Two photon bremsstrahlung

Ronald Kleiss
TH Division, CERN, Geneva

Steven van der Marck
Instituut-Lorentz, University of Leiden, The Netherlands

Abstract

We present compact expressions for the matrix elements for double initial-state bremsstrahlung in the process $e^+e^- \rightarrow f\bar{f}$ at LEP energies. Special attention is given to kinematical situations where one or both of the bremsstrahlung photons are collinear with the incoming electrons: this entails a careful description of the transition region between collinear and noncollinear kinematics. The resulting expressions allow for simple and straightforward computer implementation. We describe a Monte Carlo based on these matrix elements, which contains two-loop corrections to the initial state and one-loop corrections to the final state. Its results are compared with other analytical and Monte Carlo studies.

1 Introduction

During the last years there has been considerable progress in the field of high-precision phenomenology of the Standard Model. In particular the computation of radiative corrections for the muon/quark pair production process at LEP1, $e^+e^- \rightarrow f\bar{f}$, has been completely performed to the one-loop, or $\mathcal{O}(\alpha)$, level [1]. These corrections can be divided into ‘purely weak’ ones and QED ones; the QED corrections can be described by all Feynman diagrams where a virtual or real photon has been added to the Born level graphs, while weak corrections consist of all other one-loop diagrams. Due to the possibility of having additional particles (i.e. bremsstrahlung photons) in the final state, the QED effects are for practical applications best described with the use of Monte Carlo techniques (although extremely important and useful analytical results have also been obtained [2,3]).

It is well known that, at least around the Z resonance, one-loop corrections are in general not sufficient for a reliable and accurate phenomenological description, and higher-order corrections have to be taken into account, at least in the QED sector. It has been established that the weak corrections are typically small, provided we express the Born level cross section in terms of judiciously chosen parameters; this results in the use of the so-called ‘improved’ or ‘modified’ Born formulae, for which many different forms have been proposed [4]. Assuming that we write the cross section with the one-loop weak corrections in this way, the purely weak higher-order corrections can be neglected at the level of accuracy aimed for in the LEP experiments. This leaves us with the problem of dealing with the higher-order virtual photon and bremsstrahlung effects. The most important and well-known way of treating these higher-order QED effects is the so-called soft-photon exponentiation. First treated in a consistent and thorough manner in the classical paper by Yennie, Frautschi and Suura (YFS) [5], this amounts to a resummation of the leading terms in all higher-order virtual and real-photon graphs, where the total amount of energy radiated off by the bremsstrahlung photons is restricted to be below some value $\Delta E \ll E_b$, where $E_b = \sqrt{s}/2$ is the beam energy.

While theoretically attractive, this approach to the radiative corrections can nowadays not be considered adequate. The use of an energy cutoff ΔE is only an attempt to implement the influence of experimental cuts, however crudely: in typical experimental situations, ΔE is not a fixed number but may for instance depend on the production angles of the fermion pair¹. Also, the bremsstrahlung photons cannot always be treated in the soft-photon approximation: in particular when a photon is emitted close to the beam direction, or collinear with one of the produced fermions, its energy may be a quite sizeable fraction of E_b under realistic experimental conditions. So, in the summation of the higher-order effects, nonleading terms may become relevant, of which the precise exponentiation prescription is not at present known, although a number of promising Ansätze exist [6]. The last drawback in the use of YFS exponentiation is the fact that the formulae as they appear in [5] concern only *inclusive* cross sections, that is, cross sections in which integration has been performed over the photon momenta. If one is only interested in the total cross section σ or the fermion angular distribution $d\sigma/d\Omega_f$ this is of course desirable, but for a Monte Carlo treatment it is less adequate since in a Monte Carlo simulation (where every event must have a well-defined, positive probability) we want to know the *exclusive*, multidifferential cross section $d\sigma/d\Omega_f dk_1 d\Omega_1 dk_2 d\Omega_2 \dots dk_n d\Omega_n$ (where k_i and Ω_i denote the energy and solid angle of

¹Here and in the following, ‘fermion’ stands for muon, tau lepton, their associated neutrino types, or any of the light quarks.

bremsstrahlung photon i), for arbitrary n .

In an extremely important and beautiful paper, Jadach and Ward [7] have shown that the YFS approach of [5] can in fact be transformed so as to give information on the above complete multidifferential distribution, and they have presented an algorithm for actually generating such distributions. Whereas in this way one of the main obstacles to ‘exclusive exponentiation’ has been overcome, a number of nontrivial issues remain to be settled. In the first place, the Jadach-Ward approach [7] is only valid at present for initial-state radiation. The radiative corrections from the final-state fermions as well as the interference between initial- and final-state radiation remains to be adequately implemented in a similar algorithm. In the second place, the use of exclusive exponentiation enables us to use the full matrix elements for each order in the QED perturbation series: however, in all existing implementations at most only the $\mathcal{O}(\alpha)$ level is treated exactly, and the higher-order matrix elements are implemented only approximately.

It is the purpose of the present paper to describe the implementation of the two-loop bremsstrahlung matrix element, that is, the matrix elements for $e^+e^- \rightarrow f\bar{f}\gamma\gamma$, where the photons are emitted from the initial state. For massless fermions and electrons, these matrix elements have been given by the CALKUL collaboration [8] for the case of photon exchange and are therefore not appropriate for LEP physics: we generalize this treatment to the case of photon and Z exchange. In addition, the CALKUL group has indicated how mass effects are to be taken into account [9]: however this treatment is not really suited to computer implementation, and we also extend and improve on that. An additional feature of our result is that we give separate helicity amplitudes so that the possibility of polarized e^+e^- beams is included.

The layout of this paper is as follows. In section 2 we present all the necessary formulae for the matrix elements, based on the use of spinor techniques [10]. A good deal of the section is devoted to a discussion on how these formulae are to be used in an actual Monte Carlo program: in particular the question of how to compute the mass effects turns out to be far from trivial. In section 3 we describe the implementation of our results in the Monte Carlo program FPAIR. This program simulates carefully the emission of up to two hard bremsstrahlung photons from the initial state, and up to one from the final state. Not complete yet in the sense that we have not included exponentiation so far, its strength is rather in the precise description of explicitly hard-photon events. We describe all algorithms leading to the full simulation. In section 4 we compare our results with some other formulae and algorithms present in the literature. In particular, we are in a good position to compare FPAIR with the Monte Carlo program DYMU2 by Campagne and Zilouni [11]. Since DYMU2 also generates up to 2 photons from the initial-, and up to 1 from the final state, it is ideally suited to this kind of comparison. It should be noted that in DYMU2 the alternative structure function approach [12] to the higher-order QED effects has been adopted. This provides an interesting possibility of checking the structure function approach against a more exact treatment, which both have their advantages and drawbacks. Finally, in section 5 we draw our conclusions and indicate the directions of further development.

2 Matrix elements

In this section we present the matrix elements for the process

$$e^+(p_+, \lambda_+) e^-(p_-, \lambda_-) \rightarrow f(q_1, \lambda_f) \bar{f}(q_2, \lambda_{\bar{f}}) \gamma(k_1, \lambda_1) \gamma(k_2, \lambda_2) \quad (1)$$

where the two photons are emitted from the initial-state electron and positron. In Eq.(1) we have indicated the momenta as well as the helicity indices $\lambda = \pm$ of the various particles. Since we are also going to consider collinear photon radiation, where the masses of the e^+e^- cannot be neglected, the helicities λ_+ and λ_- are to be taken as independent of each other; on the other hand, we shall take the final-state fermions to be massless in this section, and helicity conservation along massless fermion lines implies that in our convention (which is described in detail in appendix A) the helicity indices for both f and $\bar{f}$ must be equal. Since it will turn out that we also need the zero- and one-photon amplitudes we shall also give them: these have of course been given many times in the literature, but we include them for completeness and in order to be able to work in a consistent manner in the case of collinear radiation.

For computational tools we use the so-called spinor techniques which are described in great detail in [10]. We present a summary of the most important results in appendix A for completeness. The basic quantity is the so-called spinor product which we define as

$$s_{\lambda_1, \lambda_2}(p_1, p_2) = \bar{u}_{\lambda_1}(p_1) u_{\lambda_2}(p_2) ; \quad (2)$$

for massless fermions (i.e. $p_1^2 = p_2^2 = 0$) the product vanishes if $\lambda_1 = \lambda_2$ but if one or both momenta are massive it is also nonzero. Another point of importance is the choice of the polarization vector $\epsilon_{\lambda_i}^\mu(k_i)$ of photon $i = 1, 2$. It turns out to be most convenient to use also a spinorial form:

$$\epsilon_{\lambda_i}^\mu(k_i) = \frac{\lambda_i}{\sqrt{2}} \frac{\bar{u}_{\lambda_i}(k_i) \gamma^\mu u_{\lambda_i}(q_i)}{s_{-\lambda_i, \lambda_i}(k_i, q_i)} . \quad (3)$$

As demonstrated in [10], this satisfies the usual requirements for definite-helicity states ($\epsilon_{\pm} \cdot k_i = 0$, $(\epsilon_{\pm} \cdot \epsilon_{\pm}) = 0$, $(\epsilon_{+} \cdot \epsilon_{-}) = -1$ and $(\epsilon_{\pm}^\mu)^* = \epsilon_{\mp}^\mu$). The vector q_i^μ is an auxiliary massless vector with $(q_i, k_i) \neq 0$, a choice of which fixes a gauge for the external photon i . The representation (3) also has the nice property that in each gauge-invariant subset of diagrams q_i^μ can be chosen completely arbitrarily, without the need to take into account relative complex phases which plagued the original CALKUL results [8]. The freedom of choice of q_i^μ allows an optimization of the calculation, which we employ as follows. In initial-state radiation, all bremsstrahlung photon lines are attached to the e^+e^- fermion line which starts with $\bar{u}_{\lambda_+}(p_+)$ and ends with $u_{\lambda_-}(p_-)$. In case we consider the electrons to be massless, we have to have $\lambda_+ = \lambda_- \equiv \lambda_e$. We can then make the following helicity-dependent choice for q_i for photon i :

$$q_i^\mu = \begin{cases} p_+^\mu & \text{if } \lambda_i = \lambda_e \\ p_-^\mu & \text{if } \lambda_i = -\lambda_e \end{cases} , \quad (4)$$

which leads to a considerable simplification of the calculation since many Feynman diagrams are made to vanish. A few remarks are in order here. In the first place, although this may not be completely obvious from the above formulation, the choice (4) is in fact equivalent to the original CALKUL gauge choice. There is however a difference due to the fact that the CALKUL group only considered Abelian gauge theories (i.e. QED) with all particles strictly massless, while the above definition is completely applicable to nonabelian bremsstrahlung (i.e. QCD) with arbitrary fermion masses as well. Secondly, it should be noted that we are allowed to make a choice for q_i for each bremsstrahlung photon independently, which is necessary to make the definition (4) workable for the case of more than one photon. In the last place, there is a price to be paid for the simplification of the calculation in the use of Eq.(4): as can be seen from Eq.(3), the polarization vector ϵ introduces an additional denominator into the matrix element, which goes like $\sqrt{(p_+ \cdot k)}$ or $\sqrt{(p_- \cdot k)}$. Hence when

the photon becomes collinear with either p_+ or p_- the matrix element will superficially diverge more strongly than the expected $1/\sqrt{(p_{\pm} \cdot k)}$. Of course this is compensated for by cancellations in the denominator (as described in detail by the CALKUL group in [13]) but it indicates that in collinear situations the choice (4) is less adequate. Since anyway in such situations we cannot neglect the electron mass and therefore λ_+ and λ_- do not have to be equal, we shall employ a different choice for collinear photons:

$$q_i^\mu = \begin{cases} p_+^\mu & \text{if } k_i \text{ is collinear with } p_+ \\ p_-^\mu & \text{if } k_i \text{ is collinear with } p_- \end{cases} \quad (5)$$

This way, the individual components of ϵ_i^μ will remain of order unity even in collinear situations. This implies that under this alternative choice (5) only those Feynman diagrams have to be taken into account in which the photon line is actually attached to the external leg of the fermion with which it is collinear [9]. With 'collinear' in the above we mean a kinematical configuration where, say, $(p_+ \cdot k)$ is much smaller than all the other momentum transfers in the problem, but not necessarily so small that it is comparable with m_e^2 , where m_e is the electron mass. In other words, our approach assumes that there is a possibility of kinematical configurations where the strong ordering $m_e^2 \ll (p_+ \cdot k) \ll s$ holds: in that case, the two gauge choices (4) and (5) will lead to results that are smoothly connected, although in the choice (4) we neglect all mass terms, and in the choice (5) we neglect many of the Feynman diagrams. Below, we will study this question in more detail and show that our approach is in fact possible at LEP energies.

Let us start with the no-bremsstrahlung case. It turns out to be convenient to define the combination of Z and photon propagators and their associated couplings to the incoming and outgoing fermions as

$$G_{\lambda, \lambda_f}(s) = \frac{Q_e Q_f}{s} + \frac{(v_e - \lambda_e a_e)(v_f - \lambda_f a_f)}{s - M_Z^2 + i M_Z \Gamma_Z} , \quad (6)$$

which is appropriate when the fermions are considered massless. Here, Q_e , v_e and a_e stand for the electron charge, vector and axial-vector coupling, and similarly for Q_f , v_f and a_f . The $\lambda_{e,f}$ are the helicity indices of the electrons and fermions. The argument s is the momentum transfer in the photon and Z propagator, and M_Z , Γ_Z are the mass and width of the Z . The above definition can of course be trivially extended to take into account the effects of the photon and Z self-energy correction effects as amply described in the literature [4].

The helicity amplitudes $\mathcal{M}_0(\lambda_+, \lambda_-; \lambda_f)$ for the no-bremsstrahlung case and for massless e^+e^- , where λ_+ , λ_- and λ_f denote the helicity indices of the e^+ , e^- , and f , respectively, can now be written quite simply as follows:

$$\begin{aligned} \mathcal{M}_0(\lambda, \lambda_f) &= 2i G_{\lambda, \lambda_f}(s) s_{\lambda, -\lambda}(p_+, p_-) s_{-\lambda, \lambda}(r_1, r_2) \\ &= 2i G_{\lambda, \lambda_f}(s) \frac{1}{s} s_{\lambda, -\lambda}(p_+, p_-) s_{\lambda, -\lambda}(r_1, r_2) \{s_{-\lambda, \lambda}(p_-, r_2)\}^2 \\ &= 2i G_{\lambda, \lambda_f}(s) \frac{1}{s} s_{-\lambda, \lambda}(p_+, p_-) s_{-\lambda, \lambda}(r_1, r_2) \{s_{\lambda, -\lambda}(p_+, r_1)\}^2 , \\ \mathcal{M}_0(\lambda, -\lambda_f) &= 0 , \end{aligned} \quad (7)$$

where helicity conservation along the electron line is explicitly indicated, and the set of vectors r_1^μ, r_2^μ is the following helicity-dependent permutation of q_1^μ, q_2^μ :

$$(r_1, r_2) = \begin{cases} (q_1, q_2) & \text{if } \lambda_f = \lambda \\ (q_2, q_1) & \text{if } \lambda_f = -\lambda \end{cases} \quad (8)$$

In Eq.(7) we have neglected an irrelevant overall complex phase. The three equivalent forms for $M_0(\lambda, -\lambda; \lambda_f)$ follow immediately from the momentum conservation $p_+ + p_- = q_1 + q_2$; they are given here since we shall encounter them again in the bremsstrahlung case. The above formulation is obviously well suited for compact and easy-to-check computer implementation, and we shall put the results for single and double bremsstrahlung in the same form.

For single bremsstrahlung we have helicity amplitudes $M_1(\lambda_+, \lambda_-, \lambda_f; \lambda_i)$ which now depend on the photon helicity as well. It turns out that the relevant index is not the helicity index λ_i of photon 1 itself, but rather the helicity relative to that of the electrons. As before, the helicity amplitudes with $\lambda_+ \neq \lambda_-$ are zero; the 8 non-zero ones are given by

$$\begin{aligned} M_1(\lambda, \lambda; \lambda_f; \lambda) &= -2iQ_e G_{\lambda\lambda_f}(s') N_+(k_1) \times \\ &\quad s_{\lambda,-\lambda}(p_+, p_-) s_{\lambda,-\lambda}(\tau_1, \tau_2) \{s_{-\lambda,\lambda}(p_-, \tau_2)\}^2, \\ M_1(\lambda, \lambda; \lambda_f; -\lambda) &= +2iQ_e G_{\lambda\lambda_f}(s') N_-(k_1) \times \\ &\quad s_{-\lambda,\lambda}(p_+, p_-) s_{-\lambda,\lambda}(\tau_1, \tau_2) \{s_{\lambda,-\lambda}(p_+, \tau_1)\}^2, \end{aligned} \quad (9)$$

with

$$N_\pm(k_1) = \frac{\lambda\sqrt{2}}{s_{-\lambda,\lambda}(k_1, p_+) s_{\lambda,-\lambda}(p_+, p_-) s_{-\lambda,\lambda}(p_-, k_1)}, \quad (10)$$

and $s' = (q_1 + q_2)^2$ is the invariant mass of the final state fermion pair.

For double bremsstrahlung we now have of course two photon helicities entering in the amplitude $M_2(\lambda_+, \lambda_-, \lambda_f; \lambda_1, \lambda_2)$. Its 16 non-zero values are the following:

$$\begin{aligned} M_2(\lambda, \lambda; \lambda_f; \lambda, \lambda) &= -2iQ_e s G_{\lambda\lambda_f}(s') N_+(k_1) N_+(k_2) \times \\ &\quad s_{\lambda,-\lambda}(p_+, p_-) s_{\lambda,-\lambda}(\tau_1, \tau_2) \{s_{-\lambda,\lambda}(p_-, \tau_2)\}^2, \\ M_2(\lambda, \lambda; \lambda_f; -\lambda, -\lambda) &= -2iQ_e s G_{\lambda\lambda_f}(s') N_-(k_1) N_-(k_2) \times \\ &\quad s_{-\lambda,\lambda}(p_+, p_-) s_{-\lambda,\lambda}(\tau_1, \tau_2) \{s_{\lambda,-\lambda}(p_+, \tau_1)\}^2, \\ M_2(\lambda, \lambda; \lambda_f; \lambda, -\lambda) &= -2iQ_e s G_{\lambda\lambda_f}(s') N_+(k_1) N_-(k_2) \times \\ &\quad \left\{ \frac{1}{\Delta_+} s_{\lambda,-\lambda}(p_+, k_1) s_{-\lambda,\lambda}(k_1, p_-) s_{\lambda,-\lambda}(p_-, k_2) s_{-\lambda,\lambda}(\tau_2, p_-) \right. \\ &\quad \times (s_{-\lambda,\lambda}(k_2, p_+) s_{\lambda,-\lambda}(p_+, \tau_1) - s_{-\lambda,\lambda}(k_2, k_1) s_{\lambda,-\lambda}(\tau_1, \tau_2)) \\ &\quad + \frac{1}{\Delta_-} s_{\lambda,-\lambda}(p_+, \tau_1) s_{-\lambda,\lambda}(k_1, p_+) s_{\lambda,-\lambda}(p_+, k_2) s_{-\lambda,\lambda}(k_2, p_-) \\ &\quad \times (s_{-\lambda,\lambda}(\tau_2, p_-) s_{\lambda,-\lambda}(p_-, k_1) - s_{-\lambda,\lambda}(\tau_2, k_2) s_{\lambda,-\lambda}(k_2, k_1)) \\ &\quad - (s_{\lambda,-\lambda}(p_+, p_-) s_{-\lambda,\lambda}(p_-, \tau_2) - s_{\lambda,-\lambda}(p_+, k_2) s_{-\lambda,\lambda}(k_2, \tau_2)) \\ &\quad \times (s_{\lambda,-\lambda}(\tau_1, p_+) s_{-\lambda,\lambda}(p_+, p_-) - s_{\lambda,-\lambda}(\tau_1, k_1) s_{-\lambda,\lambda}(k_1, p_-)) \left. \right\}, \\ M_2(\lambda, \lambda; \lambda_f; -\lambda, \lambda) &= \{M_2(\lambda, \lambda; \lambda_f; \lambda, -\lambda)\}_{k_1 \leftrightarrow k_2}, \end{aligned} \quad (11)$$

with

$$\Delta_\pm = (p_\pm - k_1 - k_2)^2. \quad (12)$$

It should be noted that if the photon helicities are equal the resulting amplitude has a quite simple form. It reduces as in the one-photon case to the Born level amplitude, multiplied by

two 'infrared factors' (as compared to one factor in the single-bremsstrahlung case): note, however, that in the presence of bremsstrahlung the various 'Born-type' forms that arise are no longer numerically equivalent. In fact this simplification has been shown by the CALCUL group to persist to arbitrary order for those helicity amplitudes where all photon helicities are equal [8]. The same is observed in QCD where a similar factorization for multigluon scattering for which the form of the amplitudes where two gluons have helicity opposite to all the other ones was first conjectured by Parke and Taylor in [15] and later proved rigorously by Berends and Giele [16]. For the cases where the photon helicities are opposite no such simplification is observed: in particular 'composite denominators' like Δ_+ are not wholly compensated by a combination of Dirac matrices in the numerator, whereas in the equal-helicity case they are. As a consequence the behaviour in the various collinear limits is more complicated in the opposite-helicity cases: still, when $\Delta_+ \rightarrow 0$ the numerator also vanishes, as can be seen from the concurrence of the products $s_{\lambda,-\lambda}(p_+, k_1)$, $s_{-\lambda,\lambda}(k_2, p_+)$, and $s_{-\lambda,\lambda}(k_2, k_1)$ with $1/\Delta_+$, which go to zero if p_-^2 , $\vec{k}_1$ and $\vec{k}_2$ are all collinear. This has a consequence for the case of triple bremsstrahlung which we shall discuss below.

We now turn to the issue of the mass effects. As mentioned above, mass effects become relevant when for instance $(p_+ \cdot k_1)$ becomes comparable to m_e^2 . In that case, $(p_+ \cdot k_1) \ll s$ and we are allowed to neglect many Feynman diagrams, namely those that do not contain $(p_+ \cdot k_1)^{-1}$ explicitly as a propagator. These considerations lead us to the following strategy for computing the mass effects. Let us consider a Feynman diagram where one or more photons are emitted collinearly with p_+^2 : all these diagrams start with the e^+ external leg, to which the collinear photons are attached. In this part of the relevant diagrams, which is equal for all these diagrams that we have to take into account, all occurring momentum transfers are comparable to each other and much smaller than the large momentum transfers like s (by our definition of collinearity). After this 'collinear' part of the diagrams, we encounter the first vertex where a large momentum transfer takes place. It can easily be seen that after this vertex, no more small denominators can occur in the electron propagator until we reach the other extreme end of the diagram, where a collinear part may occur due to those photons that are collinear with p_- . We see that the diagrams are decomposed naturally into collinear parts where all momentum transfers are small (and we may have to keep the electron mass), and a 'hard scattering' part where we have to take into account all diagrams but where we can certainly neglect the electron mass. It is obvious that this decomposition is not strictly gauge-invariant and we therefore have to be careful in choosing the gauge of the collinear photons, for which as we have argued above the choice (5) is appropriate.

We shall now derive the expressions for collinear photon radiation. This is most simply described by the following explicit example. Consider a process where a photon with momentum k_1^μ and helicity $+$ is radiated collinearly with the electron, with momentum p_-^μ and helicity $+$. As argued above, in the gauge of Eq.(5) we only have to consider diagrams which contain the collinear part

$$C^{(-)}(p_-, +, k_1, +) \equiv \frac{1}{-2(p_- \cdot k_1)} (\not{p}_- - \not{k}_1 + m_e) \not{\epsilon}(k_1) u_+(p_-), \quad (13)$$

By using the definition of $\not{\epsilon}$ of Eq.(3) and the identities

$$\not{p}_- + m_e = u_+(p_-) \bar{u}_+(p_-) + u_-(p_-) \bar{u}_-(p_-),$$

$$\bar{V}_i = u_+(k_1)\bar{u}_+(k_1) + u_-(k_1)\bar{u}_-(k_1), \quad (14)$$

we can immediately decompose the above collinear part to write

$$\begin{aligned} \mathcal{C}^{(-)}(p_-, +; k_1, +) &= -\frac{1}{\sqrt{2}(p_- \cdot k_1)} s_{+,+}(k_1, q) \\ &\quad \{s_{+,+}(p_-, q)s_{+,+}(p_-, k_1)u_+(p_-) \\ &\quad - s_{+,-}(p_-, k_1)s_{+,-}(p_-, q)u_+(p_-) \\ &\quad + s_{-,-}(q, k_1)s_{-,-}(p_-, k_1)u_-(k_1)\}, \end{aligned} \quad (15)$$

where already 5 out of the 8 terms have dropped out because of the symmetry properties of the spinor products and the fact that the $s_{+,+}$ and $s_{-,-}$ vanish for two massless vectors. We will now further simplify Eq.(15) by considering the order of magnitude of the various terms. In the collinear limit, the denominator is of order $\mathcal{O}(m_e^2)^2$, and this would give rise to a cross section behaving like $\mathcal{O}(m_e^{-4})$. As is well known, however, the actual behaviour of the cross section is rather $\mathcal{O}(m_e^{-2})$. Therefore, the numerator of Eq.(15) has to exhibit cancellations that brings its peaking behaviour down by $\mathcal{O}(m_e)$. Thus, we may expand the numerator of (15) and neglect terms that are of order $\mathcal{O}(m_e^2)$, which will not give a noticeable contribution to the cross section (in the same sense in which we neglect a number of Feynman diagrams in the collinear situation). We have the following orders of magnitude for the various quantities appearing in this numerator:

$$\begin{aligned} s_{+,+}(p_-, k_1) &= \mathcal{O}(m_e), \\ s_{+,+}(p_-, q) &= \mathcal{O}(m_e), \\ s_{+,-}(q, k_1) &= y_1 s_{+,-}(q, p_-) + \mathcal{O}(m_e), \\ u_+(k_1) &= y_1 u_+(p_-) + \mathcal{O}(m_e). \end{aligned} \quad (16)$$

Here we have introduced

$$y_i \equiv \sqrt{k_i^0/p_-^0} \sim \sqrt{(k_i \cdot q)/(p_- \cdot q)}. \quad (17)$$

With this information we find that Eq.(15) can be rewritten as follows, where we have dropped terms of order $\mathcal{O}(m_e^2)$ and higher in the numerator:

$$\begin{aligned} \mathcal{C}^{(-)}(p_-, +; k_1, +) &= \frac{1}{\sqrt{2}(p_- \cdot k_1)} y_1 \times \{ s_{+,-}(p_-, k_1)u_+(p_-) \\ &\quad + y_1^2 s_{+,+}(p_-, k_1)u_-(p_-) \}. \end{aligned} \quad (18)$$

Note that the spinor products with q^μ as an argument have cancelled: this is in fact a proof that to the order in m_e that we are working in, our result is indeed gauge-invariant. We see from Eq.(18) that the effect of the collinear bremsstrahlung can be viewed as turning the definite-helicity state $u_+(p_-)$ into a linear combination of two helicity states $u_+(p_-)$ and $u_-(p_-)$, each with a different weight. In fact, because we have argued that all mass effects have been properly taken into account and from this point on we can treat the remainder of the amplitude as describing the scattering of massless particles, we can assume that $(p_- - k_1)^\mu$ is a massless vector as well, and we have

$$u_\pm(p_-) \sim \frac{u_\pm(p_- - k_1)}{\sqrt{1 - y_1^2}}. \quad (19)$$

²Note that, since we use the gauge choice (5), we take q^μ to be explicitly not collinear to p_-^μ or k_1^μ so that the product $s_{+,-}(k_1, q)$ remains of order $\mathcal{O}(1)$ in the collinear situation.

Hence we are allowed to use

$$\begin{aligned} \mathcal{C}^{(-)}(p_-, +, k_1, +) &= A_+^{(-)}(p_-, +, k_1, +)u_+(p_- - k_1) \\ &\quad + A_-^{(-)}(p_-, +, k_1, +)u_-(p_- - k_1), \end{aligned} \quad (20)$$

with

$$\begin{aligned} A_+^{(-)}(p_-, +, k_1, +) &= \frac{s_{+,-}(p_-, k_1)}{\sqrt{2}y_1\sqrt{1 - y_1^2}(p_- \cdot k_1)}, \\ A_-^{(-)}(p_-, +, k_1, +) &= \frac{m_e y_1^2}{\sqrt{2}\sqrt{1 - y_1^2}(p_- \cdot k_1)}. \end{aligned} \quad (21)$$

Here we have used the explicit expression $s_{+,+}(p_-, k_1) = m_e y_1$ in order to illustrate the fact that for strictly massless electrons the coefficient $A_-^{(-)}(p_-, +, k_1, +)$ vanishes. It is of course straightforward to derive similar expressions for the other helicity cases, and for the case of collinear radiation from the positron as well. In general, we have

$$\begin{aligned} \mathcal{C}^{(-)}(p_-, \lambda_-, k_1, \lambda_1) &\equiv \frac{(\not{p}_- - \not{k}_1 + m_e)\not{p}_\lambda(k_1)u_{\lambda_-}(p_-)}{(p_- - k_1)^2 - m_e^2} \\ &= A_+^{(-)}(p_-, \lambda_-, k_1, \lambda_1)u_+(p_- - k_1) + \\ &\quad A_-^{(-)}(p_-, \lambda_-, k_1, \lambda_1)u_-(p_- - k_1), \\ \mathcal{C}^{(+)}(p_+, \lambda_+, k_1, \lambda_1) &\equiv \frac{\bar{v}_{\lambda_+}(p_+)\not{p}_\lambda(k_1)(-\not{p}_+ + \not{k}_1 + m_e)}{(p_+ - k_1)^2 - m_e^2} \\ &= A_+^{(+)}(p_+, \lambda_-, k_1, \lambda_1)\bar{v}_+(p_+ - k_1) + \\ &\quad A_-^{(+)}(p_+, \lambda_-, k_1, \lambda_1)\bar{v}_-(p_+ - k_1). \end{aligned} \quad (22)$$

The explicit expressions for $A_\pm^{(\pm)}$ are given in appendix B. The case of double collinear radiation can be treated in the same way: in this case, terms up to order $\mathcal{O}(m_e^2)$ in the numerator must be kept, and also we have to take into account two diagrams instead of only one. We have

$$\begin{aligned} \mathcal{C}^{(-)}(p_-, \lambda_-, k_1, \lambda_1, k_2, \lambda_2) &\equiv \\ &\equiv \frac{(\not{p}_- - \not{k}_1 - \not{k}_2 + m_e)\not{p}_{\lambda_2}(k_2)(\not{p}_- - \not{k}_1 - \not{k}_2)u_{\lambda_-}(p_-)}{[(p_- - k_1)^2 - m_e^2][(p_- - k_1 - k_2)^2 - m_e^2]} + (k_1, \lambda_1 \leftrightarrow k_2, \lambda_2) \\ &= A_+^{(-)}(p_-, \lambda_-, k_1, \lambda_1, k_2, \lambda_2)u_+(p_- - k_1 - k_2) + \\ &\quad A_-^{(-)}(p_-, \lambda_-, k_1, \lambda_1, k_2, \lambda_2)u_-(p_- - k_1 - k_2), \\ \mathcal{C}^{(+)}(p_+, \lambda_+, k_1, \lambda_1, k_2, \lambda_2) &\equiv \\ &\equiv \frac{\bar{v}_{\lambda_+}(p_+)\not{p}_{\lambda_2}(k_2)(-\not{p}_+ + \not{k}_1 + m_e)\not{p}_{\lambda_1}(k_1)(-\not{p}_+ + \not{k}_1 + \not{k}_2 + m_e)}{[(p_+ - k_1)^2 - m_e^2][(p_+ - k_1 - k_2)^2 - m_e^2]} + (k_1, \lambda_1 \leftrightarrow k_2, \lambda_2) \\ &= A_+^{(+)}(p_+, \lambda_+, k_1, \lambda_1, k_2, \lambda_2)\bar{v}_+(p_+ - k_1 - k_2) + \\ &\quad A_-^{(+)}(p_+, \lambda_+, k_1, \lambda_1, k_2, \lambda_2)\bar{v}_-(p_+ - k_1 - k_2), \end{aligned} \quad (23)$$

where again the explicit expressions are given in appendix B. We now turn to the implementation of these results in the computation of collinear amplitudes. In the case of single

bremstrahlung collinear with the incoming electron, it is clear that we can write the amplitude as

$$\begin{aligned}
\mathcal{M}_1(p_+, \lambda_+, p_-, \lambda_-, k_1, \lambda_1; q_1, q_2, \lambda_f) &= \\
&\mathcal{M}_0(p_+, \lambda_+, p_-, k_1, +; q_1, q_2, \lambda_f) A_+^{(-)}(p_-, \lambda_-, k_1, \lambda_1) \\
&+ \mathcal{M}_0(p_+, \lambda_+, p_-, k_1, -; q_1, q_2, \lambda_f) A_-^{(-)}(p_-, \lambda_-, k_1, \lambda_1) \\
&= \mathcal{M}_0(p_+, \lambda_+, p_-, k_1, \lambda_+; q_1, q_2, \lambda_f) A_{\lambda_+}^{(-)}(p_-, \lambda_-, k_1, \lambda_1),
\end{aligned} \tag{25}$$

where in the last line we have used the conservation of helicity along the massless electron-positron line in $\mathcal{M}_0$. Note that the momentum argument p_-^μ in $\mathcal{M}_0$ now has changed into $p_-^\mu - k_1^\mu$. This is important for the following reason. In any physical amplitude or cross section, overall momentum conservation holds, and consequently there are infinitely many different ways to write the cross section as a function of the external momenta. If the actual arguments used in $\mathcal{M}_{0,1,2,\dots}$ do in fact satisfy momentum conservation, all these alternative formulations will give the same numerical value for the cross section; but if they do not satisfy momentum conservation differing formulations will lead to different results. In Eq.(25), any of the equivalent formulations will give the same result: if we would have used $u(p_-)$ instead this would not be the case.

For single-photon bremsstrahlung collinear with the incoming positron, we of course have

$$\begin{aligned}
\mathcal{M}_1(p_+, \lambda_+, p_-, \lambda_-, k_1, \lambda_1; q_1, q_2, \lambda_f) &= \\
&\mathcal{M}_0(p_+ - k_1, +, p_-, \lambda_-; q_1, q_2, \lambda_f) A_+^{(+)}(p_+, \lambda_+, k_1, \lambda_1) \\
&+ \mathcal{M}_0(p_+ - k_1, -, p_-, \lambda_-; q_1, q_2, \lambda_f) A_-^{(+)}(p_+, \lambda_+, k_1, \lambda_1) \\
&= \mathcal{M}_0(p_+ - k_1, \lambda_-, \lambda_-, p_-, \lambda_-; q_1, q_2, \lambda_f) A_{\lambda_-}^{(-)}(p_+, \lambda_+, k_1, \lambda_1).
\end{aligned} \tag{26}$$

In the case of double bremsstrahlung there are more possible collinear situations: we can have one photon collinear with one beam, two photons collinear with the same beam, or one photon collinear with one of the beams each. If only one of the photons, say k_1 is collinear with, say p_- , we can immediately write the extension of Eq.(25):

$$\begin{aligned}
\mathcal{M}_2(p_+, \lambda_+, p_-, \lambda_-, k_1, \lambda_{1,2}, \lambda_2; q_1, q_2, \lambda_f) &= \\
&= \mathcal{M}_1(p_+, \lambda_+, p_-, k_1, \lambda_+, k_2, \lambda_2; q_1, q_2, \lambda_f) A_{\lambda_+}^{(-)}(p_-, \lambda_-, k_1, \lambda_1),
\end{aligned} \tag{27}$$

with obvious changes for radiation collinear with the positron. In case the two photons are both collinear with the electron, we of course find

$$\begin{aligned}
\mathcal{M}_2(p_+, \lambda_+, p_-, \lambda_-, k_1, \lambda_{1,2}, \lambda_2; q_1, q_2, \lambda_f) &= \\
&= \mathcal{M}_1(p_+, \lambda_+, p_-, k_1 - k_2, \lambda_+; q_1, q_2, \lambda_f) A_{\lambda_+}^{(-)}(p_-, \lambda_-, k_1, \lambda_1, k_2, \lambda_2).
\end{aligned} \tag{28}$$

Note that as in the case of single bremsstrahlung, in Eqs.(27) and (28) only one of the helicities survives in the final answer, since always either the positron or electron can be taken as massless. In the case where each of these has its own collinear photon this does no longer hold, and we have, when k_1 is collinear with p_- and k_2 with p_+ :

$$\begin{aligned}
\mathcal{M}_2(p_+, \lambda_+, p_-, \lambda_-, k_1, \lambda_{1,2}, \lambda_2; q_1, q_2, \lambda_f) &= \\
&= \mathcal{M}_0(p_+ - k_2, +, p_-, k_1, +; q_1, q_2, \lambda_f) \times A_+^{(+)}(p_+, \lambda_+, k_2, \lambda_2) A_+^{(-)}(p_-, \lambda_-, k_2, \lambda_2) \\
&+ \mathcal{M}_0(p_+ - k_2, -, p_-, k_1, -; q_1, q_2, \lambda_f) \times A_-^{(+)}(p_+, \lambda_+, k_2, \lambda_2) A_-^{(-)}(p_-, \lambda_-, k_2, \lambda_2).
\end{aligned} \tag{29}$$

In this expression we have a coherent superposition of the two opposite helicity combinations of the incoming fermions: since we evaluate the cross section at the level of helicity amplitudes this is not a problem but if one would rather use the more standard techniques using traces in the squared matrix element extremely cumbersome expressions would arise. This can be easily seen from the fact that for instance in Eq.(29) we have a nontrivial complex linear combination of G_{+, λ_f} and G_{-, λ_f} , whereas in Eq.(29) we have a nontrivial complex linear combination of G_{+, λ_f} and G_{-, λ_f} in every amplitude.

The above expressions suffice for a completely well-defined algorithm to evaluate up to the two-photon bremsstrahlung matrix elements in all possible kinematical configurations. The extension to the case of final-state collinear radiation is of course straightforward. The only thing left for a computer implementation of the above formulae (in which we have indicated all arguments explicitly in order to show how the actual computer code looks in practice) is a way to determine, for a given event, whether or not to use the massless expression, or any of the appropriate collinear ones. Obviously, we need to ensure a smooth transition from the noncollinear, massless case with all diagrams but fewer helicity combinations, and the collinear, massive case with fewer diagrams but all helicity combinations. Due to our definition of collinearity such a smooth transition is in fact possible: if we let a photon become more and more collinear with a beam the point where we can start neglecting diagrams comes before the point where we start having to account for m_e . Some experimentation has led us to the following algorithm: for each photon i we define the quantity

$$f_i = \log \frac{(p_+ \cdot k_i)}{(p_- \cdot k_i)}, \tag{30}$$

which for massless electrons is just the rapidity of the photon. If for a given event (i.e. configuration of generated momenta) $f_1 > 15$ we call k_1 collinear with p_+ ; for $f_1 < -15$ we call it collinear with p_- . Note that for $f_1 = 15$ the angle between $\vec{k}_1$ and $\vec{p}_+$ is about 1 mrad, which is still a lot larger than m_e^2/E_1^2 . Next, if $|f_2| > 10$ we also call k_2 collinear (obviously, we also have to check this with k_1 and k_2 interchanged). The reason to relax the condition on the second photon is the following. If k_2 is only slightly less collinear than k_1 , say, $f_2 = 14.5$, one may run into problems: calling k_1 collinear and k_2 noncollinear, one would neglect terms like $(p_+ \cdot k_1)$ with respect to terms like $(p_+ \cdot k_2)$, which are in fact of comparable magnitude: so we would have to call both of them either collinear or noncollinear. This dilemma is best solved empirically: considering events with $15 < f_1 < 16$ and $10 < f_2 < 11$ we found that the 'collinear' and the 'noncollinear' treatment give results agreeing to within a fraction of 10^{-3} . This justifies the smoothness of the above procedure.

Before finishing this section we want to remark that of course Eqs.(25) and (27) are not strictly necessary. Because only one helicity contribution is nonnegligible, we may also consider the squared form of the amplitude, implicitly summed over the spins: for instance if k_1 is collinear with p_- , this reads

$$|\mathcal{M}_1(p_+, p_-, k_1)|^2 = |\mathcal{M}_0(p_+, p_- - k_1)|^2 \times B(p_-, k_1), \tag{31}$$

with

$$\begin{aligned}
B(p_-, k_1) &= \sum_{\lambda_+, \lambda_1} |A_{\lambda_+}^{(-)}(p_-, \lambda_-, k_1, \lambda_1)|^2 \\
&= \frac{1 + (1 - x_1)^2}{x_1 (p_- \cdot k_1)} - \frac{m_e^2 (1 - x_1)}{(p_- \cdot k_1)^2},
\end{aligned} \tag{32}$$

which is of course nothing but the well-known Altarelli-Parisi type splitting function for massive fermions. For the double-bremsstrahlung case the corresponding factor can be derived for the case of Eq.(28) but, as already remarked above, not for the case of Eq.(29). In both cases the amplitude-level expressions are anyway simpler than the squared ones. However we have checked that we get equivalent results by squaring and summing the explicit λ 's in the double-collinear case, or first squaring the matrix elements, and then working out the collinear situation (using, of course, the same axial gauge in both cases). That is, the following equality holds in the collinear limit up to terms of order m^2 :

$$\sum_{\lambda_+, \lambda_1, \lambda_2} \left| \lambda_+^{(-)}(p_-, \lambda_-, k_1, \lambda_1, k_2, \lambda_2) \right|^2 \mathcal{P}_- = \frac{1}{1-x^2-x_2^2} \times \quad (33)$$

$$\times \left[\frac{\mathcal{P}_- - \mathcal{Y}_1 - \mathcal{Y}_2 + m}{(p_- - k_1 - k_2)^2 - m^2} \gamma^\mu \frac{\mathcal{P}_- - \mathcal{Y}_1 + m}{(p_- - k_1)^2 - m^2} \gamma^\mu + (k_1 \leftrightarrow k_2) \right] (\mathcal{P}_- + m) \quad (34)$$

$$\times \left[\gamma^\mu \frac{\mathcal{P}_- - \mathcal{Y}_1 + m}{(p_- - k_1)^2 - m^2} \gamma^\mu \frac{\mathcal{P}_- - \mathcal{Y}_1 - \mathcal{Y}_2 + m}{(p_- - k_1 - k_2)^2 - m^2} + (k_1 \leftrightarrow k_2) \right] \quad (35)$$

$$\times \left(-g_{\mu\alpha} + \frac{k_{1\mu} q_\alpha + k_{1\mu} q_\alpha}{(k_1 \cdot q)} \right) \left(-g_{\mu\alpha} + \frac{k_{2\mu} q_\alpha + k_{2\mu} q_\alpha}{(k_2 \cdot q)} \right). \quad (36)$$

Another remark is in order here. We have been able to express the collinear amplitudes for single-bremsstrahlung in that for the nonradiative process, and the collinear one for double-bremsstrahlung in that for the single-bremsstrahlung process. Likewise, in the case of *triple* bremsstrahlung we might envisage reducing the collinear amplitude to the one for double-bremsstrahlung with a collinear factor. However we think that at that point the quality of the approximation may start to deteriorate for the following reason. The nonradiative and single-bremsstrahlung amplitudes can be written in a *robust* form, that is, the numerical result of their evaluation can be seen to be stable if the argument momenta p_+ or p_- are not strictly massless; therefore we are allowed to make the approximation that for instance $p_+ - k_1$ is proportional to p_+ if k_1 is sufficiently collinear with p_+ . As remarked above, in our expression for the double-bremsstrahlung amplitude a number of numerical cancellations are still apparent in the numerator, cancelled by similar (but not identical!) ones in the denominator. Consequently we have less control over the stability of the final answer when photons become slightly collinear (but not collinear enough to warrant going over to the collinear expression). It is therefore not clear yet how such an expression will behave when we shall replace $p_+ - k_3$, say, by p_+ . It appears that it would be extremely useful if a simple and robust expression for the double-bremsstrahlung cross section could be constructed. We doubt, however, that this is possible.

3 Generating events

In addition to expressions for the matrix elements, we also need algorithms to generate events distributed according to this matrix element. Because this distribution peaks sharply, generating such events efficiently is not trivial when two bremsstrahlung photons occur.

Let us first discuss the phase space, $\mathcal{P}S_4$. For the four-particle final state $f(q_2)f(q_1)\gamma(k_1)\gamma(k_2)$ we have

$$\mathcal{P}S_4 = \frac{1}{2} \frac{1}{2s} \frac{1}{(2\pi)^8} d^4 k_1 \delta^+(k_1^2) d^4 k_2 \delta^+(k_2^2) \times d^4 q_2 \delta^+(q_2^2 - m_f^2) d^4 q_1 \delta^+(q_1^2 - m_f^2)$$

$$\times \delta^{(4)}(p_+ + p_- - q_2 - q_1 - k_1 - k_2) \quad (37)$$

The factor $\frac{1}{2}$ is the statistical factor for two identical bosons in the final state, whereas the factor $1/2s$ is the flux factor and $s = (p_+ + p_-)^2$ is the C.M. energy squared of the incoming particles. As basic variables we will choose $s' = (q_2 + q_1)^2$, the invariant mass squared of the final state fermion pair, k_1^0 , the energy of one of the photons, Ω_1 and Ω_2 , the solid angles of both the photons and Ω_f^* , the solid angle of q_2^* , where the $*$ indicates that it is defined in the C.M. frame of the final state fermion pair. These variables have been chosen because the matrix element peaks in most of them. We have, then,

$$d\mathcal{P}S_4 = \frac{1}{2} \frac{1}{2s} \frac{1}{(2\pi)^8} \frac{1}{2s} \frac{k_1^0(k_2^0)^2}{s' - s - 2k_1^0\sqrt{s}} dk_1^0 d\Omega_1 d\Omega_2 \sqrt{1 - m_f^2/q_2^{*2}} d\Omega_f^*, \quad (38)$$

where

$$k_2^0 = \frac{s - s' - 2k_1^0\sqrt{s}}{2\sqrt{s} - 2k_1^0(1 - \cos\angle(k_1, k_2))}. \quad (39)$$

We will treat the final state fermions as massless, $m_f = 0$, since we also did this for the matrix element. We approximate the square of the matrix element by

$$\frac{1}{4} \sum_{\lambda_j} |M|_{\text{app}}^2 = \frac{e^8 s^3 s' G(s')}{(p_+ \cdot k_1)(p_+ \cdot k_2)(p_- \cdot k_1)(p_- \cdot k_2)}, \quad (40)$$

where we used

$$G(s') = \sum_{\lambda_+, \lambda_f = -1, 1} |G_{\lambda_+, \lambda_f}(s')|^2. \quad (41)$$

With the notation

$$d\Omega_4 = d\epsilon_1 d\epsilon_2, \quad (42)$$

$$p = \frac{1}{\sqrt{1 - m^2/(p_+^0)^2}}, \quad (43)$$

we have for the approximant in terms of our basic variables:

$$\frac{1}{4} \sum_{\lambda_j} |M|_{\text{app}}^2 = \frac{2^{1/2} \pi^4 \alpha^4 s' s' G(s')}{(k_1^0)^2 (k_2^0)^2 (p^2 - c_1^2)(p^2 - c_2^2)} \quad (44)$$

The integration over Ω_f^* is trivial. The next integral we would like to perform is that over k_1^0 integral, from $k_1^0 = k_0$ up to $k_1^0 = k_{\text{max}}$, where k_0 is the distinction between soft and hard photons and

$$k_{\text{max}} = \frac{s - s' - 2k_0\sqrt{s}}{2\sqrt{s} - 2k_0(1 - \cos\angle(k_1, k_2))}. \quad (45)$$

This upper boundary introduces an extra dependence on the angle between the two photons. Performing the integral with this boundary therefore would leave us with a distribution for the photon angles that we would not be able to generate. Instead, exploiting the $k_1 \leftrightarrow k_2$ symmetry of the problem, we integrate over k_1^0 from k_0 up to

$$k_{\text{symm}} = \frac{1}{2}(\sqrt{s} - \sqrt{s'})$$

and multiply by two. As can be seen from the Dalitz plot 1 we now have overestimated area II by a factor 2 and we have added a contribution from area I, which was not there before.

We correct for this when generating k_1^0 values, by assigning a weight equal to 1 for events in area II, equal to $\frac{1}{2}$ for events in area I and equal to 0 for events in area I, and then applying weight rejection. This way we are able to generate the distribution of Eq. (40). Having done the k_1^0 integration we have

$$\int dP S_1 \frac{1}{4} \sum_{\lambda_j} |M|^2_{app} = \frac{2^2 \alpha^4}{(2\pi)^3} \int d\Omega_1 d\Omega_2 ds' \frac{G(s')}{s-s'} \times \frac{1}{(p^2 - c_1^2)(p^2 - c_2^2)} \log \frac{s-s'-2k_0\sqrt{s}}{2k_0\sqrt{s'}} \quad (46)$$

The advantage of the procedure followed so far is that the integration over Ω_1 and Ω_2 is now extremely simple, yielding

$$\int dP S_1 \frac{1}{4} \sum_{\lambda_j} |M|^2_{app} = \frac{4\alpha^4}{\pi} \log^2 \frac{s}{m^2} \int ds' \frac{s' G(s')}{s-s'} \log \frac{s-s'-2k_0\sqrt{s'}}{2k_0\sqrt{s'}} \quad (47)$$

The approximate total cross section resulting from the s' integration is given in appendix C for completeness. At this point we will assume that this s' distribution (47) can be generated in some way. As it still is a rather complicated distribution, due to the behaviour of $G(s')$, we used a general purpose-routine. The algorithm, then, is as follows.

1. Generate s' according to the distribution (47).
2. Generate c_1 according to $1/(p-c_1)$ and flip the sign in 50% of the cases. Generate φ according to a flat distribution. Repeat this procedure for Ω_2 .
3. Generate k_1^0 according to $\frac{1}{k_1^0} + \frac{1}{2\sqrt{s}(1-s'/s) - k_1^0}$

between the boundaries k_0 and $\frac{1}{2}(\sqrt{s} - \sqrt{s'})$. Notice that both terms are positive definite over this k_1^0 range. The energy of the other photon is then given by Eq. (39).

4. Apply weight rejection with the following weight:

$$w = \begin{cases} 0 & \text{if } k_2^0 < k_0 \\ \frac{1}{2} & \text{if } k_0 < k_2^0 < k_{symm} \\ 1 & \text{if } k_2^0 > k_{symm} \end{cases}$$

5. Generate Ω_F , that is $\cos\theta^*$ and φ^* , both according to a flat distribution. The resulting momenta q_2^0 and q_1^0 have to be boosted to the lab frame.

6. Apply weight rejection with as weight

$$w = \frac{\sum_{\lambda_j} |M|^2}{\sum_{\lambda_j} |M|^2_{app}} \quad (48)$$

The last step ensures that in the end we generate the distribution as given by the complete matrix element. Since it is clear that steps 1–5 can be done very fast, the efficiency of this algorithm is determined by the efficiency of the last step. For reasonable efficiency we need

a weight distribution that is sharply peaked and bounded from above by a value not too far from the peak. For a typical set of values for the input parameters ($M_Z = 91.1$ GeV, $\sin^2\theta_w = 0.2309$) we show the result for this weight distribution in figures 2 and 3. From these histograms it appears the weight rejection is in fact quite efficient at all energies.

Being able to generate events with two hard photons (with $k_1^0 > k_0$), in order to have an event generator that is complete up to and including order α^2 , we also need events with one hard photon or even no hard photon at all. From the Monte Carlo point of view these are not very difficult to generate. The only difficulty is including the two loop virtual and soft photon corrections. But since these have been calculated analytically [2], we can simply use those results. We employ the following algorithm.

- Approximate the no hard photon distribution by

$$\frac{1}{4} \sum_{\lambda_j} |M^{0\gamma}|^2_{app} = (1 + \delta^{VS}) 2 e^4 s^2 G(s), \quad (49)$$

which can be generated trivially, and weigh this with the full lowest matrix element, divided by the above approximation. The factor δ^{VS} contains the complete order α and order α^2 virtual and soft photon corrections as calculated in [2]. The above approximation gives rise to an approximate total cross section

$$\sigma_{app}^{0\gamma} = (1 + \delta^{VS}) 2 \pi \alpha^2 s G(s), \quad (50)$$

which will be modified to the exact cross section for this type of events by multiplying by the average of the weights.

- Approximate the 1 hard photon distribution by

$$\frac{1}{4} \sum_{\lambda_j} |M^{1\gamma}|^2_{app} = \frac{e^6}{3} \frac{1}{(p_+ \cdot k_1)} \left(2 \frac{p_+^0}{k_1^0} + \frac{p_+^0}{p_+^0 - k_1^0} \right) s^2 G(s') + (p_+ \leftrightarrow p_-), \quad (51)$$

which leads to an approximate total cross section of

$$\sigma_{app}^{1\gamma} = \frac{\alpha^3}{3s} \log \frac{s}{m^2} \int ds' \left(\frac{2}{s-s'} + \frac{1}{s'} \right) s^2 G(s') \quad (52)$$

The result when integrated over s' has been listed in appendix C as well, for completeness. The distribution (51) can be generated, assuming once more we have a way of generating the resulting $d\sigma_{app}/ds'$ distribution. We then have to weigh the events with the complete matrix element, where one loop virtual and soft photon corrections have been included. To make the approximation (51) better one can include e.g. a factor

$$1 + \frac{\alpha}{\pi} \left(2 \log k_0 \left(\log \frac{s}{m^2} - 1 \right) + \frac{3}{2} \log \frac{s}{m^2} + \frac{\pi^2}{3} - 2 \right) \quad (53)$$

to take into account part of the virtual and soft photon corrections already. This makes the weight rejection step more efficient. The complete one loop virtual and soft photon corrections can once again be taken from [2]. It should be noted however that if one does so, one lacks the $\cos\theta^*$ dependence of the virtual photon corrections to the single hard bremsstrahlung matrix element. Also another remark is in order here. Because the one hard photon contribution is the only one where virtual and soft photon corrections appear up to one loop, this is the only contribution that can become negative when including such corrections and lowering the value of k_0 . The restriction that this contribution remains positive turns out to restrict k_0 to $k_0 > 10^{-4} p_+$. This seems to be low enough for all practical purposes.

We have implemented the generation of the no photon distribution and the one hard photon distribution as well. In the next section we will present some numerical results.

4 Numerical results

In the Monte Carlo program **FPAIR** we have implemented up to double photon bremsstrahlung from the initial state using the methods described in the previous sections. Weak corrections have been included by using the library of the program **ZSHAPE** (see [6]). We also incorporated one loop final state QED corrections using a somewhat modified version of the techniques described in [18]. In this section we present some results. As the input parameters we use $f = \mu^-$, $M_Z = 91.1$ GeV, $M_H = 100$ GeV, $m_{top} = 100$ GeV, which leads to $\sin^2 \theta_w = 0.2309$ and $I_2^2 = 2.478$ GeV. We will consider the distributions of $\cos \theta = \cos \angle(q_2, p_+)$, the angle between e^+ and μ^+ , and $E_\gamma = \sum k_i^0$, the total energy carried away by photons. There are several calculations that we can compare our results with. In the first place, there is the analytical result incorporated in **ZSHAPE**, which computes the total cross section to high precision and serves as a benchmark for Monte Carlo event generators. For the $\cos \theta$ distribution there is the semianalytical program **MUCUT** presented in [3], which also gives the forward-backward asymmetry. In this program the weak corrections are approximated by using the so-called G_μ representation, which should be good enough for our purposes. Finally, we have compared to the Monte Carlo **DYB2** of [11], which is ideally suited to our purposes since it also generates up to 2 photons from the initial, and up to 1 from the final state. Throughout we impose the cut $\sqrt{s'} > 1$ GeV. This is not meant to be realistic; indeed, in [2] it was shown that a much stricter cut would be necessary to suppress the background from two-photon processes such as $e^+e^- \rightarrow e^+e^-\mu^+\mu^-$. However, since we are only interested in comparing theoretical results with each other, any cut (or none at all) would be allowed: requiring $\sqrt{s'} > 1$ allows us to treat the muons as massless for all events, thus avoiding the region at $s' \sim 0$ where all calculations are known to differ. We can also imagine an additional set of ‘canonical cuts’, which are slightly more realistic. These we choose to be

$$\begin{aligned} 15^\circ < \theta < 165^\circ, \\ 180^\circ - \angle(q_1, q_2) < 10^\circ, \\ \sqrt{s'} > 10 \text{ GeV}. \end{aligned} \quad (5.1)$$

The results for the total cross section from the runs with and without canonical cuts are given in table 1, for a number of different values for the CM energy $\sqrt{s}$. In table 2 the analogous results for the forward-backward asymmetry are given. The agreement with the analytical results from **ZSHAPE** (with its exponentiation option turned off) and **MUCUT** is seen to be quite good; in particular the line shape is well reproduced. For the residual effects of exponentiation in the line shape we refer to [6]. The influence of the cuts is identical in both **FPAIR** and **DYB2**.

Next, we plot the $\cos \theta$ and E_γ distributions when no canonical cuts are applied. The results are shown in figures 4 and 5. As can be seen from these histograms, there is agreement between the two results. Only in the E_γ distribution the two Monte Carlo programs differ slightly in the hard photon area. This need not be a surprise since it was the motivation for this article. That these differences integrate to a negligible amount for the total cross section is the statement that the structure function approach is good for inclusive quantities,

up to and including the next-to-leading log terms of order α_s^2 . Note that here we have taken $\sqrt{s} = 120$ GeV, that is, well above the Z peak. In that case, hard bremsstrahlung is one of the dominant effects, so that a comparison between **FPAIR** and **DYB2** really informs us about the behaviour of the programs when generating bremsstrahlung, not just about the behaviour of back-to-back events which dominate around the peak.

The results after canonical cuts are shown in figures 6 and 7. Again the results of the two programs compare well, but for the hard photon part in the E_γ distribution.

5 Conclusions

In this paper we have described how the matrix elements for double bremsstrahlung from the initial state can be computed. The use of spinor techniques ensures quick and compact evaluation, even in the case where photons are emitted close to the beams. The extension to bremsstrahlung from the produced fermions is of course straightforward. Special care is necessary in the transition region between collinear and non-collinear radiation to ensure smooth behaviour of the numerical result: due to the fact that a strong ordering of the type $m_e^2 \ll (p \cdot k_1) \ll (p \cdot k_2) \ll s$ is kinematically possible: if this were not the case then probably the complete set of diagrams, including a non-zero electron mass throughout, would be necessary. We have outlined an algorithm for generating double-bremsstrahlung events according to the exact distribution. Here a careful study of the structure of phase space for two soft photons is necessary: by employing the symmetry in the photon momenta an efficient algorithm can be constructed. We have presented some results from the corresponding program **FPAIR**, and compared these to other results known in the literature. The program is seen to do as well as can be expected. To finish, we want to stress again that of course a two-loop treatment is not the final answer, even if the results for the total cross section coincide with those of analytical exponentiated programs like **ZSHAPE**. For one, there is still the so-called k_0 -problem, that is we cannot put k_0 lower than a given value without having events with negative weights. Only a fully exponentiated Monte Carlo can circumvent this problem. Secondly, final state radiation is only included up to one loop, and interference between initial- and final-state radiation is not treated at all. Fortunately, these effects are small at the Z resonance but a good all-round Monte Carlo will at some time have to be constructed that does include them and will allow detailed studies of off-resonance phenomena. Such a Monte Carlo will unavoidably have to include two-loop effects as described in this paper. The present study can be seen as a step towards this goal.

6 Appendix A: Spinor techniques

In this appendix we shall present the relevant formulae necessary for the evaluation of the bremsstrahlung matrix elements using spinor techniques. More details can for instance be found in [10]. We start with two four-vectors k_0^μ and k_1^μ which satisfy

$$(k_0 \cdot k_0) = (k_0 \cdot k_1) = 0, \quad (k_1 \cdot k_1) = -1, \quad (5.5)$$

and two spinors $u_-(k_0)$ and $u_+(k_0)$ defined by

$$\begin{aligned} u_-(k_0) \bar{u}_-(k_0) &= \frac{1}{2}(1 - \gamma^5) \not{k}_0 \\ u_+(k_0) &\equiv \frac{1}{\sqrt{2}} u_-(k_0). \end{aligned} \quad (5.6)$$

We now define a spinor for any (massive or massless) fermion momentum p_i^μ as follows:

$$u_\lambda(p_i) = \frac{1}{\sqrt{2(p \cdot k_0)}} (\not{p}_i + m_i) u_{-\lambda}(k_0) \quad (57)$$

where $(p_i \cdot p_i) = m_i^2$: for particles, m_i is positive, while for antiparticles m_i is negative. The limit $m_i \rightarrow 0$ is smooth and leads to fermion states with definite helicity:

$$\lim_{m_i \rightarrow 0} u_\lambda(p_i) \bar{u}_\lambda(p_i) = \frac{1}{2} (1 + \lambda \gamma^5) \not{p}_i, \quad (58)$$

while for $m_i \neq 0$ (when the helicity is not a Lorentz-invariant concept) we have the usual form

$$u_\lambda(p_i) \bar{u}_\lambda(p_i) = \frac{1}{2} (\not{p}_i + m_i) (1 + \gamma^5 \not{p}_i), \quad (59)$$

where the fermion spin polarization s_i^μ is given by

$$s_i^\mu = \frac{1}{m_i} \left(p_i^\mu - \frac{m_i^2}{(p_i \cdot k_0)} k_0^\mu \right). \quad (60)$$

The spinor products can now be computed using the well-known trace techniques: for instance, we have

$$s_{+, -}(p_1, p_2) = \frac{\text{Tr}[(1 - \gamma^5) \not{p}_0 (\not{p}_1 + m_1) (\not{p}_2 + m_2) \not{p}_1]}{4 \sqrt{(p_1 \cdot k_0) (p_2 \cdot k_0)}}. \quad (61)$$

Explicit expressions for the spinor products are obtained by taking explicit forms for k_0^μ and k_1^μ : using

$$\begin{aligned} (k_0^0, k_0^x, k_0^y, k_0^z) &= (1, 1, 0, 0) \\ (k_1^0, k_1^x, k_1^y, k_1^z) &= (0, 0, 1, 0) \end{aligned} \quad (62)$$

we obtain the following formulae:

$$\begin{aligned} s_{+, -}(p_1, p_2) &= (p_1^y + i p_1^z) \left[\frac{p_2^0 - p_2^x}{p_1^0 - p_1^x} \right]^{\frac{1}{2}} - (p_1 \leftrightarrow p_2), \\ s_{+, -}(p_2, p_1) &= -s_{+, -}(p_1, p_2), \\ s_{-, +}(p_1, p_2) &= -s_{+, -}(p_1, p_2)^*, \\ s_{+, +}(p_1, p_2) &= m_1 \left[\frac{p_2^0 - p_2^x}{p_1^0 - p_1^x} \right]^{\frac{1}{2}} + (p_1 \leftrightarrow p_2), \\ s_{+, +}(p_2, p_1) &= s_{+, +}(p_1, p_2), \\ s_{-, -}(p_1, p_2) &= s_{+, +}(p_1, p_2). \end{aligned} \quad (63)$$

Note that $s_{+, -}$ and $s_{-, +}$ are antisymmetric, and $s_{+, +}$ and $s_{-, -}$ are symmetric in p_1^μ and p_2^μ . Moreover, in the ultrarelativistic limit $m_1, m_2 \rightarrow 0$ $s_{+, +}$ and $s_{-, -}$ vanish, whereas in the nonrelativistic limit $\vec{p}_1, \vec{p}_2 \rightarrow 0$ the $s_{+, -}$ and $s_{-, +}$ go to zero. There is an interesting relation between the spinor products and the more familiar vector products. In the general massive case we are considering here it reads

$$\frac{1}{4} \sum_{\lambda_1, \lambda_2 = \pm} |s_{\lambda_1, \lambda_2}(p_1, p_2)|^2 = (p_1 \cdot p_2) + m_1 m_2, \quad (64)$$

which in fact turns out to be about the most (numerically) stable way of computing the vector product $(p_1 \cdot p_2)$, especially when the masses are zero or small. A few other useful relations exist, which we give here without derivation: again we refer to [10] for details. First, for any string S of Dirac matrices, we have the reversion identity:

$$\bar{u}_{\lambda_1}(p_1) S u_{\lambda_2}(p_2) = \lambda_1 \lambda_2 \bar{u}_{-\lambda_2}(p_2) S^R u_{-\lambda_1}(p_1), \quad (65)$$

where S^R is the string S written in reversed order. Then, there is the Chisholm identity:

$$[\bar{u}_+(q_1) \gamma^\mu u_+(q_2)] \gamma_\mu = 2 \{ u_+(q_2) \bar{u}_+(q_1) + u_-(q_1) \bar{u}_-(q_2) \}, \quad (66)$$

which holds for massless q_1^μ and q_2^μ . The so-called two-dimensional Schouten identity leads to the formula

$$\begin{aligned} s_{+, -}(p_1, p_2) s_{+, -}(p_3, p_4) &+ \\ s_{+, -}(p_2, p_3) s_{+, -}(p_1, p_4) &+ \\ s_{+, -}(p_3, p_1) s_{+, -}(p_2, p_4) &= 0, \end{aligned} \quad (67)$$

where the momenta $p_{1,2,3,4}^\mu$ may have any mass. We want to stress that the result (67) is new in the sense that it is also seen to hold for massive momenta: to our knowledge this has not been established before. Note that in the Weyl-spinor formulation constructed by Berends and Giele [14], which is completely equivalent to our spinor techniques for massless fermions, this relation follows immediately from the fact that their spinors have two components: in our case, it is derived most trivially by writing out the explicit definition of $s_{+, -}$ as given in Eq.(63). It follows that Eq.(67) also holds if we restrict the products $s_{+, -}$ to their real or imaginary parts only. Finally, another amusing identity is

$$s_{+, -}(p_1, p_2) = \frac{1}{m_3} (s_{+, +}(p_3, p_1) s_{+, -}(p_3, p_2) - s_{+, +}(p_3, p_2) s_{+, -}(p_3, p_1)), \quad (68)$$

which for instance turns out to be handy in the case of collinear double bremsstrahlung: again, here p_1^μ and p_2^μ may have any mass, and m_3 is of course the mass associated with p_3 . This finishes our review of spinor formulae: with the above tools, the matrix elements presented in section 2 can be derived immediately.

7 Appendix B: collinear factors

Here we give the collinear factors A discussed in section 2. It is immediately clear that upon flipping all helicities in any A we obtain the same result with only the replacement $s_{+, -} \leftrightarrow s_{-, +}$, so we only list half of them. For single collinear radiation we have

$$\begin{aligned} A_+^{(-)}(p_-, +, k_1, +) &= D_1 s_{+, -}(p_-, k_1), \\ A_+^{(-)}(p_-, +, k_1, -) &= -D_1 s_{-, +}(p_-, k_1) (1 - y_1^2), \\ A_+^{(-)}(p_-, -, k_1, +) &= 0, \\ A_+^{(-)}(p_-, -, k_1, -) &= -D_1 m_e y_1^2, \end{aligned} \quad (69)$$

with

$$D_1 = \frac{1}{\sqrt{2} y_1 \sqrt{1 - y_1^2} (p_- \cdot k_1)}. \quad (70)$$

Similarly,

$$\begin{aligned} A_+^{(+)}(p_+, +, k_1, +) &= -D_2 s_{+-}(p_+, k_1)(1 - y_1^2), \\ A_+^{(+)}(p_+, +, k_1, -) &= D_2 s_{+-}(p_+, k_1), \\ A_+^{(+)}(p_+, -, k_1, +) &= D_2 m_e y_1^2, \\ A_+^{(+)}(p_+, -, k_1, -) &= 0, \end{aligned} \quad (71)$$

with

$$D_2 = \frac{1}{\sqrt{2} y_1 \sqrt{1 - y_1^2} (p_+ \cdot k_1)} \quad (72)$$

Note that since $p_+^0 = p_-^0 = E_b$ the value of y_1 is the same whether defined using p_- or p_+ for final-state radiation this is of course no longer true. For double collinear radiation we have to take into account two diagrams, leading to

$$\begin{aligned} A_-^{(-)}(p_-, \lambda_-, k_1, \lambda_1, k_2, \lambda_2) &= F_-^{(-)}(p_-, \lambda_-, k_1, \lambda_1, k_2, \lambda_2) \\ &+ F_-^{(-)}(p_-, \lambda_-, k_2, \lambda_2, k_1, \lambda_1), \\ F_-^{(-)}(p_-, +, k_1, +, k_2, +) &= D_3 s_{+-}(p_-, k_1) (s_{+-}(p_-, k_1) y_1 y_2 \\ &+ s_{+-}(p_-, k_2)(1 - y_1^2)), \\ F_-^{(-)}(p_-, +, k_1, +, k_2, -) &= -D_3 (m_e^2 y_1^2 y_2 + s_{+-}(p_-, k_1) s_{+-}(p_-, k_2)(1 - y_1^2 - y_2^2)), \\ F_-^{(-)}(p_-, +, k_1, -, k_2, +) &= -D_3 s_{+-}(p_-, k_1) (s_{+-}(p_-, k_1) y_1 y_2 (1 - y_1^2) \\ &+ s_{+-}(p_-, k_2)(1 - y_1^2)^2), \\ F_-^{(-)}(p_-, +, k_1, -, k_2, -) &= D_3 s_{+-}(p_-, k_1) (s_{+-}(p_-, k_1) y_1 y_2 (1 - y_1^2 - y_2^2) \\ &+ s_{+-}(p_-, k_2)(1 - y_1^2)(1 - y_2^2 - y_2^2)), \\ F_-^{(-)}(p_-, +, k_1, +, k_2, +) &= D_3 m_e (s_{+-}(p_-, k_1) y_1^2 y_2^2 + y_2^2 + y_1^2 y_2^2) \\ &+ s_{+-}(p_-, k_2) y_1^2 (1 - y_1^2 - y_2^2), \\ F_-^{(-)}(p_-, +, k_1, +, k_2, -) &= -D_3 m_e y_1^2 (s_{+-}(p_-, k_1) y_1 y_2 + s_{+-}(p_-, k_2)(1 - y_1^2)), \\ F_-^{(-)}(p_-, +, k_1, -, k_2, +) &= -D_3 m_e y_2^2 s_{+-}(p_-, k_1), \\ F_-^{(-)}(p_-, +, k_1, -, k_2, -) &= 0, \end{aligned} \quad (73)$$

with

$$D_3 = \frac{1}{2 y_1 y_2 \sqrt{1 - y_1^2 - y_2^2} (p_- \cdot k_1) [(p_- \cdot k_1) + (p_- \cdot k_2) - (k_1 \cdot k_2)]} \quad (74)$$

Finally, for double-collinear radiation from the positron we have

$$\begin{aligned} A_+^{(+)}(p_+, \lambda_+, k_1, \lambda_1, k_2, \lambda_2) &= F_+^{(+)}(p_+, \lambda_+, k_1, \lambda_1, k_2, \lambda_2) \\ &+ F_+^{(+)}(p_+, \lambda_+, k_2, \lambda_2, k_1, \lambda_1), \\ F_+^{(+)}(p_+, +, k_1, +, k_2, +) &= D_4 s_{+-}(p_+, k_1) (s_{+-}(p_+, k_1) y_1 y_2 (1 - y_1^2 - y_2^2) \\ &+ s_{+-}(p_+, k_2)(1 - y_1^2)(1 - y_2^2 - y_2^2)), \\ F_+^{(+)}(p_+, +, k_1, +, k_2, -) &= -D_4 s_{+-}(p_+, k_1) (s_{+-}(p_+, k_1) y_1 y_2 (1 - y_1^2) \\ &+ s_{+-}(p_+, k_2)(1 - y_1^2)^2), \end{aligned} \quad (75)$$

$$\begin{aligned} F_+^{(+)}(p_+, +, k_1, -, k_2, +) &= -D_4 (m_e^2 y_1^2 y_2 + s_{+-}(p_+, k_1) s_{+-}(p_+, k_2)(1 - y_1^2 - y_2^2)), \\ F_+^{(+)}(p_+, +, k_1, -, k_2, -) &= D_4 s_{+-}(p_+, k_1) (s_{+-}(p_+, k_1) y_1 y_2 \\ &+ s_{+-}(p_+, k_2)(1 - y_1^2)), \end{aligned}$$

$$\begin{aligned} F_-^{(+)}(p_+, +, k_1, +, k_2, +) &= 0, \\ F_-^{(+)}(p_+, +, k_1, +, k_2, -) &= D_4 m_e y_2^2 s_{+-}(p_+, k_1), \\ F_-^{(+)}(p_+, +, k_1, -, k_2, +) &= D_4 m_e y_1^2 (s_{+-}(p_+, k_1) y_1 y_2 + s_{+-}(p_+, k_2)(1 - y_1^2)), \\ F_-^{(+)}(p_+, +, k_1, -, k_2, -) &= -D_4 m_e (s_{+-}(p_+, k_1) y_1 y_2 (y_1^2 + y_2^2 + y_1^2 y_2^2) \\ &+ s_{+-}(p_+, k_2) y_1^2 (1 - y_1^2 - y_2^2)), \end{aligned} \quad (76)$$

with

$$D_4 = \frac{1}{2 y_1 y_2 \sqrt{1 - y_1^2 - y_2^2} (p_+ \cdot k_1) ((p_+ \cdot k_1) + (p_+ \cdot k_2) - (k_1 \cdot k_2))} \quad (76)$$

8 Appendix C: The approximate total cross sections

For completeness we list our results for the approximate total cross sections corresponding with the one-hard photon contribution and the two-hard photon contribution respectively. These expressions can be obtained by integrating Eq. (52) and (47) over s' . It must be stressed that these are not 'approximate cross sections' in the sense that their result should coincide with the exact one to some acceptable degree of accuracy, but rather that they just serve to normalize the event weights: indeed, $\sigma_{app}^{1\gamma}$ and $\sigma_{app}^{2\gamma}$ coincide with their exact counterparts only up to a factor $< w >$ (the mean weight) which can differ appreciably from 1.

$$\begin{aligned} \sigma_{app}^{1\gamma} &= \frac{\alpha^3}{3s} \log \frac{s}{m^2} \left[s (B_1 I_1 + 2 B_2 I_2) + (B_3 + B_4 (s + M_Z^2)) I_4 \right. \\ &\quad \left. + (B_2 (s + M_Z^2) - M_Z^2 I_2^2 B_4) I_3 \right] \end{aligned} \quad (77)$$

$$\begin{aligned} \sigma_{app}^{2\gamma} &= \frac{4\alpha^4}{\pi} \log^2 \frac{s}{m^2} \left[-\log(2k_0) (B_1 I_1 + B_2 I_2 + B_3 I_3 + B_4 I_4) \right. \\ &\quad + B_1 \left(-\frac{1}{2} I_5 + I_6 \right) + B_2 \left(-\frac{1}{2} I_7 + I_8 \right) \\ &\quad + B_4 \operatorname{Re} \left(-\frac{1}{2} I_9 + I_{10} \right) - \frac{B_3}{M_Z^2 I_2^2} \operatorname{Im} \left(-\frac{1}{2} I_9 + I_{10} \right) \left. \right] \end{aligned} \quad (78)$$

If one has included a factor like the one in Eq. (53) in the approximation (51) for the one hard photon distribution, one should multiply $\sigma_{app}^{1\gamma}$ by the same factor.

We have defined the following integrals

$$\begin{aligned} I_1 &= \log \frac{s'_{\max}}{s'_{\min}} \\ I_2 &= \log \frac{s - s'_{\min}}{s - s'_{\max}} \\ I_3 &= \frac{1}{M_Z^2 I_2^2} \left(\arctan \frac{s'_{\max} - M_Z^2}{M_Z I_2^2} - \arctan \frac{s'_{\min} - M_Z^2}{M_Z I_2^2} \right) \end{aligned}$$

$$\begin{aligned}
I_4 &= \frac{1}{2} \log \left(\frac{s'_{\max} - M_Z^2}{s'_{\min} - M_Z^2} \right)^2 + \frac{M_Z^2 V_Z^2}{2} \\
I_5 &= \frac{1}{2} \log^2 s'_{\max} - \frac{1}{2} \log^2 s'_{\min} \\
I_6 &= \log \left(s - 2k_0 \sqrt{s} \right) \log \frac{s'_{\max}}{s'_{\min}} - \text{Li}_2 \left(\frac{s'_{\max}}{s - 2k_0 \sqrt{s}} \right) + \text{Li}_2 \left(\frac{s'_{\min}}{s - 2k_0 \sqrt{s}} \right) \\
I_7 &= \log s \log \frac{s - s'_{\min}}{s - s'_{\max}} + \text{Li}_2 \left(1 - s'_{\max}/s \right) - \text{Li}_2 \left(1 - s'_{\min}/s \right) \\
I_8 &= -\frac{1}{2} \log^2 (s - s'_{\max}) - \text{Li}_2 \left(\frac{2k_0 \sqrt{s}}{s - s'_{\max}} \right) \\
&\quad + \frac{1}{2} \log^2 (s - s'_{\min}) + \text{Li}_2 \left(\frac{2k_0 \sqrt{s}}{s - s'_{\min}} \right) \\
I_9 &= + \log \left(s'_{\max} - M_Z^2 + iM_Z V_Z \right) \log s'_{\max} - \frac{1}{2} \log^2 s'_{\max} - \text{Li}_2 \left(\frac{M_Z^2 - iM_Z V_Z}{s'_{\max}} \right) \\
&\quad - \log \left(s'_{\min} - M_Z^2 + iM_Z V_Z \right) \log s'_{\min} + \frac{1}{2} \log^2 s'_{\min} + \text{Li}_2 \left(\frac{M_Z^2 - iM_Z V_Z}{s'_{\min}} \right) \\
I_{10} &= + \log \left(s - 2k_0 \sqrt{s} - M_Z^2 + iM_Z V_Z \right) \log \left(s'_{\max} - M_Z^2 + iM_Z V_Z \right) \\
&\quad - \log \left(s - 2k_0 \sqrt{s} - M_Z^2 + iM_Z V_Z \right) \log \left(s'_{\min} - M_Z^2 + iM_Z V_Z \right) \\
&\quad - \text{Li}_2 \left(\frac{s'_{\max} - M_Z^2 + iM_Z V_Z}{s - 2k_0 \sqrt{s} - M_Z^2 + iM_Z V_Z} \right) + \text{Li}_2 \left(\frac{s'_{\min} - M_Z^2 + iM_Z V_Z}{s - 2k_0 \sqrt{s} - M_Z^2 + iM_Z V_Z} \right)
\end{aligned}$$

and the following constants

$$\begin{aligned}
H_1 &= \frac{4Q_e Q_l}{s} \\
B_2 &= B_1 + \frac{4Q_e Q_f s + 8Q_e Q_l v_e v_f (s - M_Z^2)}{(s - M_Z^2)^2 + M_Z^2 V_Z^2} \\
B_3 &= \frac{4(v_e^2 + a_e^2)(v_f^2 + a_f^2)M_Z^2(s - M_Z^2 - V_Z^2) - 8Q_e Q_l v_e v_f M_Z^2 V_Z^2}{(s - M_Z^2)^2 + M_Z^2 V_Z^2} \\
B_4 &= B_2 - B_1
\end{aligned}$$

The value for the lower bound on the invariant mass squared of the final state fermion pair as dictated by the phase space is $4m_f^2$. But in the given procedure this value can be set arbitrarily, with $4m_f^2$ as the minimum possible value. The upper boundary $s'_{\max}$ is determined by how 'soft' hard photons can become and by the number of hard photons. To be precise, for one hard photon one has

$$s'_{\max} = s \left(1 - \frac{k_0}{p_0^0} \right)^2, \quad (79)$$

whereas for two hard photons

$$s'_{\max} = s \left(1 - \frac{k_0}{p_0^0} \right)^2 \quad (80)$$

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$\sqrt{s}$ (GeV)	ZSHAPE	FPAIR (pb)	DYMU2 (pb)
		no cuts	with cuts
81.1	38.2	38.1(1)	38.3(1)
86.1	99.1	98.7(3)	99.2(1)
88.1	221.5	223.0(6)	224.7(2)
89.1	410.6	408.4(8)	410.4(6)
90.1	873.7	869(2)	873(1)
91.1	1478	1473(2)	1483(3)
92.1	1060	1058(2)	1071(2)
93.1	611.3	611(1)	618(1)
94.1	395.2	394.5(8)	399(1)
96.1	217.9	217.5(6)	218.8(6)
101.1	96.9	97.0(3)	96.8(4)

Table 1: Monte Carlo and semi-analytical results for the total cross section. The numbers in brackets denote the Monte Carlo error estimate in the last digit.

$\sqrt{s}$ (GeV)	MUCUT	FPAIR	DYMU2
		no cuts	with cuts
81.1	-0.706	-0.679(3)	-0.680(2)
86.1	-0.459	-0.457(2)	-0.459(2)
88.1	-0.286	-0.286(2)	-0.290(2)
89.1	-0.190	-0.192(2)	-0.192(2)
90.1	-0.093	-0.092(2)	-0.093(2)
91.1	-0.001	0.000(2)	-0.002(2)
92.1	0.071	0.072(2)	0.070(2)
93.1	0.121	0.125(2)	0.124(2)
94.1	0.157	0.159(2)	0.152(2)
96.1	0.207	0.201(2)	0.202(2)
101.1	0.341	0.255(2)	0.253(2)

Table 2: Monte Carlo and semi-analytical results for the forward-backward asymmetry. The numbers in brackets denote the Monte Carlo error estimate in the last digit. The flags used in MUCUT were $igs=1$ $icxp=1$ $interf=0$ $ifinal=1$ $inorm=1$ $isof2=1$ $dela=0.2$

Figure 1: The Dalitz plot for the $f\bar{f}\gamma\gamma$ phase space. The z and y coordinates are the energies of the two photons.

Figure 2: The weight distribution in 1000 events per bin at $\sqrt{s} = 91.1$ GeV.

Figure 3: The weight distribution in 1000 events per bin at $\sqrt{s} = 87.1$ GeV (solid line) and 95.1 (dashed line).

Figure 4: The $\cos \theta$ distribution at $\sqrt{s} = 120\text{GeV}$ for FPAIR (solid line) and DYMU2 (dashed line).

Figure 5: The E_γ distribution at $\sqrt{s} = 120\text{GeV}$ for FPAIR (solid line) and DYMU2 (dashed line).

Figure 6: The distributions of $\cos \theta$ for $\sqrt{s} = 92$ GeV by the programs FPAIR (solid line) and DYM02 (dashed line). Canonical cuts as in Eq.(54).

Figure 7: The distribution of E_γ for $\sqrt{s} = 92$ GeV by the programs FPAIR (solid line) and DYM02 (dashed line). Canonical cuts as in Eq.(51).