the effects of chiral symmetry breaking induced by M_Q we conclude that these coefficients are insensitive to the realization of chiral symmetry at long distances in QCD, at least at $\mathcal{O}(\alpha^0)$ in the weak field expansion around the respective vacua.

ii) The fact that we find $l_4 = l_6 = 0$ is not surprising, since these constants can only get contributions from non-leading terms in the $\frac{1}{N_c}$ expansion and in our calculation we have only retained leading terms. The constant l_7 deserves special attention. We shall come back to it later.

iii) The constants l_5 and l_9 are sensitive to the regularization of the logarithmic UV divergence and so is the constant H_1 . The constant H_2 is sensitive to the regularization of both the quadratic as well as the logarithmic UV divergences. One can, however, find

combinations which are insensitive to regularization ambiguities. They are

$$\frac{1}{4} \frac{f_\pi^2}{M_Q^2} - \frac{1}{x} l_5 = \frac{N_c}{16\pi^2} (1 + \dots) \quad (4.21)$$

$$\frac{1}{4} \frac{f_\pi^2}{M_Q^2} x(1-2x) - 2l_8 = \frac{N_c}{16\pi^2} \frac{1}{12} (1 + \dots) \quad (4.22)$$

and those involving the constants H_1 and H_2

$$\frac{1}{4} \frac{f_\pi^2}{M_Q^2} + 6H_1 = \frac{N_c}{16\pi^2} \frac{1}{2} (1 + \dots) \quad (4.23)$$

$$-v \frac{x^2}{M_Q^3} + \frac{1}{4} \frac{f_\pi^2}{M_Q^2} x(1+2x) + H_2 = \frac{N_c}{16\pi^2} \frac{1}{12} (1 + \dots) \quad (4.24)$$

The dots in the r.h.s. of these equations indicate the effects of uncalculated gluonic terms. With neglect of these gluonic terms, and using $M_Q = 320$ MeV, we find

$$\begin{aligned} l_5 &= 0.43 \times 10^{-3} & (1.4 \pm 0.5) \times 10^{-3} \\ l_8 &= 0.46 \times 10^{-3} & (0.9 \pm 0.3) \times 10^{-3} \\ H_1 &= -2.0 \times 10^{-3} \\ H_2 &= 0.04 \times 10^{-3} \end{aligned} \quad (4.25)$$

where we have indicated in brackets the values extracted from the experiment by Gasser and Leutwyler [1] (at the M_ρ scale). Given the uncertainties from uncalculated gluonic terms, these results are rather satisfactory. The values for H_1 and H_2 do not really mean much, as these coefficients are scheme dependent.

iv) We wish to comment on the value for v which has emerged from the calculations of our regularized effective action. The result in eq. (4.20) is precisely the piece corresponding to the term $\langle \bar{q}q \rangle_{\mu=0}$ in eq. (3.9), i.e. the term that has to be subtracted.

v) It is instructive to compute the effective action induced by the first gluonic invariants which appear in $H_2(x, x)$ and $H_3(x, x)$

$$\Gamma = \frac{1}{16\pi^2} \int d^4x \operatorname{tr} \left[-\frac{2x^2}{3} \left(\frac{\alpha}{\pi} G_{\mu\nu}^a G_{\mu\nu}^a \right) \left(\ln \frac{\mu^2}{M_Q^2} I - \frac{\sum_i}{M_Q} \right) \right] \quad (4.26)$$

The first term corresponds to a renormalization of the gluonic Yang-Mills piece in the QCD lagrangian and we recognize indeed the coupling constant renormalization due to

the quark loop (n_f = number of flavours = $\text{tr} t$). The second term in (4.26) leads to an effective action

$$\Gamma = \frac{1}{24\pi} \frac{\langle GG \rangle}{M_Q} \int d^4x \text{tr}(\mathcal{M}^\dagger U + U^\dagger \mathcal{M}) \quad (4.27)$$

from which we read the first gluonic contribution to v

$$v|_G = \frac{1}{24\pi} \frac{\langle GG \rangle}{M_Q} \quad (4.28)$$

In agreement with the calculation of ref. [14].

vii) We have obtained values for the L_i which do not contain any $\log \mu^2$. In other words, at the leading order in the large N_c expansion they are scale independent. To recover the $\log \mu^2$ dependence one should include higher terms in the cumulant expansion (2.13). This brings about divergent gluon integrals, at order α^2 and at $\mathcal{O}(1)$ in the $1/N_c$ expansion, which should reproduce the $\log \mu^2$ dependence induced by chiral loops. This is quite interesting since it provides a nice and unambiguous way of matching short and long distances in QCD; in other words, it tells us which is the meaning of the chiral logs in relation to the ultraviolet divergences in the underlying theory.

viii) Of course the L_i 's have also gluonic corrections in this scheme. Using the formulae in Appendix A it is quite trivial to pass from the regulator Λ to M_Q for the constants associated to operators that do not vanish in the chiral limit, i.e. for L_1, L_2, L_3, L_9 and L_{10} . We find

$$8L_1 = 4L_2 = L_9 = \frac{N_c}{48\pi^2} \left(1 + \mathcal{O}\left(\frac{1}{M_Q^6}\right)\right) \quad (4.29)$$

$$L_3 = L_{10} = -\frac{N_c}{96\pi^2} \left(1 + \frac{\pi^2}{5N_c} \frac{\langle GG \rangle}{M_Q^4} + \mathcal{O}\left(\frac{1}{M_Q^6}\right)\right) \quad (4.30)$$

Numerically the coefficients for the gluonic corrections we obtain in both schemes are different, but this is not surprising because the expectation values of the local operators are also regularization dependent. Furthermore the fact that we get the same values for the L_i at leading order in α does not necessarily imply that they are numerically equal in a theory with or without chiral symmetry breaking.

As we mentioned in section 2, for some coefficients it is possible to go beyond the $(\frac{\alpha}{\pi} GG)$ condensate. In particular, from the results of [19] one can easily obtain the $(g^3 GGG)/M_Q^6$ contributions to f_π (and also to L_{10} and H_1). This allows us to have

an idea of the size of the terms we are neglecting, even though the precise value of the triple gluon condensate is not known. We find that, for example, (2.26) gets modified to

$$\frac{f_\pi^2}{4} = \frac{N_c M_Q^2}{16\pi^2} \left(y + \frac{\pi^2}{6N_c} \frac{\langle \frac{\alpha}{\pi} GG \rangle}{M_Q^4} + \frac{1}{360N_c} \frac{\langle g^3 GGG \rangle}{M_Q^6} + \dots \right) \quad (4.31)$$

While the first gluonic correction is at the 50% level, the new term contributes only around 5% (if we believe the dilute instanton gas estimate of the triple condensate).

viii) Finally we would like to briefly discuss L_7 . As we have seen, L_7 does not have a contribution at $\mathcal{O}(N_c)$. At $\mathcal{O}(1)$ it gets a contribution from (among other things) η' interchange. This differs from the $\mathcal{O}(N_c^2)$ counting which is found in ref. [1b]. Our claim is based on the following argument. After computing the fermion determinant we get a piece of the form

$$\frac{\alpha}{4\pi} \frac{k}{M_Q^2} \text{tr}(\chi U^\dagger - \chi^\dagger U) G_{\mu\nu} G_{\mu\nu}^* \quad (4.32)$$

where k is a constant and $G_{\mu\nu}^*$ is the gluonic dual field strength. The above term does not originate from the modulus of the determinant (eq. (4.7)), but from its phase, as the Wess-Zumino term does. In fact to determine k one basically has to compute the standard $U(1)$ axial anomaly graph.

From (4.32) and the interchange of GG^* we immediately get a contribution to L_7 which is proportional to

$$\left(\frac{\alpha}{4\pi}\right)^2 \frac{1}{M_Q^4} \langle GG^*(p) GG^*(-p) \rangle|_{p=0} \quad (4.33)$$

The correlator in (4.33) plays a crucial role in the resolution of the $U_A(1)$ problem in QCD. From [20] (see also [21]) we know that it picks a contribution from the η' which is $\mathcal{O}(N_c^2)$. However, the large N_c counting rules require the redefinition $\alpha \rightarrow \frac{\alpha}{N_c}$, so the final value for L_7 is indeed $\mathcal{O}(1)$.

5. Comments on Chiral Power Counting

In chiral perturbation theory calculations it is customary to estimate the order of magnitude of uncalculated higher order terms by normalizing the extra powers of derivatives and/or quark masses. At a fixed constant Λ_{CSB} , usually Λ_{CSB} is taken to be

$$\Lambda_{CSB}^2 \simeq 16\pi^2 f_\pi^2 = (1.2\text{GeV})^2 \quad (5.1)$$

The rationale behind this “naïve” chiral power counting is based on the fact that chiral loops indeed provide such a normalization factor (up to logarithms). However, as we already mentioned in the introduction, the local counterterm structure induced by chiral loops of lower order terms is a particular subset of all the possible invariants which can appear in the QCD effective action. For example, in the $\mathcal{O}(p^1)$ terms, this is reflected by the fact that the combination of coupling constants

$$\begin{aligned} L_4 - \frac{1}{2}L_2, \quad L_3, \quad L_9 + L_{10}, \\ L_4 + \frac{1}{3}L_5, \quad L_7, \quad L_8 - \frac{11}{15}L_9, \\ \text{and} \quad H_2 + \frac{5}{3}H_1 \end{aligned}$$

are independent of the scale used to renormalize the effective action generated from the lowest order terms at the order of one chiral loop. Therefore a reliable chiral power counting estimate for coupling constants of local terms in the effective action should be made without reference to chiral loop arguments. As it stands, the approach to the QCD effective action developed in the preceding sections provides a framework for such an “improved” chiral power counting which is, at least, compatible with the large N_c counting rules[22].

To illustrate our claim let us first discuss the $\mathcal{O}(p^1)$ terms. The constants

$$L_1, \quad L_2, \quad L_3, \quad L_4, \quad L_5, \quad L_8, \quad L_9 \quad \text{and} \quad L_{10}$$

appear to leading order in the $1/N_c$ expansion, and they all have the common factor

$$\frac{1}{2} \frac{N_c}{16\pi^2}$$

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(The overall factor $1/2$ appears in the definition of eq. (4.7).) To identify “extra” suppression factors we look for the index n of the lowest Seeley coefficient $H_n(x, x)$ which can give contributions to these constants. In the heat kernel approach, $n = 2$ and 3 for L_5 and L_8 ; $n = 3$ and 4 for the others). We then conclude that an average estimate for these L_i should be

$$|L_i| \simeq \frac{1}{2} \frac{N_c}{16\pi^2} \frac{1}{3!} \quad (5.2)$$

Of course a closer look will tell us that in fact the $H_2(x, x)$ contribution to L_3 and L_8 is UV regularization dependent and the estimate above should be restricted to $i = 1, 2, 3, 9$ and 10 . When compared to the results in eq. (4.20) we see that the worse estimate is for L_{10} which is off by a factor of four.

In terms of a scale Λ_x which normalizes the $\mathcal{O}(p^4)$ contributions compared to the leading $\mathcal{O}(p^2)$ terms, the estimate above corresponds to

$$\Lambda_x^2 \simeq 16\pi^2 f_\pi^2 \times \frac{1}{N_c} \times 3 \quad (5.3)$$

clearly showing to $\mathcal{O}(p^4)$ that although the chiral symmetry breaking scale Λ_{CSB} which appears in chiral loops is $\mathcal{O}(\sqrt{N_c})$, the scale that governs the chiral expansion in the effective action is $\mathcal{O}(1)$ in the $1/N_c$ expansion. (Let us recall that f_π is $\mathcal{O}(\sqrt{N_c})$.) Of course for $N_c = 3$ at $\mathcal{O}(p^4)$ the distinction is immaterial, but it will not be so at higher orders in the chiral expansion.

For terms in the effective action of $\mathcal{O}(p^{1+2p})$ the “improved” chiral power counting that we propose goes as follows. First, find the lowest index n in the Seeley coefficient which can give a contribution to the term in question. The corresponding coupling constant, which must have the dimensions of an inverse mass to the power $2p$, is then estimated to be of order

$$\frac{N_c}{16\pi^2} \frac{1}{2} \frac{(n-3)!}{M_Q^{2p}} \frac{1}{n!} \quad (5.4)$$

As an example let us consider the $\mathcal{O}(p^6)$ effective coupling

$$\frac{1}{9} \frac{1}{16\pi^2 f_\pi^2} \text{tr}(F_R^{\mu\nu} \partial^2 F_{R\mu\nu} + F_L^{\mu\nu} \partial^2 F_{L\mu\nu}) \quad (5.5)$$

This coupling, for $F_R^{\mu\nu} = F_L^{\mu\nu} = eQ F^{\mu\nu}$ where $F_{\mu\nu}$ is the electromagnetic field strength and Q is the quark charge matrix $Q = \text{diag}(2/3, -1/3, -1/3)$ is the effective coupling

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which governs the hadronic vacuum polarization effects in low energy phenomenology [23]. It is related to the slope at zero momentum transfer of the hadronic vacuum polarization function

$$\frac{\alpha}{\pi} \frac{4}{3} \frac{1}{f_\pi^2} g = \frac{\partial \Pi(q^2)}{\partial q^2} \Big|_{q^2=0} = \int_0^\infty \frac{dt}{t^2} \frac{1}{\pi} \text{Im} \Pi(t) \quad (5.6)$$

where $\frac{1}{\pi} \text{Im} \Pi(t)$ is the hadronic spectral function which can be determined by the one photon e^+e^- annihilation total cross-section.

$$\sigma_{e^+e^- \rightarrow \text{hadrons}} = \frac{4\pi^2 \alpha}{t} \frac{1}{\pi} \text{Im} \Pi(t)$$

The integral in the r.h.s. of (5.6) is dominated by the low energy region ($\text{Im} \Pi(t) \rightarrow \text{constant}$ as $t \rightarrow \infty$), and for our purposes it is sufficient to consider its VMD evaluation[23]

$$\frac{1}{\pi} \text{Im} \Pi(t) = \frac{\alpha}{\pi} 16\pi^2 f_\pi^2 \frac{1}{3} \delta(t - M_V^2) \quad (5.7)$$

which corresponds to a coupling constant

$$g = 4\pi^2 \left(\frac{f_\pi^2}{M_V^2} \right)^2 = 8.5 \times 10^{-3} \quad (5.8)$$

This phenomenological determination is to be compared with the naïve chiral power counting estimate which, with the $16\pi^2 f_\pi^2$ already factored out, tells that g should be of the same order as the $\mathcal{O}(p^4)$ couplings L_i , i.e. $g \simeq 1.6 \times 10^{-3}$ according to eq. (5.2). That is a factor of five smaller than the phenomenological determination.

Let us now see which is the “improved” chiral power estimate of the constant g . The lowest order Seeley coefficient which can produce the desired term in (5.5) is $n = 3$. We therefore have

$$\frac{g}{16\pi^2 f_\pi^2} \simeq \frac{N_c}{16\pi^2} \frac{1}{2} \frac{1}{2} \frac{1}{3!} \quad (5.9)$$

i.e. $g \simeq 0.7 \times 10^{-2}$ (for $M_Q = 320$ MeV), indeed a much better estimate. In fact the exact calculation, with neglect of gluon corrections, gives

$$\frac{g}{16\pi^2 f_\pi^2} = \frac{N_c}{16\pi^2} \frac{1}{15} \frac{1}{M_Q^2} \quad (5.10)$$

A result rather close to the “improved” chiral power counting estimate (5.9)

Another interesting example is the $\mathcal{O}(p^6)$ effective coupling (see eq. (4.14) and Appendix B for notation)

$$\frac{h}{16\pi^2 f_\pi^2} \{ \xi_\lambda, f_{\mu\nu}^{(+)} \} \{ \xi^\mu, f^{(+)\nu\lambda} \} \quad (5.11)$$

which contributes, in particular, to the process $\gamma\gamma \rightarrow \pi^0 \pi^0$ via the term

$$\frac{h}{16\pi^2 f_\pi^2} e^2 \frac{4}{f_\pi^2} \frac{5}{9} F_{\mu\nu} F^{\nu\lambda} \partial_\lambda \pi^0 \partial^\mu \pi^0 \quad (5.12)$$

The naïve chiral power counting estimate of the coupling constant h , with the $16\pi^2 f_\pi^2$ factor pulled out, is the same as a typical $\mathcal{O}(p^4)$ coupling, i.e. $h \simeq 1.6 \times 10^{-3}$. By contrast, a VMD-model of V -exchange between two $V\pi^0\gamma$ vertices leads to the result[24] $h \simeq 6.5 \times 10^{-3}$, again rather close to the “improved” chiral power counting estimate (the same as for the constant g of eq. (5.9)) $h \simeq 7 \times 10^{-3}$.

6. Conclusions and Outlook

The previous sections describe an attempt to derive a low energy effective lagrangian of the strong interactions between the octet of pseudoscalar states from QCD. We have found useful to organize our work within the framework of the large N_c expansion. To leading order in this expansion, but with $\mathcal{O}(\alpha)$ terms and higher powers neglected, the coupling constants of the $\mathcal{O}(p^4)$ effective chiral lagrangian in the chiral limit $\mathcal{M} \rightarrow 0$ are unambiguously determined. Our results, obtained with two different regularizations, coincide with those of previous calculations[8] (see also [25] within a more phenomenological context). They are given in eqs. (2.24), (2.25) and (4.25). The numerical agreement with the experimental values obtained by Gasser and Leutwyler[1] is quite remarkable.

We have extended the determination of the coupling constants L_1, L_2, L_3, L_9 and L_{10} to the lowest order where non trivial gluonic interactions appear. These terms are already sensitive to the dynamics of chiral symmetry breaking. In the lack of a theory for this mechanism, we proceed by adding a term to the QCD effective action, which simulates the effect of quark condensation by triggering a non zero value for $\langle \bar{q}q \rangle$. The parameter M_Q we add in eq. (3.1) can be thought of as a “constituent”-like quark mass and the effective chiral lagrangian appears then as an expansion in inverse powers of M_Q when the quark and gluon degrees of freedom are integrated out. An adequate formal framework to carry out this expansion is provided by the heat kernel technique and has been explained in detail in section 4. The $\mathcal{O}(\alpha)$ terms appear then as ratios of the SVZ[9] gluon vacuum ($\langle \bar{G}G \rangle$) condensate to M_Q^4 . Their contribution to L_i , $i = 1, 2, 3, 9$ and 10 and to H_1 are shown in

eqs. (4.29), (4.30) and (4.31). There is no correction to L_1 , L_2 and L_9 ; and the corrections to L_3 and L_{10} preserve the relation $L_3 = L_{10}$. The relation $2L_1 = L_2$ is a property of the leading large N_c expansion and has already been noticed by Casser and Leutwyler[1]. The relations $4L_2 = L_9$ and $L_3 = L_{10}$ are more intriguing. It turns out that they are also satisfied by the L_i 's determined from the minimal axial and vector model discussed in [1]. The gluonic corrections to L_3 and L_{10} are negative and bring these constants to a better agreement with the experimental values.

We have also studied the possible determination of the other $\mathcal{O}(p^4)$ coupling constants which appear when explicit chiral symmetry breaking via the quark mass matrix M is introduced. To leading order in $1/N_c$ only two new constants, L_5 and L_8 , and the contact term $H_2 \text{tr} \chi$ appear. These constants, however, are sensitive to the ultraviolet regularization procedure, much the same way as f_2 and v . (As discussed in section 4, v picks up a spurious piece that has to be subtracted.) We can, however, find combinations which are insensitive to the ultraviolet regularization, and obtain the results shown in eq. (4.25). No attempt to include the $\mathcal{O}(\alpha)$ corrections here has been made. (They are however ultraviolet finite.) The agreement with the experimental values is less striking than for some of the other L_i 's, but still satisfactory.

We find that the heat kernel approach discussed above to derive an effective chiral lagrangian provides a useful chiral power counting procedure to estimate unknown coupling constants. Certainly an improvement with respect to the naive rule which counts powers of $16\pi^2 f_\pi^2$ only, and which has been shown to fail in various cases. The simple "improved" chiral power counting rule suggested in section 5 gives much better estimates.

We have various suggestions for progress along the lines explored in this paper.

The first of course is the calculation of the $\mathcal{O}(\alpha)$ terms to L_5 and L_8 . It would be nice if this helps towards a better agreement with experiment (for L_5 in particular).

Some $\mathcal{O}(p^6)$ terms have been shown to be important in low-energy phenomenology. In particular those coming from possible vector exchange between two $\mathcal{O}(p^3) \bar{1} \pi \gamma$ effective vertices. The "improved" chiral power counting made in section 5 suggests the same order of magnitude as the values from VMD-phenomenology. However the calculation of the relevant coupling constants analogous to our calculation of the L_i 's of $\mathcal{O}(p^4)$ has not yet

been done.

We are particularly interested in extending this program to the calculation of weak matrix elements. (Here, of course, one has to go to next to leading contributions in the large N_c expansion and technically the book-keeping of terms becomes much more complicated.) There is, however, a systematic framework and we foresee the way to reach an unambiguous matching of the μ^2 dependence from short distances and long distances. This is under investigation.

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Appendix A

This is a summary of the steps of the computation of section 2. $\text{tr}_D \gamma_5(x | e^{s\Delta^2} | x)$ is the interesting object to evaluate in powers of s . tr_D means the trace over Dirac indices. Δ is defined in (2.3). We have

$$(x | e^{s\Delta^2} | x) = \frac{c}{s^2} + \frac{a_0}{s} + a_1 + a_2 s + a_3 s^2 + \dots \quad (\text{A.1})$$

where c is a numeric matrix. Using this information, we have

$$\begin{aligned} \text{tr}_D \gamma_5(x | \frac{1}{2}(\frac{1}{\Delta + M} - \frac{1}{\Delta - M}) | x) &= -\text{tr}_D \gamma_5 M(x | \frac{1}{\Delta^2 - M^2} | x) \\ &= \text{tr}_D \gamma_5 M \int_0^\infty ds e^{-sM^2} (x | e^{s\Delta^2} | x) \\ &= \text{tr}_D \gamma_5 (M L a_0 + \frac{a_1}{M} + \frac{a_2}{M^3} + 2 \frac{a_3}{M^5} + \dots + (n-1)! \frac{a_n}{M^{2n-1}} + \dots) \end{aligned} \quad (\text{A.2})$$

where L is a logarithmically divergent integral $L = \int_0^\infty \frac{dx}{x} e^{-x} x$, which has to be regularized. M is an auxiliary parameter with dimensions of mass. We can relate the regulator used in section 2 with the M_Q of sections 3 and 4, at least for the a_k with $k \geq 1$, for the terms that survive in the chiral limit, by elementary manipulations in (A.1) and (A.2) and setting $M = M_Q$ and $s = 1/\Lambda^2$. For us here M is just an auxiliary variable.

On the other hand one can evaluate

$$\begin{aligned} \text{tr}_D \gamma_5(x | \frac{1}{\Delta + M} | x) &= \text{tr}_D \gamma_5 \int \frac{d^4 k}{(2\pi)^4} \frac{1}{i \not{k}_\mu \gamma_\mu + M + \Delta} \\ &= - \int \frac{d^4 k}{(2\pi)^4} \text{tr}_D \gamma_5 (S + S \Delta S + S \Delta S \Delta S + \dots) \end{aligned} \quad (\text{A.3})$$

where $S = -\frac{1}{i \not{k}_\mu \gamma_\mu + M}$. The result will be a series in powers of M . By comparing equal odd powers of M , one can read off the coefficients $\text{tr}_D \gamma_5 a_n$ from (A.3). For $\text{tr}_D \gamma_5 a_0$, which multiplies the logarithmic divergence, taking into account that, formally

$$\int \frac{d^4 k}{(2\pi)^4} \frac{1}{(k^2 + M^2)^2} = \int \frac{d^4 k}{(2\pi)^4} \int_0^\infty dt t e^{-(k^2 + M^2)t} = \frac{1}{(4\pi)^2} \int_0^\infty \frac{ds}{s} e^{-s} \quad (\text{A.4})$$

the comparison can be also done since it is always possible to regularise this integral in such a way that the comparison is meaningful. (This is true because the quadratic divergence does not play any role.)

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If we consider the gluonic corrections from (A.3) we get a bunch of terms of the form $\langle D_\alpha D_\beta D_\mu D_\nu \rangle$; although the final result is gauge invariant, each term is not. Due to Lorentz invariance one can write

$$\langle D_\alpha D_\beta D_\mu D_\nu \rangle = A_1 \delta_{\alpha\beta} \delta_{\mu\nu} + A_2 (\delta_{\alpha\mu} \delta_{\beta\nu} + \delta_{\alpha\nu} \delta_{\beta\mu}) + A_3 (\delta_{\alpha\mu} \delta_{\beta\nu} - \delta_{\alpha\nu} \delta_{\beta\mu}) \quad (A.5)$$

On the other hand, using the identity

$$D_\alpha D_\beta = \frac{1}{2} \{D_\alpha, D_\beta\} + \frac{1}{2} G_{\alpha\beta} \quad (A.6)$$

(with $G_{\alpha\beta} = [D_\alpha, D_\beta]$), we have

$$\langle D_\alpha D_\beta D_\mu D_\nu \rangle = \frac{1}{4} \{ \{D_\alpha, D_\beta\} \{D_\mu, D_\nu\} \} + \frac{1}{4} \langle G_{\alpha\beta} G_{\mu\nu} \rangle \quad (A.7)$$

The crossed terms cannot contribute due to Lorentz invariance again, since there is no way to build a combination of tensors δ with the symmetries required by $\langle \{D_\alpha, D_\beta\} G_{\mu\nu} \rangle$: symmetric in α, β and antisymmetric in μ, ν . Since

$$\langle G_{\alpha\beta} G_{\mu\nu} \rangle = \frac{1}{12} \langle G_{\alpha\sigma} G_{\rho\sigma} \rangle (\delta_{\alpha\mu} \delta_{\beta\nu} - \delta_{\alpha\nu} \delta_{\beta\mu}) \quad (A.8)$$

comparing (A.5) and (A.7) one finds

$$A_3 = \frac{1}{12} \langle G_{\alpha\beta} G_{\alpha\beta} \rangle \quad (A.9)$$

For the trace

$$\text{tr}_{color} A_3 = \frac{1}{12} \langle \text{tr}_{color} G_{\mu\nu} G_{\mu\nu} \rangle = -\frac{\pi^2}{24} \langle F^2 \rangle \quad (A.10)$$

where $G_{\alpha\beta} = -ig \frac{\lambda^a}{2} G_{\mu\nu}^a$, $G_{\mu\nu}^a$ as previously defined below equation (2.2) and $G^2 = G_{\mu\nu}^a G_{\mu\nu}^a$. This is the gauge invariant part from (A.5), and it is the only one that should survive at the end, providing a good check of the computer code used to generate the gluonic corrections. The method could be easily generalized to include higher gluonic corrections.

Appendix B

Here we collect some useful formulae with the conventions used in sections 3 and 4. The left and right components of the quarks are defined as

$$q_L = \frac{1 - \gamma_5}{2} q \quad q_R = \frac{1 + \gamma_5}{2} q \quad (B.1)$$

$$\bar{q}_L = \bar{q} \frac{1 + \gamma_5}{2} \quad \bar{q}_R = \bar{q} \frac{1 - \gamma_5}{2}$$

Apart from the quarks, the NG-bosons appear through the unitary unimodular matrix $U \in SU(3)$. Let us also combine the external sources v_μ and a_μ as used in section 2 for the vector and axial-vector currents into left and right fields L_μ and R_μ in the following manner

$$-iL_\mu = v_\mu - a_\mu$$

$$-iR_\mu = v_\mu + a_\mu \quad (B.2)$$

which will transform as Yang-Mills connections under local $SU(3)_L \otimes SU(3)_R$.

Under a $SU(3)_L$ local transformation we have

$$q_L \rightarrow g_L q_L$$

$$q_R \rightarrow q_R$$

$$L_\mu \rightarrow i g_L \partial_\mu g_L^{-1} + g_L L_\mu g_L^{-1} \quad (B.3)$$

$$R_\mu \rightarrow R_\mu$$

$$U \rightarrow U g_L^{-1}$$

where $g_L(x) \in SU(3)_L$. Similarly, for a $SU(3)_R$ transformation

$$q_L \rightarrow q_L$$

$$q_R \rightarrow g_R q_R$$

$$L_\mu \rightarrow L_\mu \quad (B.4)$$

$$R_\mu \rightarrow i g_R \partial_\mu g_R^{-1} + g_R R_\mu g_R^{-1}$$

$$U \rightarrow g_R U$$

where $g_\mu(x) \in SU(3)_R$. Now we introduce

$$D_\mu U = \partial_\mu U - iR_\mu U + iUL_\mu$$

$$D_\mu U^\dagger = \partial_\mu U^\dagger + iU^\dagger R_\mu - iL_\mu U^\dagger \quad (B.5)$$

(This has nothing to do with the covariant derivative of QCD which acts on colour indices and involves the gluon fields; D_μ in (B.5) is only concerned with the flavour indices and always appears acting on U or $U^\dagger$; no confusion is possible, though). With this definition

$$[D_\mu, D_\nu]U = i(U F_{\mu\nu} - F_{\mu\nu} U)$$

$$F_{C\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu - i[C_\mu, C_\nu], \quad C = L, R \quad (B.6)$$

Under a combined $SU(3)_L \otimes SU(3)_R$ transformation both U and $D_\mu U$ transform linearly according to

$$U \rightarrow g_R U g_L^{-1} \\ D_\mu U \rightarrow g_R (D_\mu U) g_L^{-1} \quad (B.7)$$

We list some useful relations that have been used in passing from (4.13) to (4.16) so as to leave the final expression only in terms of $U = \xi^2$.

$$\begin{aligned} \xi_\mu &= i\xi^\dagger (D_\mu U) \xi^\dagger = -i\xi (D_\mu U^\dagger) \xi \\ d_\mu \xi_\nu &= i\xi^\dagger (D_\mu D_\nu U) \xi^\dagger + \frac{i}{2} \{\xi_\mu, \xi_\nu\} \\ &= -i\xi (D_\mu D_\nu U^\dagger) \xi - \frac{i}{2} \{\xi_\mu, \xi_\nu\} \\ \xi_{\mu\nu} &= d_\mu \xi_\nu - d_\nu \xi_\mu = \xi^\dagger F_{R\mu\nu} \xi - \xi F_{L\mu\nu} \xi^\dagger \\ \Gamma_{\mu\nu} - \frac{1}{4} [\xi_\mu, \xi_\nu] &= -\frac{i}{2} (\xi^\dagger F_{L\mu\nu} \xi + \xi F_{R\mu\nu} \xi^\dagger) \\ [d_\mu, d_\nu] \xi_\alpha &= [\Gamma_{\mu\nu}, \xi_\alpha] \end{aligned} \quad (B.8)$$

We have also used the equations of motion in the next-to-leading terms we want to keep. From (euclidean space)

$$\delta \int d^4x \frac{f_\pi^2}{4} \text{tr} \{ D_\mu U^\dagger D_\mu U - (X^\dagger U + U^\dagger X) \} = 0 \quad (B.9)$$

together with the constrain $UU^\dagger = 1$ one can derive

$$(D^2 U) U^\dagger - U (D^2 U^\dagger) = U X^\dagger - X U^\dagger \quad (B.10)$$

where $D^2 = D_\mu D_\mu$. This, combined with the identities

$$(D_\mu U) U^\dagger + U (D_\mu U^\dagger) = 0 \\ (D^2 U) U^\dagger + U (D^2 U^\dagger) = -2(D_\mu U) (D_\mu U^\dagger) \quad (B.11)$$

enable us to get rid of the $D^2 U$ terms. These are

$$\begin{aligned} \text{tr} (i\Delta d_\mu \xi_\mu) &= \text{tr} (\Sigma \xi_\mu \xi_\mu) + \text{tr} (\mathcal{M} D^2 U^\dagger + \mathcal{M}^\dagger D^2 U) = \\ &= -\frac{1}{2} \text{tr} (\mathcal{M} X^\dagger + \mathcal{M}^\dagger X) + \frac{1}{2} \text{tr} (\mathcal{M} U^\dagger X U^\dagger + \mathcal{M}^\dagger U X^\dagger U) \\ \text{tr} (d_\mu \xi_\mu)^2 &= \text{tr} \{ (D^2 U) (D^2 U^\dagger) - [(D_\mu U) (D_\mu U^\dagger)]^2 \} = \\ &= \frac{1}{2} \text{tr} (X^\dagger X) - \frac{1}{4} \text{tr} (U^\dagger X U^\dagger X + U X^\dagger U X^\dagger) \end{aligned} \quad (B.12)$$