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FERMION MASSES FROM DIMENSIONAL REDUCTION

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Abstract

We consider the fermion masses in gauge theories obtained from ten dimensions through dimensional reduction on coset spaces. We calculate the general fermion mass matrix and we apply the mass formula in illustrative examples.

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1. Introduction

In recent years the attempts to unify all the observed interactions led to speculations that the fundamental theory should be defined in higher dimensions. Then the four-dimensional theories should be considered only as effective theories obtained from those defined in higher dimensions through some kind of dimensional reduction.

The obvious way to dimensionally reduce a theory is to demand that the field dependence on the extra coordinates is such that the Lagrangian is independent of the extra dimensions. A crude way to achieve that is to set to zero the field dependence on the extra coordinates, while an elegant way is to allow a field dependence and employ a symmetry of the Lagrangian. In the latter case a natural guess is to use the gauge symmetry. Then the dependence on the extra coordinates is required to be a gauge transformation. Therefore the Lagrangian is independent of the extra coordinates just because it is gauge invariant. This is the basis of the Coset Space Dimensional Reduction (CSDR) scheme [1-14], which dimensionally reduces down to four dimensions a gauge theory defined on a manifold $M^D = M^4 \times S/R$.

A fundamental test of any unified theory is its ability to accommodate small fermion masses. The observed fermion mass matrix (masses and mixings) which remains unexplained within the standard model has vanishingly small matrix elements as compared to the fundamental scale of any unified theory defined in four or higher dimensions. Therefore a promising unified theory should predict or at least permit very small fermion masses. For instance very interesting theories such as Kaluza-Klein theories fail to pass this test and are therefore rejected as candidates for a fundamental unified theory [15]. An alternative criterion for a unified theory is the ability to accommodate chiral fermions, a property that guarantees the masslessness of fermions. This criterion is not usually enough since the spontaneous symmetry breaking could produce superheavy fermion masses in a theory with vanishing fermion masses in the symmetric mode.

The theories that can be dimensionally reduced via CSDR are very interesting since they admit chiral fermions in four dimensions even in the case of the higher dimensional theory being vectorlike. On the other hand, in a large class of theories derived after dimensional reduction on coset spaces, fermions become superheavy after the spontaneous symmetry breaking of the four-dimensional theory. Therefore it is mandatory to examine in general the resulting fermion mass matrix after CSDR of a gauge theory defined in higher dimensions.

In the present paper we consider the general fermion mass matrix obtained via CSDR from ten-dimensional theories. The generality lies in the fact that we do not restrict the gauge group, we permit the coset space to be non-symmetric, with various radii when possible, and we

leave open the option to introduce torsion. Special cases which have been examined before are obtained by restricting our general formula.

2. The Coset Space Dimensional Reduction (CSDR) scheme.

Let us first recall the CSDR scheme [1-14]. In this scheme one starts with a pure gauge field and Dirac theory based on a gauge group G and defined on a manifold M^D which compactifies to $M^4 \times S/R$.

We assume that S/R is reductive but non-symmetric in general, so we can divide the generators of S , Q_A in two sets: the generators of R , Q_i ($i=1, \dots, \dim R$), and the generators of S/R , Q_a ($a=\dim R+1, \dots, \dim S$). Then the commutation relations for S are the following:

$$\begin{aligned} [Q_i, Q_j] &= f_{ij}^k Q_k \\ [Q_i, Q_a] &= f_{ia}^b Q_b \\ [Q_a, Q_b] &= f_{ab}^i Q_i + f_{ab}^c Q_c \end{aligned} \quad (1)$$

In the case where S/R is symmetric, then $f_{ab}^c = 0$. In order to define the vielbein let y^a be local coordinates on S/R . A representative $L(y)$ for each left coset of S with respect to R is chosen in a neighbourhood of the origin. Then the vielbeins are given by:

$$e_A^\alpha dy^\alpha = L^{-1}(y) dL(y) \quad (2)$$

From this relation we have that at $y=0$ $e_a^\alpha = \delta_a^\alpha$, $e_i^\alpha = 0$ a fact that we shall use later.

The vielbeins satisfy the Maurer-Cartan equations on the coset:

$$de_a^\alpha = -\frac{1}{2} f_{bc}^a e^{bc} e^\alpha, \quad de_i^\alpha = -\frac{1}{2} f_{bc}^i e^{bc} e^\alpha - \frac{1}{2} f_{jk}^i e^{jk} e^\alpha \quad (3)$$

where $e^{AB} = e^A_\lambda e^B_\lambda$.

The action of the theory in $D=4+d$ dimensions is the following:

$$A = \int d^4x d^{D-4}y \sqrt{-g} \left\{ -\frac{1}{4} \text{Tr}(F_{MN} F_{KL}) g^{MK} g^{NL} + \frac{1}{2} \bar{\psi} \Gamma^M D_M \psi \right\} \quad (4)$$

We use Greek letters to denote curved indices (while Latin letters will denote flat indices); K, L, M, N run over the D -dimensional space. As we discussed before we assume that transformations of the fields under symmetries of S/R are compensated by gauge transformations. So

let ξ_A^α be the symmetry generators of S (Killing vectors satisfying $[\xi_A, \xi_B]^\alpha = f_{AB}^\alpha C^\alpha$) and W_A be the compensating gauge transformation associated with each ξ_A^α . Therefore for the scalar, gauge (on the coset) and spinor fields we have respectively:

$$\begin{aligned}\delta_A(\phi) &= -\xi_A^\alpha \partial_\alpha \phi = D(W_A)\phi \\ \delta_A(A_\mu) &= -\xi_A^\alpha \partial_\alpha A_\mu - \partial_\mu \xi_A^\alpha A_\alpha = \partial_\mu W_A + [W_A, A_\mu] \\ \delta_A \psi &= -\xi_A^\alpha \partial_\alpha \psi + \frac{1}{4} \Omega_{ABC} \Sigma^{BC} \psi = D(W_A)\psi\end{aligned}\quad (5)$$

where $D(W_A)$ represents a gauge transformation in the appropriate representation of the fields, Ω_{ABC} represents the tangent space rotation of the spinor fields when they transform on a curved space and $\Sigma^{AB} = \frac{1}{2}[\Gamma^A, \Gamma^B]$.

The W 's must also satisfy the cocycle condition $([\delta_A, \delta_B] = f_{AB}^\alpha C_\alpha)$

$$\xi_A^\alpha \partial_\alpha W_B - \xi_B^\alpha \partial_\alpha W_A + [W_A, W_B] + f_{AB}^\alpha W_C = 0$$

These constraints can be solved at $y=0$ since the invariance conditions imply that the Lagrangian is independent of y .

Writing $\phi_i = W_i$ and ϕ_a for $e_a^\alpha A_\alpha$, as these fields become Lorentz scalars after dimensional reduction, we have the conditions (among others):

$$\begin{aligned}\alpha. \partial_a A_\mu &= 0 = \partial_a \psi & \delta. [\phi_i, A_\mu] &= 0 \\ \beta. [\phi_i, \phi_j] &= f_{ij}^k \phi_k & \epsilon. [\phi_i, \phi_a] &= f_{ia}^b \phi_b \\ \gamma. \partial_\mu \phi_i &= 0 & \sigma. \frac{1}{4} f_{\mu bc} \Sigma^{bc} \psi &= \phi_i \psi\end{aligned}\quad (6)$$

Solutions of these constraints provide us with the following results:

$\alpha)$ From 6α we see that the gauge fields A_μ and the spinor fields ψ are independent of the extra coordinates. Also 6γ and 6β indicate that there exist constants ϕ_i which generate an R subalgebra of G.

$\beta)$ From 6δ we see that A_μ commutes with the R subalgebra of G, so we have to embed R into G. The surviving symmetry in four dimensions is H, the centralizer of R in G.

$\gamma)$ Using Schur's lemma we can solve the constraints 6ϵ . The scalar fields ϕ_a transform under R as vectors, as specified by the embedding $\text{adS} = \text{adR} + \mathbf{y}$ and $\mathbf{y} = \sum_i s_i$ where each s_i is an irreducible representation of R. Then the multiplets under H of independent Higgs fields can be found by decomposing the adjoint representation of G under $R_G \times H$, $\text{adG} = \sum_k (r_k, h_k)$ where r_k and h_k are some irreducible representations of R_G and H respectively. For each pair of (r_k, s_i) where r_k and s_i are identical irreducible representations, there is a Higgs multiplet h_k in the four-dimensional theory.

$\delta)$ In the case of fermions we have to analyze the constraint 6σ . Using again Schur's lemma, first embed R in the Lorentz group $\text{SO}(N)$ of S/R in such a way that the N of $\text{SO}(N)$ has the branching rule $N = \mathbf{y}$ where $\mathbf{y} = \sum_i s_i$. Then we decompose σ , the spinor of $\text{SO}(N)$, into irreps of R ($\sigma = \sum_i \sigma_i$) and the representation F of the gauge group G, to which the fermions are assigned, under $R_G \times H$, $F = \sum_k (r_k, h_k)$. It turns out that for each pair of (r_k, σ_i) where r_k and σ_i are identical irreps, there is an h_k multiplet of spinor fields in the four-dimensional theory.

Taking into account all the constraints and integrating out the extra coordinates we obtain in 4 dimensions, the following Lagrangian:

$$\begin{aligned}A = C \int d^4 x \{ & \frac{1}{4} \text{Tr}(F_{\mu\nu} F^{\mu\nu}) + \frac{1}{2} \text{Tr}(D_\mu \phi_a D^\mu \phi_a) - V(\phi) \} \\ & + C \int d^4 x \frac{1}{2} \{ \bar{\psi} \Gamma^\mu D_\mu \psi + \bar{\psi} \Gamma^a D_a \psi \}\end{aligned}\quad (7)$$

where C is the volume of the coset space and the potential $V(\phi)$ is given by the formula

$$V(\phi) = \frac{1}{4} g^{ac} g^{bd} \text{Tr} \{ (f_{ab}^D \phi_D - f_{ab}^E \phi_E) (f_{cd}^E \phi_c - f_{cd}^D \phi_d) \} \quad (8)$$

Furthermore D_μ, D_a are the appropriate covariant derivatives:

$$\begin{aligned}D_\mu &= \partial_\mu - i A_\mu \\ D_a &= \partial_a - \frac{1}{4} \theta_{abc} \Sigma^{bc} - i \phi_a\end{aligned}\quad (9)$$

where θ_{abc} is the connection of S/R.

3. Geometry of non-symmetric spaces

It is evident from eq.(8) that the potential depends on the metric of the space S/R. In general the metric of the coset space is not restricted to be the usual Cartan-Killing metric of S on S/R. The most general, S-invariant metric on S/R satisfies:

$$f_{la}^b g_{db} + f_{ld}^b g_{ab} = 0 \quad (10)$$

The condition in eq.(10) allows the introduction of different scales (radii) for each R-irrep of the coset. To see this, note that $(f_i)_a^b$ serve as a matrix representation of R (one can prove that using the Jacobi identities of the coset), which is generally reducible. One can write $(f_i)_a^b$ in block form, which in a way represents R-irreps and consequently eq.(10) forces the radii to be the same in each block. Then the metric for the six-dimensional coset spaces, that we consider from here on, will be of the form:

$$g_{ab} = \text{diag}(a_1, \dots, a_6) \quad (11)$$

where some of the a_i 's are possibly the same. It should be noted that the coset indices a, b, c, etc., which are raised and lowered employing the metric g_{ab} , change in general the antisymmetry properties of the structure constants when different radii are involved. We choose the structure constants defined by eq.(1) to be fully antisymmetric.

For generality we allow the coset space to have torsion which is defined through the equation:

$$T^a = \frac{1}{2} T^a_{bc} \theta^{bc} = d\theta^a + \theta^a_b \wedge \theta^b \quad (12)$$

In the presence of torsion the curvature two-form Ω^a_b is given by:

$$\Omega^a_b = \frac{1}{2} \Omega^a_{bcd} \theta^{cd} = d\theta^a_b + \theta^a_c \wedge \theta^c_b \quad (13)$$

and we assume metricity so $\theta_{ab} = -\theta_{ba}$. In addition the connection in the presence of torsion is given by

$$\theta^a_b = \omega^a_b + \tau^a_b; \quad \tau^a_b = \frac{1}{2} \Sigma^a_c \theta^c_b \quad (14)$$

where ω^a_b is the connection in the absence of torsion and Σ^a_b is the contorsion tensor which is given in terms of the torsion tensor by $\Sigma_{abc} = T_{abc} + T_{bac}$.

Then using the Maurer-Cartan equations (3) we can calculate the torsionless part of the connection:

$$\omega^a_b = f_{lb}^a e^l + D_{cb}^a e^c \quad (15)$$

where

$$D_{cb}^a = g^{ad} \frac{1}{2} [f_{db}^e g_{ec} + f_{cb}^e g_{de} - f_{cd}^e g_{be}] \quad (16)$$

Substituting eq.(15) in (14) we obtain:

$$\theta^a_b = f_{lb}^a e^l + (D_{cb}^a + \frac{1}{2} \Sigma_{cb}^a) e^c = f_{lb}^a e^l + G_{cb}^a e^c \quad (17)$$

or, in component form, $\theta_c^a = e^l_c e^a_l + G_{cb}^a$.

Now the curvature two-form becomes:

$$\begin{aligned} \Omega^a_b = e^{jk} [& \frac{1}{2} f_{lb}^a f_{jk}^i + f_{jc}^a f_{kb}^c \\ & + e^{di} [G_{cb}^a f_{di}^c + G_{dc}^a f_{lb}^c - G_{db}^c f_{lc}^a] \\ & + e^{de} [\frac{1}{2} f_{lb}^a f_{de}^i - \frac{1}{2} G_{cb}^a f_{de}^c + \frac{1}{2} (G_{dc}^a G_{eb}^c - G_{ec}^a G_{db}^c)] \end{aligned} \quad (18)$$

The first term in eq.(18) vanishes due to the Jacobi identity, while the second term (a term with mixed indices in R and S/R which obviously we do not want) vanishes if we use eq.(10) and choose the contorsion tensor to be:

$$\Sigma_{abc} = 2\tau[D_{abc} + D_{bca} - D_{cba}] \quad (19)$$

where τ is a free parameter. We note that it is fully antisymmetric as it should be [14]. From eq.(18) we find that the curvature scalar is:

$$\Omega = \Omega^{ab}_{ab} = g^{be} f_{lb}^a f_{ae}^i - G_{cb}^a f_{ae}^c + G_{ac}^a G_{eb}^c - G_{ec}^a G_{ab}^c. \quad (20)$$

4. The fermion mass-matrix

The term $\bar{\psi} \Gamma^a D_a \psi$ in eq.(7) provides the four-dimensional fermions with Yukawa couplings and geometrical masses. In order to analyze it further we have to specify the Γ matrices. These obey the Dirac algebra:

$$\begin{aligned} \{\Gamma^\mu, \Gamma^\nu\} &= 2\eta^{\mu\nu} I_{10} \\ \{\Gamma^a, \Gamma^b\} &= 2g^{ab} I_{10} \end{aligned} \quad (21)$$

where $\eta^{\mu\nu}$ is the usual Minkowski metric and g^{ab} is the metric of the coset space S/R with radii. The Dirac algebra has a representation in terms of $2[D/2] \times 2[D/2]$ matrices. One can conveniently express the Dirac matrices in terms of the usual γ and σ -matrices. In ten dimensions we choose the following representation:

$$\Gamma^{p+3} = \frac{1}{\sqrt{a_p}} \gamma_5 \otimes \begin{pmatrix} 0 & -i\bar{\gamma}^p \\ i\bar{\gamma}^p & 0 \end{pmatrix} \Gamma^6 = \frac{1}{\sqrt{a_4}} \gamma_5 \otimes \begin{pmatrix} 0 & I_4 \\ I_4 & 0 \end{pmatrix} \quad (22)$$

where γ^μ, γ_5 are the usual four-dimensional γ -matrices and $\bar{\gamma}^p$ are:

$$\begin{aligned} \bar{\gamma}^1 &= \sigma^1 \otimes \sigma^2, & \bar{\gamma}^2 &= \sigma^2 \otimes \sigma^2, & \bar{\gamma}^4 &= -I_2 \otimes \sigma^3 \\ \bar{\gamma}^5 &= \sigma^3 \otimes \sigma^2, & \bar{\gamma}^6 &= -I_2 \otimes \sigma^1 \end{aligned} \quad (23)$$

In this representation Γ^{11} is:

$$\Gamma^{11} = \sqrt{\prod a_i} \Gamma^0 \dots \Gamma^{10} = \gamma_5 \otimes \begin{pmatrix} -I_4 & 0 \\ 0 & I_4 \end{pmatrix} \quad (24)$$

and the charge conjugation matrix C_{10} is:

$$C_{10} = C \otimes \begin{pmatrix} 0 & -\bar{\theta} \\ \bar{\theta} & 0 \end{pmatrix} \quad (25)$$

where C is the usual four-dimensional charge conjugation matrix and $\bar{\theta} = \sigma^2 \otimes iI_2$.

We write the ten-dimensional spinor ψ as eight four-dimensional Dirac spinors $\psi = (\psi_1, \dots, \psi_8)^T$. Next we impose the Weyl and Majorana conditions on the ten-dimensional spinor since in this way we obtain chiral fermions [2,13] at four dimensions even if we start with a vectorlike representation F for the fermions. The Weyl $\Gamma^{11}\psi = \psi$ and

Majorana $\psi = C_{10}(\bar{\psi})^T$ conditions force ψ to take the form

$$\psi = (\psi_1, \psi_2, \psi_3, \psi_4, \tilde{\psi}_2, \tilde{\psi}_1, \tilde{\psi}_4, \tilde{\psi}_3)^T \quad (26)$$

where $\tilde{\psi}_i = C(\bar{\psi}_i)^T$, for $i=1,3$, $\tilde{\psi}_j = C(\bar{\psi}_j)^T$, for $j=2,4$, $R, L = \frac{1}{2}(1 \pm \gamma_5)$. As was analyzed in sect.2, the constraint $6\sigma\tau$ determines the physical spinors. From eq.(26) we can extract the surviving four-dimensional spinors as follows. First we determine in which R -irrep of $SO(6)$ belongs each one of the ψ_i 's, and then we disregard the non-physical ones according to the rules of CSDR.

Substituting eq.(26) into eq.(7) we see that the first term of the Dirac Lagrangian $\frac{1}{2} \bar{\psi} \Gamma^\mu D_\mu \psi$ is just the kinetic interaction with the gauge-field term of the Dirac spinors ψ_i , $i=1, \dots, 4$, $i\bar{\Sigma} \bar{\psi}_{iL} \Gamma^\mu D_\mu \psi_{iL}$. The second term gives the geometrical mass matrix elements and the Yukawa couplings of the fermions with the scalars that potentially could appear in four dimensions.

$$\begin{aligned} \frac{1}{2} \bar{\psi} \Gamma^a D_a \psi &= \frac{1}{2} \bar{\psi} \Gamma^a \left[\partial_a - \frac{1}{4} e^j{}_a \omega^i{}_{jcb} \Sigma^{bc} - \frac{i}{4} G_{acb} \Sigma^{bc} i\phi_a \right] \\ &= \frac{1}{2} \bar{\psi} \Gamma^a D_a \psi + \bar{\psi} V \psi \end{aligned} \quad (27)$$

$$\text{where } \nabla_a = \partial_a - \frac{1}{4} e^j{}_a \omega^i{}_{jcb} \Sigma^{bc} i\phi_a; \quad (28)$$

$$\text{and } V = \frac{1}{8} G_{acb} \Gamma^a \Sigma^{bc} \quad (29)$$

Employing the CSDR constraints then as we saw in sect. 2, we have $e^j{}_a = 0$ and $\partial_a \psi = 0$. Then $V_a = -i\phi_a$ and the first term of eq.(27) becomes:

$$\begin{aligned}
& \bar{\psi}_{1L} \left(\frac{\phi_3}{\sqrt{a_3}} + i \frac{\phi_4}{\sqrt{a_4}} \right) \psi_{2L}^c - \bar{\psi}_{1L} \left(\frac{\phi_5}{\sqrt{a_5}} - i \frac{\phi_6}{\sqrt{a_6}} \right) \psi_{4L}^c + \bar{\psi}_{1L} \left(\frac{\phi_1}{\sqrt{a_1}} - i \frac{\phi_2}{\sqrt{a_2}} \right) \psi_{3L}^c \\
& - \bar{\psi}_{2L} \left(\frac{\phi_1}{\sqrt{a_1}} + i \frac{\phi_2}{\sqrt{a_2}} \right) \psi_{4L}^c - \bar{\psi}_{2L} \left(\frac{\phi_5}{\sqrt{a_5}} + i \frac{\phi_6}{\sqrt{a_6}} \right) \psi_{3L}^c + \bar{\psi}_{3L} \left(\frac{\phi_3}{\sqrt{a_3}} - i \frac{\phi_4}{\sqrt{a_4}} \right) \psi_{4L}^c + \text{h.c.} \quad (30)
\end{aligned}$$

This term is the Yukawa term and can give masses to fermions after spontaneous symmetry breaking. The second term in eq.(27) depends only on the geometry of the coset space. We shall explicitly calculate this term in a specific example in order to obtain some insight. However, a few remarks are in order. The operator V anticommutes with the six-dimensional helicity operator and commutes with the generators of R written in the form $Q_i = \frac{1}{2} f_{iab} \Sigma^{ab}$. One way to show this is by forming the rescaled algebra $Q'_a = \lambda_a Q_a$ (no summation) which has as structure constants the D_{abc} 's.

$$\begin{aligned}
[Q_i, Q_j] &= f_{ij}^k Q_k \\
[Q_i, Q'_a] &= f_{ia}^b Q'_b \\
[Q'_a, Q'_b] &= f_{ab}^c Q'_c + D_{ab}^c Q'_c
\end{aligned} \quad (31)$$

where $D_{ab}^c = \frac{\lambda_a \lambda_b}{\lambda_c} f_{ab}^c$, $f_{ab}^c = \lambda_a \lambda_b f_{ab}^c$ and the λ 's depend only on radii. Then we can use the Jacobi identities of this algebra to show that V commutes with Q_i . In turn we can draw the conclusion that the non-vanishing elements of V are those which are singlets under the decomposition of 4 and $\bar{4}$ of $SO(6)$ under R .

In order to contact previous investigations on the subject [7-10], we calculate the square mass operator:

$$M^2 \psi = [-D^2 D_a + \frac{1}{4} \Omega - i \Sigma^{ab} F_{ab}] \psi \quad (32)$$

Using the constraints of CSDR this becomes

$$\begin{aligned}
M^2 \psi &= \frac{1}{4} (G_{pq}^a \Sigma^{pq}) (G_{abc} \Sigma^{bc}) \psi - i G_{abc} \Sigma^{bc} \phi^a \psi - \phi^a \phi_a \psi \\
&+ \frac{1}{4} \Omega \psi - i \Sigma^{ab} F_{ab} \psi
\end{aligned} \quad (33)$$

The features that make eqs. (33) and (27) general are the possibilities of a non-symmetric coset space with torsion and non-equal radii. It is obvious that when we select $\tau=0$ in eq.(19) we have zero torsion, while

when the radii are equal, the metric is analogous to δ_{ab} , i.e., analogous to the Cartan-Killing metric on the coset. On the other hand the limit of a symmetric space can be obtained when $f_{abc}=0$, in which case the radii have to be equal.

Let us start by examining the case [7] where torsion is zero and the radii are equal. In this case eq.(33) becomes:

$$\begin{aligned}
M^2 \psi &= \left[\frac{3}{32} f_{abc} f_{abc} - \frac{1}{64} f_{pqabc} \gamma^{pqbc} \right] \psi - \frac{1}{2} f_{abc} \Sigma^{bc} \phi^a \psi - \phi^a \phi_a \psi \\
&+ \frac{1}{4} f_{iab} f_{iab} \psi - i \Sigma^{ab} F_{ab} \psi
\end{aligned} \quad (34)$$

Using the constraint $6\sigma\tau$ and the Jacobi identities we arrive at:

$$\begin{aligned}
M^2 \psi &= \left[\frac{3}{32} f_{abc} f_{abc} + \frac{7}{64} f_{pqabc} \gamma^{pqbc} \right] \psi - \frac{1}{2} f_{abc} \Sigma^{bc} \phi^a \psi \\
&- [\phi^a \phi_a + 2\phi^i \phi_i] \psi - i \Sigma^{ab} F_{ab} \psi
\end{aligned} \quad (35)$$

Let us assume first that the theory is not spontaneously broken. This in turn means that $\langle \phi^a(x) \rangle = 0$ and $F_{ab} = -f_{ab}^i \phi_i$. In addition the coset space should be non-symmetric, given that for symmetric coset spaces the theory is necessarily spontaneously broken [9]. Then some fermions acquire geometrical masses. These fermions are more specifically those which are singlets under R and give non-vanishing matrix elements to the operator V in eq. (29). This observation has very interesting phenomenological implications. For example, in supersymmetric theories defined in higher dimensions, it means that the gauginos obtained in four dimensions after dimensional reduction obtain super-heavy masses i.e. supersymmetry is broken at the compactification scale.

Next we turn to the case in which the theory is spontaneously broken. Then some of the ϕ 's acquire vev's and give masses to fermions comparable to the compactification scale (this point will be elaborated more in an example). Note that some of the fermion and scalar fields appearing in eq. (30) do not survive in four dimensions if they do not satisfy the constraints 6 ϵ . Therefore the specific fermion mass matrix, after spontaneous symmetry breaking, is very model dependent. However there exist a class of models in which the square fermion mass matrix can be expressed in terms of Casimirs. This is the class of models that satisfy $S \subset G$ resulting in $V(\phi)|_{\phi=\phi_{\min}} = 0$ or $F_{ab}=0$. Thus for a symmetric coset space

$$M^2 \psi = -[\phi^a \phi_a + 2\phi^i \phi_i] \psi = (C_S + C_R) \psi \quad (36)$$

where C_S and C_R are the quadratic Casimirs of S and R respectively, having non-negative eigenvalues for all representations.

If we abandon the CSDR constraints, we have to use harmonic analysis for the fermion fields since now they can be in every R -irrep. This is the case examined by the authors of refs. [8] and [10] for non-symmetric spaces and [9] for symmetric ones. These authors choose as gauge group $G=R$ and $\phi_a = e^i \gamma_a^\dagger Q_i$, $F_{ab} = -f_{ab} \phi_i$, a choice which considerably simplifies the subsequent analysis. Their result is that there are many massless modes occurring in higher R -irreps. We should stress that we obtain the same results in the much more general frame that we examine here where we have introduced torsion and non-equal radii, provided we make the same choice for the gauge group. The reason is that the generalized operator V given in eq.(29) has the same properties as the operator V in ref.[8], namely it anticommutes with the six-dimensional helicity operator and commutes with the generators of R . Finally if we set $\phi_a = 0$, as we should do when the theory is not spontaneously broken, eq.(33) is identical to the formula obtained in Kaluza-Klein theories [15]. Then the analysis of ref.[16] which uses torsion and harmonic analysis to determine the zero fermion modes is immediately applicable.

5. Examples

We can illustrate the above general formula given in eq. (27) in specific examples. Let us first consider the case where $G=SU(5)$ and $S/R=Sp(4)/(SU(2) \times U(1))_{\text{non-max}}$ and the fermions of the ten-dimensional theory transform according to the adjoint. The fully antisymmetric structure constants of $Sp(4)$ are given by:

$$\begin{aligned} f_{13}^6 &= -f_{14}^5 = f_{23}^5 = f_{24}^6 = 1 \\ f_{71}^6 &= -f_{72}^5 = f_{81}^5 = f_{82}^6 = f_{91}^2 = -f_{95}^6 = f_{101}^2 = f_{105}^6 = 1 \\ f_{78}^9 &= f_{103}^4 = 2 \end{aligned}$$

The indices 7, 8, 9 and 10 refer to $SU(2) \times U(1)$ and the rest to the coset.

Solving eq.(10) we find

$$g_{ab} = \text{diag}(a, a, b, b, a, a) \quad (37)$$

So $a_1 = a_2 = a_5 = a_6 = a$ and $a_3 = a_4 = b$ in eq.(11). The non-zero D 's (up to their symmetries) are given by:

$$\begin{aligned} D_{36}^1 &= D_{13}^6 = -D_{45}^1 = -D_{54}^6 = D_{23}^5 = D_{35}^2 = D_{46}^2 = D_{24}^6 = -\frac{b}{2a} \\ D_{61}^3 &= -D_{51}^4 = D_{52}^3 = D_{42}^6 = \frac{a}{b} \left(1 - \frac{b}{2a}\right) \end{aligned}$$

We note here that we can form the rescaled algebra introduced in eq.(35) by choosing $\lambda_1 = \lambda_2 = \lambda_5 = \lambda_6 = \sqrt{\frac{1}{2} \left(1 - \frac{b}{2a}\right)}$ and $\lambda_3 = \lambda_4 = \frac{b}{2a}$. We can now easily calculate the tensor $G_{abc} = D_{abc} + \frac{1}{2} \Sigma_{cab}$

$$\begin{aligned} G_{136} &= G_{613} = -G_{145} = -G_{514} = G_{235} = G_{523} = G_{246} = G_{426} = -\frac{b + \tau(b+2a)}{2} \\ G_{361} &= -G_{451} = G_{352} = G_{462} = -\frac{2a - b + \tau(b+2a)}{2} \end{aligned}$$

Then we can calculate the operator V defined in eq.(29):

$$\bar{\psi} V \psi = (1 + 6\tau) \left(\frac{b+2a}{8a\sqrt{b}} \right) \bar{\psi}_{1L} \psi_{1L}^c + \text{h.c.} \quad (38)$$

To contact the CSDR scheme we have to decompose the adjoint of $Sp(4)$ under $(SU(2) \times U(1))_{\text{non-max}}$

$$10 = 3(0) + 1(0) + 1(2) + 1(-2) + 2(1) + 2(-1) = \text{ad} R + \chi \quad (39)$$

Then we embed $(SU(2) \times U(1))_{\text{non-max}}$ in $SO(6)$ in such a way that $6 = \chi$. With this embedding, the spinor and the antispinor of $SO(6)$ decompose under $(SU(2) \times U(1))_{\text{non-max}}$ as

$$4 = 1(0) + 1(2) + 2(-1); \quad \bar{4} = 1(0) + 1(-2) + 2(1) \quad (40)$$

In order to identify under which R -irrep the four-dimensional spinors in eq.(26) transform we have to calculate the action of the generators of $SU(2) \times U(1)$ on the spinors. We saw that the embedding of $SU(2) \times U(1)$ in $SO(6)$ is determined from the generators $Q_i = \frac{1}{4} f_{iab} \Sigma^{ab}$. So calculating

$$\delta(Q_i) \psi = \frac{1}{4} f_{iab} \Sigma^{ab} (\psi_1, \dots, \psi_8)^T$$

we find $Q_7 = \text{diag}(0, \sigma^1, 0, \sigma^1)$, $Q_8 = \text{diag}(0, \sigma^2, 0, \sigma^2)$, $Q_9 = \text{diag}(0, -\sigma^3, 0, -\sigma^3)$

for $SU(2)$ and $Q_{10} = \text{diag}(0, 2, -1, -1, -2, 0, 1, 1)$ for $U(1)$. So we see that ψ_1 transforms under R as $1(0)$, ψ_2 as $1(2)$ and (ψ_3, ψ_4) as $2(-1)$. Under this identification eq.(38) just confirms what we remarked earlier, i.e. the fact that the matrix elements of V are different from zero only for the spinors that appear in the decompositions of 4 and $\bar{4}$ ($1(0)$).

Examining the scalar sector, we have to decompose the adjoint of $SU(5)$ under $(SU(2) \times U(1))_{\text{non-max}}$. Let us first consider the case $S \subset G$. In that case we have:

$$\begin{aligned} SU(5) &\supset (SU(2) \times U(1)) \times U(1) \\ 24 &= 1(0, 0) + 2(1, -1) + 2(-1, -1) + 2(-1, 1) + 2(1, 1) \\ &\quad + 1(2, 0) + 1(0, 0) + 1(-2, 0) + 3(0, 0) \\ &\quad + 3(1, 0) + 3(0, 0) + 3(-1, 0) \end{aligned} \quad (41)$$

In order to specify the transformation properties under R of the scalar fields we follow ref.[12]. We choose the generators (Hermitian) of $Sp(4)$ according with this decomposition

$$T_{Sp(4)} = \{T_1(0), T_+(2), T_0(0), T_-(-2), T_-(1), T_+(-1)\}$$

In correspondence with the above choice we write for the fields

$$\{\phi_A, A=1, \dots, 10\} = \{\phi_1, \phi_+, \phi_0, \phi_-, \phi_c, \phi_{+c}\}$$

Using the commutation relations given in ref.[11] we can translate these back to our original basis

$$\begin{aligned} \phi_+ &= \frac{1}{2}(\phi_3 + i\phi_4); & \phi_- &= \frac{1}{2}(\phi_3 - i\phi_4); & \phi_1 &= \frac{1}{2}(\phi_1 + i\phi_2) \\ \phi_1 &= \frac{1}{2}(\phi_1 - i\phi_2); & \phi_2 &= \frac{1}{2}(\phi_5 + i\phi_6); & \phi_2 &= \frac{1}{2}(\phi_5 - i\phi_6) \end{aligned} \quad (42)$$

Using eqs. (38) and (42), eq.(27) becomes:

$$\begin{aligned} \bar{\psi} M \psi &= -\frac{2i}{\sqrt{b}} \bar{\psi}_{1L} \phi_+ \psi_{2L}^c - \frac{2i}{\sqrt{a}} \bar{\psi}_{1L} \phi_- \psi_{4L}^c - \frac{2i}{\sqrt{a}} \bar{\psi}_{1R} \phi_1 \psi_{3R}^c + \frac{2i}{\sqrt{a}} \bar{\psi}_{2L} \phi_1 \psi_{4L}^c \\ &\quad + \frac{2i}{\sqrt{a}} \bar{\psi}_{2L} \phi_{2+} \psi_{3L}^c - \frac{2i}{\sqrt{b}} \bar{\psi}_{3L} \phi_- \psi_{4L}^c + (1+6\tau) \left(\frac{b+2a}{8a\sqrt{b}} \right) \bar{\psi}_{1L} \psi_{1L}^c + \text{h.c} \end{aligned} \quad (43)$$

The fermions transform under $R \times H$ as follows $2(-1, \pm 1)$, $1(2, 0)$, $2 \times 1(0, 0)$. Next we must calculate the potential of the theory and minimize it in order to determine the vev of the scalars. This has been done in

ref.[11] and we merely quote the result. The minimum of the potential occurs when all the fields acquire vacuum expectation value one. We find by calculating the determinant of the mass matrix in eq.(43) that there are no zero eigenvalues even at the cost of adjusting radii and the torsion of the coset space. Thus we confirm the result of ref. [7] that when $S \subset G$, fermions become superheavy, even in the more general context examined here.

Next let us examine a model in which S cannot be embedded in G . For simplicity we choose again $G = SU(5)$, $S/R = Sp(4)/(SU(2) \times U(1))_{\text{non-max}}$ and the fermions are assigned to the adjoint. However now the R is chosen according to the following decomposition:

$$\begin{aligned} SU(5) &\supset SU(3) \times SU(2) \times U(1) \\ 24 &= (8, 1)(0) + (1, 3)(0) + (\bar{3}, 2)(1) + (3, 2)(-1) + (1, 1)(0) \end{aligned}$$

Therefore the four-dimensional gauge group is $H = SU(3) \times U(1)$. Applying the CSDR rules, we find the following surviving in four dimensions scalars and fermions decomposed under H , a) scalars: $(\bar{3}, 1)$, $(3, -1)$ b) fermions: $(8, 0)$, $(1, 0)$, $(\bar{3}, 1)$, $(3, -1)$. Then eq.(43) becomes in this case:

$$\bar{\psi} M \psi = -\frac{2i}{\sqrt{a}} \bar{\psi}_{1L} \phi_{2-} \psi_{4L}^c - \frac{2i}{\sqrt{a}} \bar{\psi}_{1R} \phi_{1-} \psi_{3R}^c + (1+6\tau) \left(\frac{b+2a}{8a\sqrt{b}} \right) \bar{\psi}_{1L} \psi_{1L}^c + \text{h.c}$$

As was stressed in ref. [11], the theory remains unbroken if $a > b$ and breaks down if $a < b$. In the broken case only gauginos transforming as $1(0)$ and $8(0)$ obtain superheavy masses. In the broken case all fermion fields acquire masses and depending on the choice of the radii one eigenvalue is small as compared to the compactification scale.

Finally let us consider the extreme case of a theory based on the gauge group $SU(8)$ and dimensionally reduced over the coset space $SU(4)/SU(3) \times U(1)$. We choose the embedding of $SU(3) \times U(1)$ to be the one that appears in the decomposition $SU(8) \supset SU(3) \times SU(5) \times U(1)$. Then the branching rule of the adjoint of $SU(8)$ under $SU(3) \times SU(5) \times U(1)$ is:

$$63 = (1, 1)_0 + (8, 1)_0 + (3, \bar{5})_g + (\bar{3}, 5)_g + (1, 24)_0 \quad (44)$$

From eq.(44) we find that a) the four-dimensional gauge group is $H = SU(5)$ and b) (considering the decomposition of $SU(4)$ under $SU(3) \times U(1)$ $6 = 3(-2) + \bar{3}(-2)$), there are no surviving scalar fields in four

dimensions. For the fermion fields in this case we have that $4=1(3)+3(-1)$ and we see that there is no singlet under the R field which should acquire the large geometrical mass. So all fermions in this theory remain massless. In fact this model has been considered in ref. [13] where $SU(5)$ can break down to the standard model due to the Wilson flux breaking mechanism. The breaking of the $SU(2)_L \times U(1)$ was suggested to be dynamical, given the lack of surviving scalar fields in four dimensions.

In conclusion, we have obtained a general formula for the Yukawa couplings obtained from CSDR of gauge theories defined in ten dimensions. We showed that the fermion fields which are singlets under R acquire large geometrical masses comparable to the compactification scale at four dimensions. The fermion mass matrix after the spontaneous symmetry breaking of the potential is model dependent and depends on the surviving scalar fields and the vev that they acquire. The analysis allows non-symmetric coset spaces with different radii and torsion. The results of previous authors are deduced from of the general formula.

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