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On the c-theorem in Higher Genus

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ABSTRACT

We study the extension of the c-theorem to arbitrary genus Riemann surfaces. We analyze the breakdown of conformal invariance caused by the need of cutting off regions of moduli space to regulate divergences and argue how these can be absorbed in the bare couplings on the sphere. An extension of the c-theorem then follows. We also discuss the relationship between the c-theorem and the effective action when corrections from higher genera are accounted for.

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1. Introduction

Zamolodchikov's c -theorem^[1] is one of the most important tools for studying 2-dimensional quantum field theories away from their critical points, at which they become conformally invariant. The theorem allows for the construction of a generalized central charge coefficient, $C(g)$, as a function of the renormalized coupling constants $\{g^i\}$, which in some sense interpolates between the various fixed points of the theory. This function is expressed in terms of two-point correlators of components of the stress-tensor of the theory, and equals the central charge at the conformal points, which are identified with the fixed points of the renormalisation group. It reads,

$$C(g) = 2z^{-1}(T(z, \bar{z})T(0)) - 4z^3\bar{z}(T(z, \bar{z})\Theta(0)) - 6z^2\bar{z}^2(\Theta(z, \bar{z})\Theta(0)) \quad (1)$$

where $\Theta = T_{z\bar{z}}$ and $T = T_{zz}$. The Green functions appearing in (1) are taken at the arbitrary normalization point $|z| = 1$. On dimensional grounds, $C(g)$ is then a function of the renormalization scale μ only. It satisfies the following scaling equation^[1]

$$\mu \frac{d}{d\mu} C(g) = -12z^2\bar{z}^2(\Theta(z, \bar{z})\Theta(0))|_{|z|=1} \quad (2)$$

The importance of the theorem becomes apparent in unitary theories where the two point function of the stress tensor trace is positive definite by reflection positivity. In this case the theorem implies the existence of an entropy function in the space of 2d-unitary theories which never increases under the action of the renormalization group.

When applied to string-related σ -models^[2], Zamolodchikov's c -theorem allows for expressing the conformal invariance conditions as background equations of motion obtained from a local action. An important issue is the equivalence of this action with the generating functional of string scattering amplitudes at tree level, i.e. the σ -model free energy. Recently arguments^[3] for this equivalence have been given.

An interesting question concerns the extension of these results to higher genus Riemann surfaces, thereby allowing for quantum corrections to the classical string effective actions. In the sigma model language, this would imply loop corrections to the corresponding string coupling β -functions. But in what sense the renormalization group β -function, which is insensitive to the topology of the space-time being related only to the local[†] ul-

[†] We ignore, for our purposes, the complications due to the total derivative terms that lead to the replacement of the β -functions by the $\bar{\beta}$ in the renormalization group equations.^[2]

traviolet infinities, receives corrections from higher genus? The key point, suggested by Fischler and Susskind^[4], is the existence of additional infinities that appear in summing up Riemann surfaces. When discussing loop corrections to the string sigma model, in a Polyakov path-integral language, one sums over all possible metrics, and therefore, in the one loop approximation discussed in [4] this means summing over tori, which in turn implies an integration over Teichmüller space^[5]. This integration contains additional divergences, up and above the usual ultraviolet infinities, which come from some boundary of the Teichmüller integration. Cutting it off introduces an additional scale in the problem. The idea behind the Fischler and Susskind mechanism is the absorption of this infinity into the renormalized coupling on the sphere (for closed strings, or, in general, the lowest genus Riemann surface). In their work they considered one loop corrections, and restricted themselves only to infinities coming from tadpoles associated with the exchange of a massless scalar string mode (the dilaton). The use of a common cut-off to regulate ultraviolet and modular divergences as well as its precise form in the latter case was postulated somewhat *ad hoc*.

These ideas have been generalized to include other couplings in the one loop case for the open string in [6] and for the closed string in a variant of the operator formalism in [7], providing also some basic ingredients for arbitrary genus Riemann surfaces. (The tadpoles have also been analyzed in [8].) Again, loop corrections to the β -functions come by demanding the scale independence of the total partition function over the summed up surfaces for both sorts of divergences. This requires the use of a specific parametrization of moduli space and relating a cut off in the moduli to the more familiar ultraviolet divergences. One might argue that modular infinities are of infrared nature, because they arise e.g. from a region where the momentum of a scalar mode approaches zero. Equivalently, due to conformal invariance, one could argue that they arise when the "size" of the handle (or hole in the case of open strings) becomes smaller than the cut-off. Therefore a question arises in connection with the consistency of a procedure in which two completely different classes of divergences are regularized by the same *ad hoc* cut-off.

Clearly, it would be quite interesting to understand the issues just discussed on a firmer setting; otherwise it seems very difficult to incorporate in a systematic way loop corrections to the equations of motion (with the possible exceptions of the dilaton and tachyon tadpoles^[8]). Such general discussion appears to be missing in the literature and one purpose of this paper is to fill in this gap.

So far all the work on higher genera has been concentrated “on-shell”, i.e. the loop corrections to the β -functions computed at the points where they vanish. In this article we shall do our analysis away from the critical point by constructing a generalized ϵ -function à la Zamolodchikov, which however incorporates loop corrections. Off shell relations between this function and the loop corrected β -functions will be obtained, and arguments for the connection with the generating functional of loop corrected scattering amplitudes (to arbitrary genus Riemann surfaces) will be provided.

2. Modular Corrections to Sigma Models

In this section we will discuss the appearance of a conformal factor dependence in sigma models defined on a surface of genus $g \geq 1$ due to the modular infinities. As it is well known[5], the sum over all the 2-dimensional metrics on a given Riemann surface leads to an integration over a finite number of complex parameters -- the moduli. The partition function is

$$Z_g(g^i) = \int d\tau \mu(\tau) \int [d\phi] \exp(-I^*(\phi) - g^i O_i) \quad (3)$$

Since summing up Riemann surfaces is probably of interest only for string-related σ -models, we will equivalently use the term string generating functional for (3). $\mu(\tau)$ is the measure in moduli space, which is conformally invariant[6], and $I^*(\phi)$ is the gaussian fixed point action, at $g^i = g^{i*} = 0$ (i.e. the free string propagating in a trivial background, for string-related σ -models). The index i runs over all the string modes and includes an integration over the two dimensional coordinates when necessary. Eq. (3) is nothing but a two dimensional sigma model defined on a genus g surface, and it is afflicted by the usual short distance singularities. Their removal leads to the introduction of the ultraviolet scale μ discussed in the introduction. The couplings become scale dependent, $g^i \rightarrow g^i(\mu)$, leading to a renormalization group equation (r.g.e.). Because the conformal factor, $\rho(\xi)$, for covariance reasons necessarily appears in the combination $\mu \rho^6(\xi)$, this takes care of the ρ dependence due to the integral over the high frequency modes in the sigma model partition function.

However, it turns out that the integrand in (3), as a function of τ , is finite everywhere but on some boundaries of moduli space. To make sense of the sum over all metrics one thus needs to regulate the modular integral. We are forced then to introduce a second

length -- or mass -- scale that *a priori* is different from the scale associated to the ultraviolet divergences. In fact, it will be of use to us for the present discussion to keep the two scales different.

There are many ways of parametrizing moduli space. This makes writing a renormalization group equation incorporating the effects of modular infinities rather ambiguous. In which of the infinitely many parametrizations are we going to introduce the cut-off? To what extent the results can depend on this? It is not obvious at all how to write down a renormalization group equation with two separate mass scales, but even if we set both equal it is not difficult to convince ourselves that the renormalization group equations that we get by cutting off in various ways are different and unrelated by field redefinitions. For instance, cutting the integration region in, e.g., the torus (Fig. 1) by $\log \Lambda^2 A$ or by $\Lambda^2 A$ leads to very different r.g.e. with respect to Λ ($A = \int d^2 \xi e^{\rho(\xi)}$ has units of area and it is required on dimensional grounds, as the moduli are dimensionless. See below)

So the question arises as to whether there is a “natural” way of cutting off modular divergences. Certainly the regularization must be done in a way that is consistent with the symmetries of the problem. Reparametrization invariance plays a key role in Riemann surfaces so it seems necessary to adopt a regularization procedure that respects covariance on the world sheet. We are thus led to regulate the moduli infinities by the use of the invariant distance in moduli space induced by the norm in the space of traceless deformations of the metric

$$\|\delta\gamma\|^2 = \frac{1}{2} \int d^2 \xi \sqrt{\gamma} (\gamma^{\sigma\nu} \gamma^{\mu\theta} + \gamma^{\sigma\theta} \gamma^{\mu\nu} - \gamma^{\sigma\theta} \gamma^{\mu\nu}) \delta\gamma_{\sigma\theta} \delta\gamma_{\mu\nu} \quad (4)$$

Using

$$\chi_{\sigma\theta\theta} = \frac{\partial}{\partial \tau^\sigma} \gamma_{\theta\theta} - \frac{1}{2} \gamma_{\sigma\theta} \gamma^{\mu\nu} \frac{\partial}{\partial \tau^\mu} \gamma_{\nu\theta} \quad (5)$$

We get the following invariant distance in moduli space

$$\|\delta\tau\|^2 = H_{mn} \delta\tau^m \delta\tau^n \quad (6)$$

with

$$H_{mn} = \int d^2 \xi \sqrt{\gamma} \chi_{m\alpha\theta} \chi_n^{\alpha\theta} \quad (7)$$

Note that $\|\delta\tau\|^2$ has dimensions of length squared and that H_{mn} is, of course, independent of ξ , and it transforms under a change in the modular parameters as a quadratic form.

We can now give a more precise meaning to the notion of “cutting off moduli space”. With the help of the invariant distance we introduce a parameter M (an inverse length) that in the $M \rightarrow 0$ limit extends the integration area to the whole moduli space, but otherwise renders the integral finite (see Fig. 1). We are now free to use the parametrization we wish. In different parametrizations the values at which we will have to cut the integration over the moduli parameters, τ_k^c , will be of course different but since they correspond to change of coordinates in moduli space the value of the integral will be the same. We still have some freedom in picking the integration area so not all the ambiguity is removed. But this is just as well because we do expect to have some freedom in the renormalization procedure, exactly as for ultraviolet singularities. Changes in the integration area correspond necessarily to finite renormalizations, while this is not the case if we just cut off arbitrarily, as discussed above. We notice from (6) that if we place a cut off $\frac{1}{M}$ in the invariant distance, the conformal factor will *necessarily* appear in the combination Me^ρ , just as for ordinary short distance singularities. So the dependence on the conformal factor is readily identified.

In fact, the important point to notice is that the sigma model partition function on a genus g surface, eq. (3), picks an additional conformal factor dependence due to the modular regulator. Indeed

$$\begin{aligned} \frac{\delta}{\delta\rho} Z_g(g^i) &= \frac{\delta}{\delta\rho} \int_{\mathcal{M}(Me^\rho)} d^n \tau \mu^n(\tau) \int [d\phi] e^{-I} \\ &= \int_{\partial \mathcal{M}(Me^\rho)} \sum_k \frac{\delta \tau_k^c}{\delta\rho} d^{n-1} \tau \mu^n(\tau) \int [d\phi] e^{-I} \\ &\quad + \int_{\mathcal{M}(Me^\rho)} d^n \tau \mu^n(\tau) \frac{\delta}{\delta\rho} \int [d\phi] e^{-I} \end{aligned} \quad (8)$$

This expression needs some explanation. I is simply $I^* + g^i O_i$. $\mathcal{M}$ denotes the region of moduli space we integrate over. Its dimension is $n = 6g - 6$. It will be described by an implicit function of the moduli τ and, as follows from the above discussion, of the combination Me^ρ

$$\tau^c = \tau^c(\tau, Me^\rho) \quad (9)$$

The second term in (8) originates from the usual ultraviolet divergences and so is expected to be just the same as in the sphere which has been abundantly analyzed. The first term of (8) lives on the boundary of the original Riemann surface moduli space. It is natural to expect that it can be expressed as a σ -model on a surface of genus $g-1$ or, perhaps, surfaces

of even lower genera, with some shifted couplings. We want to analyze this question more carefully.

Let us recall a few relevant facts on moduli spaces and moduli space singularities^[5]. On a given surface we can draw either non-trivial (a and b) or trivial (c) cycles. Let $\{L_n\}$ be the lengths of the cycles (Fig. 2) and $\{\theta_n\}$ the angles associated to the Dehn twists. After restricting oneself to a suitable fundamental domain of the modular group, the degenerated surfaces in which the partition function diverges are obtained when the length l of either the a or c cycles vanish. Obviously, two of them can vanish simultaneously, but there are no other singularities. These are thus the regions in moduli space to regulate carefully.

Let λ be the string coupling constant. The amplitudes will be expressed as a power series in λ . In particular the total generating functional will be $Z(g^i) = \sum_g \lambda^g Z_g(g^i)$. A surface of genus g will thus have a factor λ^g in front. We want to argue now that, in perturbation theory in λ , one can absorb all the modular divergences appearing in genus g into bare couplings in surfaces of lower genera, that is, up to $g-1$. By cutting the surface along all the independent homotopically trivial cycles and inserting a complete set of states with the help of the operators $\{O_i\}$ we can separate a genus g surface into a number of tori and spheres. We write

$$\lambda^g Z_g = \lambda^g \int \frac{dt_1}{t_1} \frac{dt_2}{t_2} \dots \sum_{i_1} \sum_{i_2} \dots \langle \Sigma_1 | O_{i_1} \rangle \frac{1}{t_1^{\alpha_1}} \langle O_{i_1} | \Sigma_2 | O_{i_2} \rangle \frac{1}{t_2^{\alpha_2}} \dots \quad (10)$$

$|\Sigma\rangle$ denotes the state corresponding to either the torus or the sphere. The interested reader can find details on the definition of these states in [10] (see also [7] and [11]). Note however, that we are considering an arbitrary sigma model and not just the free string, i.e. we use the action $I = I^* + g^i O_i$, not just I^* . Since we are working in perturbation theory, the partition function has always to be understood as a power series in the g^i . The α_i are the anomalous dimensions of the fields at the gaussian fixed point. For higher genera we will need more than one partition of the surface to cover all the singularities of moduli space. The weight to give to each partition it is not known in general. It is this difficulty that prevents multiloop divergences from being just iterations of the one loop case. However this will not affect the following argument. Notice that in general we will also need the insertion of more than one operator to recover all the modular infinities (Fig. 3).

To get the ρ dependence induced by the modular infinities we act with $M \frac{\partial}{\partial M}$ on eq. (10). We get then a number of terms, which we group into those where the derivative acts

on Σ (which contains a divergent modular integral if $\Sigma = T$, a torus) and those where it acts on the trivial cycle "plumbing fixtures", i.e. on the upper limit of the t integrals. From previous work we know that the latter singularities, which are momentum independent, will be absorbed by shifting the tachyon and the dilaton by the corresponding tadpoles $\Delta_T, \Delta_D, \dots$ (see below) [6, 7, 8a]. (Recently the cancellation has been checked explicitly up to genus 3 [9b].) In addition we have the terms obtained when $\delta \equiv M_{\partial M}^2$ acts on the tori. A typical* term generated in this way will be, for instance, $\lambda^g T \delta T T \dots$. We shall now assume that we can express $\lambda \delta T = \delta S'$, where S' is a sphere (i.e. the complex plane) with shifted couplings at $\mathcal{O}(\lambda)$. That is, $\lambda^g T \delta T T \dots = \lambda^{g-1} T \delta S' T \dots$. In fact, we can replace T by T' (a torus with the shifted couplings), since the error is of higher order in λ . By integration by parts and repeated use of the trivial identities

$$S' S' = S' \quad S' T' = T' S' = T' \quad S' \delta S' = \frac{1}{2} \delta S' \quad (11)$$

it is easy to show that the dependence in M (hence in the conformal factor due to the modular divergences) of a surface of genus g can all be absorbed by shifting the couplings in genus up to $g-1$. The argument can now be iterated to reduce everything to the $g=1$ case.

To show how all divergences in the $g=1$ case can indeed be absorbed in the sphere and to illustrate some of the points made in this section we wish now to discuss in some detail the torus with a tachyonic background. Using the standard parametrization of the metric

$$\gamma_{\alpha\beta} = \begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau_1|^2 \end{pmatrix} \quad (12)$$

we obtain the following metric in Teichmüller space

$$\|\delta\tau\|^2 = 2 \int d^2 \xi e^{\phi(\xi)} \frac{1}{\tau_2} [(\delta\tau_1)^2 + (\delta\tau_2)^2] \quad (13)$$

The divergent limit in the string generating functional is, in this case, $\tau_2 \rightarrow \infty$ where there is (in this parametrization) an exponential divergence, due to the tachyon, as well as a power law divergence caused by the dilaton. We place a cut off in τ_2 using the invariant distance, as illustrated in Fig. 1. The contribution from the upper limit of the modular integral (eq. 8) is

$$\lambda \frac{\delta\tau_2^2}{8\rho} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \mu(\tau_1, \tau_2^2) \left[\int [d\phi] e^{-I' - \phi' \cdot \mathcal{O}} \right]_{\tau_2 = \tau_2^2} \quad (14)$$

* Notice, however, that not all the cuttings of a surface can be written in a "one dimensional" form. See Fig. 3.

with

$$\frac{\delta\tau_2^2}{8\rho} = \frac{1}{8} \frac{1}{M^2 \Lambda} e^\rho \quad (15)$$

where $\Lambda = \int d^2 \xi e^{\phi(\xi)}$.

In the $M \rightarrow 0$ limit, i.e. $\tau_2 \rightarrow \infty$ limit the torus degenerates into an infinite cylinder, which is conformally equivalent to the plane. This can be checked explicitly at the level of the two-point function $\langle \phi(z) \phi(w) \rangle$, which is actually all we need in perturbation theory. Expanding now in powers of g' and performing the integral over τ_1 , which is well behaved (the leading behaviour when $M \rightarrow 0$ is independent of τ_1 anyhow), we obtain a series of Green functions *on the plane*, with divergent coefficients due to the modular infinities. There may also be finite ρ -dependent parts that correspond to the regularization ambiguities pointed out above. With the prescription taken here for the torus they are absent, but, of course, there is nothing special about this prescription.

To be more precise, let us consider a sigma model with a tachyonic background $I = I^* + T(x) \int d^2 \xi \sqrt{\gamma}$. We have the following functional integral pointwise in moduli space

$$\int [d\phi] \exp[-I^* - \int d^2 \xi \sqrt{\gamma} e^{i\phi(\xi)}] \quad (16)$$

(16) has the familiar σ -model ultraviolet singularities (including normal ordering) that can be absorbed after redefining $T(k)$. Let us now consider the perturbative expansion of (16) in the torus. Taking into proper account the zero mode we have

$$\begin{aligned} & \lambda \delta(k) T(k) \langle e^{ik\phi} \rangle_T + \frac{1}{2} \lambda \delta(k_1 + k_2) T(k_1) T(k_2) \langle e^{ik_1\phi} e^{ik_2\phi} \rangle_T \\ & + \frac{1}{6} \lambda \delta(k_1 + k_2 + k_3) T(k_1) T(k_2) T(k_3) \langle e^{ik_1\phi} e^{ik_2\phi} e^{ik_3\phi} \rangle_T + \dots \end{aligned} \quad (17)$$

An integral over the momenta k , which plays the role of the summation over i , is understood. $\langle \dots \rangle_T$ is a short-hand notation for the expectation value of the operators in the torus, including the modular integration. Because of this integration each of the terms in (17) is divergent even after removing the ultraviolet singularities. We shall cut the integrals in the manner we have discussed above and we will find that, as far as the $M e^\rho$ dependence is concerned, we can replace (17) by

$$\begin{aligned} & (\lambda T)_S + \delta(k_1 + k_2) \Delta^{(2)} T(k_1) T(k_2) \langle e^{ik_1\phi} e^{ik_2\phi} \rangle_S \\ & + \delta(k_1 + k_2 + k_3) \Delta^{(3)} T(k_1) T(k_2) T(k_3) \langle e^{ik_1\phi} e^{ik_2\phi} e^{ik_3\phi} \rangle_S + \dots \end{aligned} \quad (18)$$

$\Lambda_T, \Delta^{(2)}, \bar{\Delta}^{(3)}, \dots$ are functions of $\mathcal{O}(\lambda)$ that depend only on the modular regulator M , and S denotes expectation values on the sphere. It is clear that the terms in (18) can be obtained from a σ model on the sphere by shifting the coupling T on the sphere

$$T(\phi) \rightarrow T(\phi) + \Lambda_T + \Delta^{(2)}T(\phi) + \Delta^{(3)}T(\phi)T(\phi) + \dots \quad (19)$$

Λ_T corresponds to a tadpole, while $\Delta^{(2)}, \Delta^{(3)}, \dots$ correspond to renormalization of the mass[12], three point function, etc. This is what we wanted to prove. The argument can be easily generalized to any coupling.

3. The Generalized c-theorem

In the previous section we argued how the infinities arising from corners in moduli space when summing up Riemann surfaces of genus g can be absorbed in the renormalization of coupling constants of the surface of trivial topology. Once this is achieved the whole machinery of σ -model calculus on the sphere applies, which allows for the construction of a generalized C function à la Zamolodchikov, incorporating the string loop corrections. Let us now describe the basic steps.

We consider a generic σ -model action of the form (3). The background couplings $\{g^i\}$ are taken to be slightly relevant at the gaussian fixed point, with renormalization group anomalous dimensions ϵ^i which are set to zero at the very end. From a string point of view, this is the situation encountered in the computation of scattering amplitudes in Minkowskian target space formulation of string theory[3,13]. This implies that one is dealing with renormalizable σ -model modes only, so standard renormalization group techniques are applicable[14].

Including in the sphere the additional counterterms obtained from modular infinities the following scaling relation holds

$$\frac{d}{d\rho} \mathcal{Z}(g^i) = 0 \quad (20)$$

where $d/d\rho$ denotes the *total* functional derivative with respect to the conformal factor ρ . (We assume that the theory is classically scale invariant; if not, we have to correct (20)

for this fact.) Eq. (20) is simply stating the fact that we can absorb the modular as well as ultraviolet infinities by tuning the sphere couplings.

From the analysis of the previous section it becomes clear that all ρ dependence can be absorbed in couplings g^i “renormalized” with respect to both ultraviolet and modular divergences of the theory, giving rise to loop corrected beta functions β_{LC} . Given that ρ appears in the combinations μe^ρ and $M e^\rho$ it is clear that $d/d\rho$ is equivalent to

$$\frac{\delta}{\delta\rho} + \beta_{LC}^i \frac{\partial}{\partial g^i} = \mu \frac{\partial}{\partial \mu} + M \frac{\partial}{\partial M} + \beta_{LC}^i \frac{\partial}{\partial g^i} \quad (21)$$

where

$$\beta_{LC}^i = \frac{\delta g^i}{\delta\rho} = \mu \frac{\partial g^i}{\partial \mu} + M \frac{\partial g^i}{\partial M} \quad (22)$$

Equation (20) is all one needs to prove the generalized c-theorem. As far as the conformal dependence is concerned one can start directly from a σ model defined on the sphere whose couplings g^i are already a function of ρ (in the form prescribed by the moduli corrections) at the σ -model tree level, i.e. before removing the ultra-violet singularities. This means that if we replace

$$g^i \rightarrow \bar{g}^i(M e^\rho) \quad (23)$$

we can proceed, as far as determining the breakdown of conformal invariance is concerned, as in usual σ -models[14]. It is known[15] from σ -model calculations that if one demands Weyl invariance (as opposed to simply requiring scale invariance) one needs to work in a general conformal factor $\rho(\xi)$ and, therefore, the couplings g^i are function of the surface coordinates. Eventually we will be interested in taking the flat limit $\rho = 0$, but for the moment, we allow the most general loop corrected “renormalized” action I^{LC} , including other operators which also enter the game in the renormalization procedure[2d] and that contain derivatives of the couplings

$$I^{LC} = I^* + \int d^2\xi g^i(\xi) O_i(\xi) + \int d^2\xi \sqrt{\gamma} \nabla_a g^i(\xi) G_a^i + \int d^2\xi \sqrt{\gamma} \nabla_a g^i(\xi) \nabla_\beta g^j(\xi) G_{ij}^{ab} \quad (24)$$

with $g^i(\xi) = g^i(\mu e^{\rho(\xi)}, M e^{\rho(\xi)})$. Terms with higher derivatives would correspond to irrelevant couplings. For the closed string the only coupling with the right quantum numbers to contribute to the terms in (24) with one derivative is the dilaton coupling

$$\int d^2\xi \sqrt{\gamma} \Phi R^{(2)} \quad (25)$$

(Recall that $R^{(2)}$ is a total derivative, $R^{(2)} = \nabla_\alpha K^\alpha$.) From an string S -matrix point of view it is the trace of the graviton that plays the rôle of the dilaton mode. Again for the close string, if we ignore the antisymmetric tensor, we only have $\tilde{G}_{ij}^{\alpha\beta} = \gamma^{\alpha\beta} G_{ij}(g)$ contributing to the second one. (The argument can be easily generalized when we have an antisymmetric tensor background.) It can be seen that G_{ij} is related^[2d] to the two point function $\langle O_i O_j \rangle_S$. Functionally, G_{ij} is not changed by loop corrections; the effect of loops comes only through the modified couplings.

Defining the stress tensor as

$$T_{\alpha\beta} = -\frac{1}{\sqrt{\gamma}} \frac{\delta I^{LC}}{\delta \gamma^{\alpha\beta}} \quad (26)$$

one obtains

$$\langle \Theta = \beta_{LC}^i O_i + \beta_{LC}^j R^{(2)} - T - \nabla_\alpha N^\alpha + \nabla_\alpha g^i \nabla^\alpha g^j \beta_{LC}^i \partial_k G_{ij} \quad (27)$$

with $N_\alpha = \nabla_\alpha \Phi + 2\tilde{G}_{ij} \nabla_\alpha g^i \beta_{LC}^j \equiv \nabla_\alpha \Phi + w_i \nabla_\alpha g^i$. The β_{LC}^i include the anomalous dimension piece $-e^i g^j$ for the case of marginal couplings. $\tilde{G}_{ij}$ contains poles in ϵ . Due to the finiteness of Θ in a renormalizable theory the divergent poles will cancel in (27) and only appropriate combinations of the different coefficients will appear. T is the tachyon field, which breaks conformal invariance already at the classical level. Its beta function β_{LC}^T is included in the set $\beta_{LC}^i O_i$ accompanying the unit operator. All operators are understood to be normal ordered. The last two terms in (27) come from the last two counterterms in (24), i.e. from requiring *local* scale invariance. On dimensional grounds,

$$\langle \Theta \rangle_S = \tilde{\beta}^* R^{(2)} + \beta^T - \Lambda_T - \nabla_\alpha N^\alpha + \nabla_\alpha g^i \nabla^\alpha g^j \beta_{LC}^i \partial_k G_{ij} \quad (28)$$

Only the zero momentum part of the tachyon contributes in (28), so we have replaced T by its tadpole Λ_T , which, as we know, is generated by loop corrections. Acting with $\delta/\delta\rho$ on this last expression and going to the flat limit $\rho = 0$ we immediately obtain the correlator $\langle \Theta(z) \Theta(w) \rangle_S$ and, on covariance grounds, we have the following decomposition for the components of the stress tensor

$$\langle T_{\alpha\beta}(z) T_{\gamma\delta}(w) \rangle_S = \tilde{\beta}_{LC}^* \frac{1}{\square} (\partial_\alpha \partial_\beta - \delta_{\alpha\beta} \square) (\partial_\gamma \partial_\delta - \delta_{\gamma\delta} \square) \delta(z - w) \quad (29)$$

The two point correlators of the components of the energy-momentum tensor can be expressed as functions of the “running” $\tilde{\beta}_{LC}^*$ along the generalized renormalization group

trajectory expressing the scale dependence of the renormalized couplings^[2]. This implies that these correlators are a function of $\tilde{\beta}_{LC}^*(g^i)$ and its renormalization group derivatives.

We have all the necessary ingredients to prove the generalized c-theorem. I^{LC} is reparametrization invariant. Consequently the modified energy-momentum tensor is conserved. We have assumed that in writing eq. (29). The second ingredient we need is renormalizability. This has been the whole point of section 2: all the modular infinities can be absorbed in shifts of the couplings in the sphere, just as the ultraviolet ones. Defining the loop corrected c-function as

$$C_{LC}(g) = \tilde{\beta}_{LC}^* = 2z^1 \langle T(z, \bar{z}) T(0) \rangle_S - 4z^2 \bar{z}^2 \langle T(z, \bar{z}) \Theta(0) \rangle_S - 6z^2 \bar{z}^2 \langle \Theta(z, \bar{z}) \Theta(0) \rangle_S \quad (30)$$

where all the correlators are computed, as indicated, on the sphere in a theory with the couplings modified so as to absorb the modular infinities. Since everything is expressed in terms of renormalized quantities we have taken the $\epsilon \rightarrow 0$ limit. As before we adopt the arbitrary normalization point $|z| = 1$. From the above two requirements one obtains

$$\left(\mu \frac{\partial}{\partial \mu} + M \frac{\partial}{\partial M} \right) \tilde{\beta}_{LC}^* = \beta_{LC}^* \partial_\mu \tilde{\beta}_{LC}^* = -12 \beta_{LC}^i \beta_{LC}^j (O_i O_j)_S \quad (31)$$

If we are dealing with a theory that is unitary in the sphere for any values of the coupling constant, then the r.h.s. of (31) is negative definite and $C_{LC}(g)$ never increases under the renormalization group. This is the last ingredient we needed.

Since $\langle \Theta \rangle$ is a finite physical quantity the total derivative $d/d\rho$ of $\langle \Theta \rangle$ vanishes and one obtains, after taking the $\rho = 0$ limit, $\tilde{\beta}_{LC}^* = \beta_{LC}^* + w_i \beta_{LC}^i$, and the following “off-shell” extension

$$\partial_i \tilde{\beta}_{LC}^* = \mathcal{D}_{ij} \beta_{LC}^j \quad (32)$$

where $\mathcal{D}_{ij}$ is a (non symmetric, perturbatively invertible) matrix in coupling space that depends on G_{ij} and β_{LC}^i (therefore it does not have the same functional dependence in g^i as in the sphere case). In addition one can obtain the generalization to higher loops of the Curci-Paffuti relation^[16], which guarantees that at the fixed point $g^i = g^{i*}$, where $\beta_{LC}^i(g^{i*}) = 0$, $\tilde{\beta}_{LC}^*$ is just a constant, the central charge, which is not changed by loop corrections.

Notice that the finite-size scaling arguments of [3a] would imply the equivalence of the scattering matrix for the (loop corrected) nodes g^i obtained from $C(g)$ with the one

obtained from the σ -model free energy. Given that the latter is effectively equivalent with the loop corrected string scattering amplitudes at genus g , this completes the proof of the equivalence of the generalized ϵ -theorem with the string $\mathcal{S}$ -matrix approach, when corrections from higher genera are accounted for. The generic approach presented here can be compared with explicit calculations in the case of specific backgrounds to one loop case. It is understood that in the above analysis Mobius infinities, due to the $SL(2, C)$ invariance of the sigma model (which can be maintained even off-shell), are implicitly regulated in a renormalization group invariant way. Also in the actual string case, where redundant operators corresponding to target space-time symmetries have to be taken into account, relation (32) has to be understood under appropriate space-time integration. This corresponds to the translational zero mode of the string.

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Figure Captions

- Fig. 1.- The integration over τ_2 in the torus will be cut so that the invariant distance between τ_2^i and $\tau_2 = 0$ is $1/M$. Similar cut-offs will be applied to higher genera.
- Fig. 2.- Trivial (c) and non-trivial (a, b) cycles on a surface. Divergences are obtained when either a or c shrink to a point.
- Fig. 3.- Possible partitions of a genus 3 surface. The dotted lines indicate the "plumbing fixtures" described in the text. Divergences are obtained when the lengths of these necks diverge or the cycle a in the torus shrinks (or, equivalently by a modular transformation, when the length of the b cycle in a torus diverges).

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FIGURE 1

FIGURE 2

FIGURE 3