CERN-TH.5633/90

THE QUANTUM GROUPS $GL_q(n)$ AND WEYL-HEISENBERG OPERATORS

J. Weyers^{*)}
CERN - Geneva

ABSTRACT

It is shown that for deformation parameters $|q| = 1$, the bilinear product relations which define the quantum groups $GL_q(n)$ can be recast as Weyl-Heisenberg relations, provided some invertibility conditions are met. Finite- and infinite-dimensional representations of $GL_q(n)$ are then easily constructed, as shown in a few examples.

^{*)} On leave of absence from Institut de Physique Théorique, University of Louvain, Louvain-la-Neuve, Belgium.

In recent years there has been growing interest [1], both from the physical as well as from the mathematical point of view, in studying quantum "groups" and algebras. These structures play an essential rôle, for example, in the theory of integrable systems in 1+1 dimensions [2], in some statistical models [3] and in rational conformal field theories [4].

Quantum algebras are deformations of ordinary Lie algebras with, loosely speaking, the Yang-Baxter equation [5] playing the rôle of the Jacobi identity. For deformation parameters q which are not roots of unity, the representation theory of these algebras is completely similar [3,4,6] to the usual theory.

On the other hand, quantum "groups" - which are not groups in the usual sense of the word - can be thought of [1,7] as sets of matrices with elements which, in general, do not commute but obey various bilinear product relations. These relations can be specified [7] using an R matrix which, again, satisfies the Yang-Baxter equation.

Many of the bilinear product relations defining quantum groups have the form

$$AB = q^{-1}BA \quad (1)$$

When the deformation parameter has modulus one, Eq. (1) is of course reminiscent of the relation between Weyl-Heisenberg (WH) operators [8] in ordinary one-dimensional quantum mechanics ($\hbar = 1$):

$$W_X(\alpha) = \exp 2\pi i \alpha X \quad (2a)$$

$$W_P(\beta) = \exp 2\pi i \beta P \quad (2b)$$

with X and P the usual position and momentum operators and α, β arbitrary real numbers. The canonical commutation relation between X and P is equivalent to

$$W_X(\alpha) W_P(\beta) = \chi(-\alpha\beta) W_P(\beta) W_X(\alpha) \quad (3)$$

with the "character" $\chi(z) = \exp 2\pi iz$.

Equation (3) is of the form of Eq. (1) with the character $\chi(\alpha\beta)$ playing the rôle of the deformation parameter q . When $\alpha\beta$ is a rational number, q is of course a root of unity. Finite dimensional representations of the WH operators are also well known [9]: they are readily constructed, e.g., from quantum mechanics on finite number fields (instead of $\mathbb{R}$). Equation (3) then still holds with χ a character of the corresponding number field, i.e., some root of unity.

In this note we present a simple construction which allows one to recast all product relations defining the quantum groups $GL_q(n)$ with $|q| = 1$ in a WH form [i.e., the products are commutative or as in Eq. (1) with, possibly, different powers of q], provided some invertibility conditions are met. From the preceding remarks one is then led to a straightforward procedure for building finite- and infinite-dimensional representations of $GL_q(n)$. Here we will only sketch some simple examples.

1. The Quantum Groups $GL_q(n)$

Consider the $n \times n$ matrix $M = (m_{ij})$, $1 \leq i, j \leq n$. M belongs to the quantum group $GL_q(n)$ if its (quantum) determinant does not vanish and if its elements satisfy the bilinear relations (no summation on indices):

$$m_{ij} m_{ik} = q^{-1} m_{ik} m_{ij} \quad (j < k) \quad (4a)$$

$$m_{il} m_{jl} = q^{-1} m_{jl} m_{il} \quad (i < j) \quad (4b)$$

$$m_{ij} m_{kl} = m_{kl} m_{ij} \quad (i < k \text{ and } j > l) \quad (4c)$$

$$m_{ij} m_{kl} = m_{kl} m_{ij} + (q^{-1} - q) m_{il} m_{kj} \quad (i < k \text{ and } j < l) \quad (4d)$$

An elegant derivation of these $n^2(n^2-1)/2$ relations has been given by Corrigan et al. [7].

For $GL_q(2)$ in particular, Eqs. (4) give the six algebraic relations

$$m_{11} m_{12} = q^{-1} m_{12} m_{11} \quad m_{12} m_{22} = q^{-1} m_{22} m_{12} \quad (5a)$$

$$m_{11} m_{22} = q^{-1} m_{22} m_{11} \quad m_{12} m_{22} = q^{-1} m_{22} m_{12} \quad (5b)$$

$$m_{12} m_{21} = m_{21} m_{12} \quad (5c)$$

$$m_{11} m_{22} - m_{22} m_{11} = (q^{-1} - q) m_{12} m_{21} \quad (5d)$$

The (quantum) determinant of a 2×2 matrix is given by

$$\mathcal{D}_q \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = m_{11} m_{22} - q^{-1} m_{12} m_{21} \quad (6)$$

For a 3x3 matrix, the quantum determinant reads

$$\begin{aligned} \mathcal{D}_q &= m_{11} \mathcal{D}_q \begin{pmatrix} m_{22} & m_{23} \\ m_{32} & m_{33} \end{pmatrix} - q^{-1} m_{12} \mathcal{D}_q \begin{pmatrix} m_{21} & m_{23} \\ m_{31} & m_{33} \end{pmatrix} + q^{-2} m_{13} \mathcal{D}_q \begin{pmatrix} m_{21} & m_{22} \\ m_{31} & m_{32} \end{pmatrix} \\ &= m_{11} (m_{22} m_{33} - q^{-1} m_{23} m_{32}) - q^{-1} m_{12} (m_{21} m_{33} - q^{-1} m_{23} m_{31}) \\ &\quad + q^{-2} m_{13} (m_{21} m_{32} - q^{-1} m_{22} m_{31}) \end{aligned} \quad (7)$$

and, by recurrence, one similarly defines the quantum determinant of an $n \times n$ matrix for any n .

A crucial property for what follows is that the quantum determinant of a matrix M commutes with all matrix elements of M :

$$m_{ij} \mathcal{D}_q(M) = \mathcal{D}_q(M) m_{ij} \quad (8)$$

This is easily checked for 2x2 and 3x3 matrices and can be proved in general by recurrence.

When $\mathcal{D}_q(M) = 1$, for all M , the quantum group is denoted as $SL_q(n)$. With the quantum determinant and Eq. (8), one easily constructs the (left and right) inverse of a quantum matrix. In the case of $GL_q(2)$, for example, one finds [7]

$$M^{-1} = [\mathcal{D}_q(M)]^{-1} \begin{pmatrix} m_{22} & -q^{-1} m_{12} \\ -q^{-1} m_{21} & m_{11} \end{pmatrix} \quad (9)$$

Note that M^{-1} does not belong to $GL_q(2)$ but to $GL_{-1}(2)$!

For completeness, we recall that the relevance of the algebraic structure displayed in Eqs. (4) stems from the fact that with comultiplication [1,3,4] Δ ,

$$\Delta(M) = \sum_j m_{ij} \otimes m_{jk} \quad (10)$$

it is found that $\Delta(M)$ obeys the same algebraic relations, Eqs. (4). The matrices M can thus be considered as the "basic building blocks" for this structure. As shown explicitly in Ref. [7], Eqs. (4) can be written in tensor product form

$$\sum_{k,l} R_{ij,kl} m_{kp} m_{lq} = \sum_{k,l} m_{jl} m_{ik} R_{kl,pq} \quad (11)$$

with R a matrix satisfying the Yang-Baxter equation. The explicit R matrices obtained from Eqs. (11) and (4) turn out to be particular cases of the R -matrices constructed by Jimbo [1] for the deformed A_n algebras.

2. A WH Form of the Algebras

Equations (4d) are not of WH form. Using quantum determinants we will, so to speak, "solve" Eqs. (4d) and recast all algebraic product relations in a WH form. We give the detailed construction for $GL_q(2)$ and $GL_q(3)$ - where all algebraic manipulations can easily be checked explicitly - and then indicate (without proofs) how to extend the construction from $GL_q(n-1)$ to $GL_q(n)$.

a) $GL_q(2)$ [10]

Equation (6) can be "solved" for m_{11} , provided m_{22} is invertible:

$$m_{11} = \mathcal{D}_q m_{22}^{-1} + q^{-2} m_{12} m_{21} m_{22}^{-1} \quad (12)$$

This expression is then easily seen [10] to satisfy all product relations in Eqs. (5) which involve m_{11} . In other words, we replace the product relations Eqs. (5) by

$$m_{12} \mathcal{D}_q = \mathcal{D}_q m_{12} ; \quad m_{21} \mathcal{D}_q = \mathcal{D}_q m_{21} ; \quad m_{22} \mathcal{D}_q = \mathcal{D}_q m_{22} \quad (13a)$$

$$m_{12} m_{22} = q^{-1} m_{21} m_{12} ; \quad m_{21} m_{22} = q^{-1} m_{12} m_{21} \quad (13b)$$

$$m_{12} m_{21} = m_{21} m_{12} \quad (13c)$$

i.e., effectively eliminating m_{11} . Equations (13) are all of the WH form. Given any representation of Eqs. (13), with m_{22} invertible, we obtain, using Eq. (12), a representation of the quantum group $GL_q(2)$.

b) $GL_q(3)$

Let us denote by M_{ij} the "minor matrix" of m_{ij} , namely M_{ij} is the $(n-1) \times (n-1)$ matrix obtained by dropping the i th line and j th column in the $n \times n$ matrix M .

Similarly, $M_{ij;k}$ ($k \neq i, j$) will denote the $(n-2) \times (n-2)$ matrix obtained by dropping the i^{th} and k^{th} lines as well as the j^{th} and i^{th} columns of M . $M_{ij;k;mn}$, etc., are then analogously defined.

With M a 3×3 matrix, Eq. (7) now reads

$$\mathcal{D}_q(H) = m_{11} \mathcal{D}_q(M_{11}) - q^{-1} m_{12} \mathcal{D}_q(M_{12}) + q^{-2} m_{13} \mathcal{D}_q(M_{13})$$

and simple algebra gives

$$\mathcal{D}_q(H) = \mathcal{D}_q(M_{33}) m_{33} - q^{-1} \mathcal{D}_q(M_{32}) m_{23} + q^{-2} \mathcal{D}_q(M_{13}) m_{13} \quad (14)$$

We now use $\mathcal{D}_q(M_{11})$, $\mathcal{D}_q(M_{13})$, $\mathcal{D}_q(M_{31})$ and $\mathcal{D}_q(M)$ to eliminate (or solve for) m_{11} , m_{32} , m_{23} and m_{33} respectively. This requires m_{22} , m_{21} , m_{12} and $\mathcal{D}_q(M_{33})$ itself to be invertible. Provided this is the case, one finds again that the bilinear products involving m_{11} , m_{32} , m_{23} and m_{33} are automatically satisfied with these "solutions". The 36 products given by Eqs. (4) when expressed as bilinear products among m_{12} , m_{13} , m_{21} , m_{22} , m_{31} , $\mathcal{D}_q(M_{33})$, $\mathcal{D}_q(M_{13})$, $\mathcal{D}_q(M_{31})$ and $\mathcal{D}_q(M)$ lead to 26 commutative products (which we do not list), and the following 10 relations:

$$\begin{aligned} m_{12} m_{22} &= q^{-1} m_{22} m_{12} ; & m_{12} m_{13} &= q^{-1} m_{13} m_{12} \\ m_{22} m_{22} &= q^{-1} m_{22} m_{22} ; & m_{22} m_{32} &= q^{-1} m_{32} m_{22} \\ \mathcal{D}_q(M_{33}) m_{13} &= q^{-1} m_{13} \mathcal{D}_q(M_{33}) ; & \mathcal{D}_q(M_{33}) m_{23} &= q^{-1} m_{23} \mathcal{D}_q(M_{33}) \\ m_{22} \mathcal{D}_q(M_{32}) &= q^{-1} \mathcal{D}_q(M_{32}) m_{22} ; & m_{12} \mathcal{D}_q(M_{12}) &= q^{-1} \mathcal{D}_q(M_{12}) m_{12} \quad (15) \\ \mathcal{D}_q(M_{33}) \mathcal{D}_q(M_{32}) &= q^{-1} \mathcal{D}_q(M_{32}) \mathcal{D}_q(M_{33}) \\ \mathcal{D}_q(M_{32}) \mathcal{D}_q(M_{12}) &= q^{-1} \mathcal{D}_q(M_{12}) \mathcal{D}_q(M_{32}) \end{aligned}$$

Hence, as promised, all 36 bilinear products in the "new variables" are of the WH form. There is no mystery in the proof of any of these product relations: all ten equations (15) follow from Eqs. (4) and (8) with straightforward algebraic manipulations.

It is worth pointing out that m_{12} , m_{21} , m_{22} and $\mathcal{D}_q(M_{33})$ satisfy the WH form of the GL(2) product relations, namely Eqs. (13). In going from 2×2 to 3×3 matrices, one has added m_{13} and m_{31} (whose product relations are always of the

WH form) plus the three quantum determinants $\mathcal{D}_q(M_{13})$ - which allows one to "solve" Eqs. (4) for m_{32} -, $\mathcal{D}_q(M_{31})$, which plays a similar rôle for m_{23} , and finally $\mathcal{D}_q(M)$, which allows the elimination of m_{33} . The general construction is now almost obvious.

c) From GL(3) to GL(4) and from GL(n-1) to GL(n)

Let us first consider a 4×4 matrix $M \in GL_q(4)$. The WH form of Eqs. (4) will hold for bilinear products of m_{12} , m_{13} , m_{14} , m_{21} , m_{22} , m_{31} , m_{41} and the quantum determinants $\mathcal{D}_q(M_{44})$, $\mathcal{D}_q(M_{44;13})$, $\mathcal{D}_q(M_{44;31})$, $\mathcal{D}_q(M_{44;33})$, together with $\mathcal{D}_q(M_{14})$, $\mathcal{D}_q(M_{14;23})$, $\mathcal{D}_q(M_{41})$, $\mathcal{D}_q(M_{41;32})$ and, finally, $\mathcal{D}_q(M)$.

Again, m_{12} , m_{13} , m_{21} , m_{22} , m_{31} , $\mathcal{D}_q(M_{44})$, $\mathcal{D}_q(M_{44;13})$, $\mathcal{D}_q(M_{44;31})$ and $\mathcal{D}_q(M_{44;33})$ are the nine elements through which the GL(3) bilinear relations have been re-expressed in a WH form, Eqs. (15). To go from GL(3) to GL(4) in WH form, one has added m_{14} , m_{41} , $\mathcal{D}_q(M_{14})$, $\mathcal{D}_q(M_{14;23})$, $\mathcal{D}_q(M_{41})$, $\mathcal{D}_q(M_{41;32})$ and $\mathcal{D}_q(M)$. The matrices M_{14} , $M_{14;23}$ are, respectively, the minor matrices of m_{14} in M and of m_{23} in M_{14} : they are obtained by "going down" the off-diagonal of M . Similarly, M_{41} and $M_{41;32}$ result from "going up" the same off-diagonal. With invertibility conditions which we no longer list, the quantum determinants of the matrices obtained in going down the off-diagonal allow the elimination of m_{33} and m_{42} , while those obtained in going up play the same rôle for m_{34} and m_{24} . Finally, $\mathcal{D}_q(M)$, the quantum determinant of the full matrix allows one to get rid of m_{44} . That these "solutions" for m_{41} ($1 > 2$) and m_{j4} ($j > 2$) are compatible with Eqs. (4) can still be checked by hand, but the algebra becomes quite lengthy. We remark that one can also prove this by "recurrence".

A novel feature in the WH form of the product relations of GL(4) is that higher powers of q^{-1} appear in some products, e.g.:

$$\mathcal{D}_q(M_{44;33}) \mathcal{D}_q(M_{44;32}) = q^{-2} \mathcal{D}_q(M_{44;32}) \mathcal{D}_q(M_{44;33}) \quad (16)$$

but since $|q| = 1$, this presents no particular problem from our point of view.

The general construction from GL(n-1) to GL(n) follows word for word the pattern just described for $n = 4$: to the set of elements which express the GL(n-1) product relations in WH form, one adds, to obtain the WH form of GL(n):

(A) m_{1n} and m_{n1} ;

(B) the quantum determinants $\mathcal{D}_q(M_{1n})$, $\mathcal{D}_q(M_{1n;2(n-1)}) \dots \mathcal{D}_q(M_{1n;n-1})$, $\mathcal{D}_q(M_{n1;n-1})$.

They allow one to solve for m_{nl} ($2 \leq i < n-1$) provided appropriate invertibility conditions are met;

(C) the quantum determinants $\mathcal{D}_q^{(M)}, \mathcal{D}_q^{(M_{n1};(n-1)_2)}, \dots, \mathcal{D}_q^{(M_{n1};\dots;3(n-2))}$ which, similarly, allow us to solve for m_{in} ($2 \leq i < n-1$);

(D) the quantum determinant $\mathcal{D}_q^{(M)}$ of the full $n \times n$ matrix which is used to eliminate m_{nn} .

We note that in the resulting WH form of $GL_q(n)$ all powers of q^{-1} , from 1 to $[n/2]$, will appear.

3. FINITE- AND INFINITE-DIMENSIONAL REPRESENTATIONS

To conclude this note, we sketch with some simple examples a procedure for obtaining representations of the WH form of $SL_q(2)$ and $SL_q(3)$. For $SL_q(2)$, namely Eqs. (13), take for example

$$m_{12} = W_P(\beta_{12}) = \exp 2\pi i \beta_{12} P \quad (17a)$$

$$m_{21} = W_P(\beta_{21}) W_X(\alpha_{12}) = \exp 2\pi i \beta_{21} P \cdot \exp 2\pi i \alpha_{12} X \quad (17b)$$

$$m_{11} = W_X(\alpha_{12}) = \exp 2\pi i \alpha_{12} X \quad (17c)$$

All Eqs. (13) are then satisfied, provided

$$q = X(-\alpha_{12} \beta_{12}) = X(-\alpha_{12} \beta_{21}) ; \quad X(\alpha_{12} \beta_{12}) = 1 \quad (18)$$

The last equalities amount to the congruences

$$\alpha_{12} \beta_{12} \equiv \alpha_{12} \beta_{21} \pmod{1} \quad \alpha_{12} \beta_{12} \equiv 0 \pmod{1} \quad (19)$$

With $|q| = 1$, Eqs. (18) and (19) are trivial to solve, while Eq. (12) now reads ($\mathcal{D}_q = 1$):

$$m_{12} = W_X(-\alpha_{12}) + X(\alpha_{12} \beta_{12}) W_P(\beta_{12} + \beta_{21}) W_X(\alpha_{12} - \alpha_{12}) \quad (20)$$

Equations (17) and (20) give a simple infinite dimensional representation of $GL_q(2)$, namely of Eqs. (5).

A similar procedure has been used by Floratos [10] to build finite-dimensional representations of Eqs. (13). For $q = \exp(2\pi i/p)$, p prime, with

$$W_P(\beta) = h^\beta \quad 1 \leq \beta \leq p \quad (21a)$$

$$W_X(\alpha) = g^\alpha \quad 1 \leq \alpha \leq p \quad (21b)$$

where g and h are $p \times p$ matrices [7,10],

$$g = \text{diag}(1, q, q^2, \dots, q^{p-1}) ; \quad h = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

one easily obtains from equations similar to Eqs. (17) the solutions described in Ref. [10].

The same procedure works for $SL_q(3)$ (and for higher n as well) by "tensorizing" Eqs. (2). For example, one may write the nine elements appearing in Eqs. (15) in the form

$$A = W_P(\vec{\beta}_A) W_X(\vec{\alpha}_A) = [\exp 2\pi i \vec{\beta}_A \cdot \vec{P}] \cdot [\exp 2\pi i \vec{\alpha}_A \cdot \vec{X}] \quad (22)$$

where the dimensions of $\vec{X}$ and $\vec{P}$ correspond to the number of independent degrees of freedom. With A and B written as in Eq. (22), Eq. (3) gives

$$AB = X(-\vec{\alpha}_A \cdot \vec{\beta}_B + \vec{\beta}_A \cdot \vec{\alpha}_B) BA$$

Taking $q = \exp(2\pi i/p)$, Eqs. (15), as well as the 26 commuting bilinear relations, reduce to simple congruences analogous to Eqs. (18) and (19). With $\vec{P}$ of dimension 3, say, these are again trivially solved. The same construction works for finite dimensional representations ["tensorizing" Eqs. (21)]. Details will be given elsewhere.

We end with a comment: Weyl operators can be constructed on p -adic number fields as well as on real numbers [11]. This strongly suggests that adelic representations [12] of quantum groups may be worth exploring.

ACKNOWLEDGEMENTS

It is a pleasure to thank L. Alvarez-Gaumé, J. Nuyts, Ph. Ruelle, R. Stora, E. Thiran and B. Zumino for discussions.

REFERENCES

- [1] L.D. Faddeev, N. Reshetikhin and L.A. Takhtajan, "Quantization of Lie Groups and Lie Algebras", LOMI preprint, Leningrad (1987);
V. Drinfeld, Proceedings ICM, Berkeley (1986);
M. Jimbo, Comm. Math. Phys. 102 (1986) 537;
Yu.I. Manin, Annales de l'Institut Fourier 37 (1987) 191;
J. Fröhlich, F. Gabbiani and P.A. Marchetti, ETH preprint TH/89-36 (1989) and references quoted therein.
- [2] L.A. Takhtajan, "Quantum Groups and Integrable Models", Proceedings of the Taniguchi Symposium, Kyoto (1988).
- [3] H. Saleur and V. Pasquier, Saclay preprint SPHT/89-031 (1989).
- [4] L. Alvarez-Gaumé, C. Gomez and G. Sierra, CERN preprint TH.5369/89 (1989).
- [5] C.N. Yang, Phys. Rev. Lett. 19 (1967) 1312;
R.J. Baxter, "Exactly Solved Models in Statistical Mechanics", Academic Press, London (1982);
For a recent review, see, e.g., M. Madaffi, T. Deguchi and Y. Akutsu, "Exactly Solvable Models and Knot Theory", Physics Reports 180 (1989) 247.
- [6] H. Ruegg, Geneva preprint UGVA-DPT 1989/08-625 (1989).
- [7] See, e.g., E. Corrigan, D.B. Fairlie, P. Fletcher and R. Sasaki, DTP preprint 89/29 (1989), and references therein.
- [8] H. Weyl, "The Theory of Groups and Quantum Mechanics", Dover (1950).
- [9] J.H. Hannay and M.V. Berry, Physica 1D (1980) 267;
Y. Meurice, Lectures given at the Université Catholique de Louvain (1987).
- [10] For $GL_q(2)$, this was first noticed in :
E.G. Floratos, Phys. Lett. B233 (1989) 395.
- [11] Ph. Ruelle, E. Thiran, D. Versteegen and J. Meyers, J. Math. Phys. 30 (1989) 2854.
- [12] I.M. Gelfand, M.I. Graev and I.I. Pyatetskii-Shapiro, "Representation Theory and Automorphic Functions", Saunders, London (1966).