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STOCHASTIC QUANTIZATION OF 2D GRAVITY AND ITS LINK WITH 3D GRAVITY AND TOPOLOGICAL 4D GRAVITY

Laurent Baulieu[†], Adel Bilal* and Marco Picco[†]

** Theory Division, CERN*

1211 Geneva 23, Switzerland

*† Laboratoire de Physique Théorique et Hautes Energies
de l'Université Pierre et Marie Curie*

4, place Jussieu, 75252 Paris Cedex 05, France

ABSTRACT

We apply stochastic quantization to two-dimensional gravity. The Laplace operator acting on the space of all metrics takes a particularly simple form in terms of the Beltrami parametrization. We show the equivalence between the quantum theory defined by the standard Faddeev-Popov gauge fixing of the two-dimensional diffeomorphism invariance and the one defined by stochastic quantization. We do so by using the gauge freedom left in the Langevin equation of a diffeomorphism invariant theory to adjust the drift force. Another choice of the drift force, comparable to that of Zwanziger for Yang-Mills theories seems to avoid the analogue of the Gribov ambiguity, i.e. the necessity of the by-hand restriction to one fundamental domain. We relate the two-dimensional gravity to a three-dimensional theory, based on the 3D gravitational Chern-Simons action for $SL(2, \mathbb{C})$, $ISO(3)$ or $SU(2) \times SU(2)$ (depending on the genus of the 2D Riemann surface), in which all fields of the stochastic quantization have been distributed as components of the gauge fields. To study the 3D theory, stochastic quantization can be applied once more. This gives a theory with the action of topological gravity in four dimensions, namely the Euler invariant $\int_{N_2 \times \mathbb{R} \times \mathbb{R}} \text{tr} R \wedge R$, gauge-fixed by self-duality conditions.

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1. Introduction

Stochastic quantization was introduced in Yang-Mills theories by Parisi and Wu¹⁾ in an attempt to avoid gauge fixing²⁾. They defined stochastic evolution from a drift force equal to minus the gradient of the classical action with respect to the gauge field. In this way, no reference to the gauge fixing is introduced. The ghosts, necessary in the usual path integral quantization, also seem to be absent. However, the formalism only applies to gauge-invariant quantities. It is in fact useful to introduce a sort of gauge fixing in the equations governing the stochastic quantization by adding a drift force along the orbits of the gauge group³⁾. Gauge-independent quantities are independent of the functions which parametrize this drift force. A proper choice of these functions allows one to show the equivalence of the stochastic quantization with the more standard Faddeev-Popov gauge fixing of the path integral formulation⁴⁾ and the renormalizability of the stochastically quantized gauge theory.⁵⁾ Moreover, with another choice of the gauge function, the gauge field seems confined within the first Gribov horizon, and one escapes naturally the Gribov problem^{2,3)}. It has been pointed out that the freedom in the choice of the gauge drift force in the Langevin equation of a gauge field follows from the simple geometric principle that the stochastic evolution must be compatible with the gauge symmetry.⁶⁾

In this article we apply stochastic quantization to two-dimensional gravity. Yang-Mills gauge invariance is replaced by the diffeomorphism invariance of the Riemann surface N with complex coordinates z and $\bar{z}$. The analogue of the Gribov problem lies in the necessity to restrict the modular parameter(s) of the Riemann surface to one fundamental domain when one functionally integrates over the inequivalent metrics. We will argue that stochastic quantization might circumvent this problem by confining the metric to a region corresponding to the first Gribov horizon, i.e. one fundamental domain.⁷⁾

Metrics on a Riemann surface are conveniently described by using the Beltrami parametrization. To find the evolution operator for the Langevin equation,

we first determine the Laplace operator on the space of the "gauge fields of the diffeomorphisms", i.e. on the space of metrics, using the Beltrami parametrization. It turns out to have a particularly simple form, displaying factorization between the holomorphic and antiholomorphic parts. The gauge (diffeomorphism) invariance of two-dimensional gravity allows the addition of a drift force along the orbits of the diffeomorphism group to the Langevin equation which governs the stochastic quantization. To this modified Langevin equation corresponds a modified Fokker-Planck equation. An appropriate choice of the gauge drift force gives the formal proof that the standard Faddeev-Popov gauge-fixed path integral is an equilibrium distribution solving this Fokker-Planck equation. This establishes the equivalence between the stochastic and the Polyakov path integral quantizations. We display a different choice of the gauge drift force that tends to confine the metric field to a region corresponding to one (or a finite number of) fundamental domain(s) only.

It has already been noticed in several places^{8,9,10)} that the function which parametrizes the freedom in the Langevin equation of a gauge theory can be promoted to a new component of the gauge field corresponding to the stochastic time which, in turn, is interpreted as an extra direction of space-time. This gives a link between the original gauge theory in d dimensions and a new one in $d + 1$ dimensions. By adapting this result to 2D diffeomorphism invariant theories, we relate stochastically quantized 2D gravity to three-dimensional classical (Chern-Simons) gravity: we identify the Langevin equation of the stochastic quantization as the constraints that the equations of motion of the 3D gravity are satisfied.

This is done by reinterpreting the functions parametrizing the 2D diffeomorphism drift forces as extra components of a 3D dreibein e^a , and the stochastic noises as elements of a 3D spin connection ω^a . Zero torsion and zero curvature of the classical 3D gravity are equivalent to the stochastic equations of the 2D gravity.

The 3D Chern-Simons theory, describing the 3D gravity, is that of a $SL(2, \mathbb{C})$, $ISO(3)$ or $SU(2) \times SU(2)$ connection $A \sim (e^a, \omega^a)$ (depending on whether the genus of the 2D Riemann surface is $g \geq 2$, $g = 1$ or $g = 0$). All three cases are compatible

with the same 2D Langevin equation and the difference is that the curvature on the Riemann surface can be gauge-fixed to a negative, zero or positive constant.

Let us note that recent work of H. Verlinde¹¹⁾ bears some resemblance to ours. However, although he has a three-dimensional Chern-Simons action, its gauge group is $SL(2, \mathbb{R})$ with 3 generators corresponding to the 2D zweibein e^a and the 2D Abelian spin connection ω , whereas we have a 3D dreibein e^a and a 3D spin connection ω^a . Finally we remark that the quantization of Chern-Simons theories with complex gauge groups such as $SL(2, \mathbb{C})$ was discussed recently by Witten¹²⁾ (see also ref. 13).

The equations of motion of the 3D gravity can be seen as the classical limit of the quantum field theory generated by the Chern-Simons action. Using stochastic quantization once more, we relate it to a quantum field theory generated by the four-dimensional Weyl action plus supersymmetric terms, which is nothing other than the topological four-dimensional gravity action¹⁴⁾ defined over a manifold $M_4 = N \times \mathbb{R} \times \mathbb{R}$ with two trivial directions interpreted as two stochastic (hence Euclidean) times. The equation of motion is now $F = \tilde{F}$. For $g \geq 2$, F is the curvature of the (now four-dimensional) connection A with values in $SL(2, \mathbb{C}) \simeq SO(3, 1)$, i.e. in the tangent space of the 4D manifold (after a Wick rotation to a Minkowskian signature). For $g = 0$, one has a similar situation with $SL(2, \mathbb{C})$ replaced by $SU(2) \times SU(2)$ and $SO(3, 1)$ by $SO(4)$. The 4D action is the gauge fixed version of the topological action $\int \text{tr} R \wedge R$. We have therefore a bridge between Polyakov's idea of summing over all metrics on N and the idea of choosing at will a metric on M_4 , without affecting physical quantities, thanks to the BRST invariance.¹⁵⁾

The paper is organized as follows. In sect. 2 we review stochastic quantization of Yang-Mills gauge theories, their equivalence with the Faddeev-Popov gauge-fixed path-integral, and display the choice of the gauge drift force that tends to confine the Yang-Mills gauge field within the first Gribov domain. In sect. 3, we turn our attention to two-dimensional gravity. First, we recall the Beltrami

parametrization for conformal theories¹⁶⁾. Then we work out the Laplace operator on the space of metrics, which turns out to be particularly simple in this parametrization. Next, we write down the Focker-Planck equation for the two-dimensional gravity and display a choice of the gauge drift force that demonstrates equivalence with the Faddeev-Popov path integral quantization, à la Polyakov. We discuss other choices of the gauge drift force and how they could confine the metric tensor within the first fundamental domain. In sect. 4, we relate the two-dimensional stochastically quantized gravity to classical three-dimensional (Chern-Simons) gravity. Finally, sect. 5 describes the link with the four-dimensional topological gravity.

2. Stochastic quantization of gauge theories

We will briefly recall some basic features of stochastic quantization of gauge theories. (See e.g. ref. 2.)

In a general field theory in d dimensions, $\phi = \phi(x^i)$, $i = 1, \dots, d$, one introduces a Euclidean stochastic time t and promotes the fields ϕ to stochastic variables $\phi(x^i, t)$ whose evolution is governed by a Langevin equation with noise η and drift force f . Equivalently, the probability distribution $P[\phi]$ of the fields $\phi(x^i)$ satisfies the Focker-Planck equation

$$\frac{\partial}{\partial t} P = -(\Delta P - \nabla \cdot (fP)) \equiv L^* P. \quad (2.1)$$

Here, ∇ and Δ denote the Nabla and Laplace functional derivative operators with respect to the fields. The drift force f equals minus the gradient of the classical action. In the case of pure Yang-Mills theory, $P = P[A_\mu^a(x^i)]$, and

$$L^* = - \int d^d x \frac{\delta}{\delta A_\mu^a(x)} \left(\frac{\delta}{\delta A_\mu^a(x)} + \frac{\delta S_{cl}}{\delta A_\mu^a(x)} \right). \quad (2.2)$$

Infinitesimal gauge transformations are generated by

$$\int d^d x v^a(x) G^a(x), \quad G^a(x) = -D_\mu^{ab} \frac{\delta}{\delta A_\mu^b(x)} \quad (2.3)$$

where $D_\mu^{ab} = \partial_\mu \delta^{ab} + f^{abc} A_\mu^c$ (f^{abc} being the structure constants of the gauge group). One easily sees that L^* commutes with $G^a(x)$ and hence is gauge-invariant. It follows that the stochastic evolution is gauge-invariant, i.e.

$$P[A, t] = \exp(-L_v^*(t - t_0)) P[A, t_0] \quad (2.4)$$

is a formal solution of eq. (2.1) for any function v^a , where

$$L_v^* = L^* - \int d^d x \frac{\delta}{\delta A_\mu^a(x)} (D_\mu v)^a(x). \quad (2.5)$$

The function $v^a(x) \equiv v^a(x, A)$ is an arbitrary g -valued functional of A , reflecting the full gauge invariance of the theory. We may specify it at will. One can show that the equilibrium distribution satisfying the static limit of the Fock-Planck equation (2.1) (with $L^* \rightarrow L_v^*$) is the standard Faddeev-Popov path integral distribution P_{FP} provided one chooses⁴⁾

$$v^a(x, A) = P_{\text{FP}}^{-1} \int dC d\bar{C} d^d y C^a(x) \partial_\nu \bar{C}^b(y) \frac{\delta S_{gl}}{\delta A_\nu^b(y)} \exp(-S_{tot}). \quad (2.6)$$

This shows the equivalence of the stochastic quantization of Yang-Mills gauge theories with the more familiar Faddeev-Popov gauge fixing of the Feynman path integral quantization.

However, different choices of v^a from the one in (2.6) may be more advantageous. Zwanziger has exhibited a choice of v which tends to confine the gauge field within the first Gribov horizon.³⁾ Let us switch for a moment from the

Fock-Planck equation to the equivalent Langevin equation:

$$\dot{A}_\mu = -\frac{\delta S_{cl}}{\delta A_\mu} + \alpha D_\mu v + \eta_\mu, \quad (2.7)$$

where we have introduced a real parameter α . v^a is an arbitrary g -valued function and η_μ is the stochastic noise. Let us define a norm for the gauge field (Lyapunov function) by

$$\|A\|^2 = \int \text{tr} A_\mu A^\mu. \quad (2.8)$$

Then one has

$$\begin{aligned} \frac{d}{dt} \|A\|^2 &= 2 \int \text{tr} A^\mu \dot{A}_\mu = 2 \int \text{tr} A^\mu \frac{\delta S_{cl}}{\delta A_\mu} + 2\alpha \int \text{tr} A^\mu D_\mu v + 2 \int \text{tr} A^\mu \eta_\mu \\ &= -2\alpha \int \text{tr} v D_\mu A^\mu + 2 \int \text{tr} A^\mu \left(\frac{\delta S_{cl}}{\delta A_\mu} + \eta_\mu \right) \end{aligned} \quad (2.9)$$

When we let $\alpha \rightarrow \infty$ and choose $v = D_\mu A^\mu$ we have

$$\frac{d}{dt} \|A\|^2 \rightarrow -2\alpha \|D_\mu A^\mu\|^2 \leq 0 \quad (2.10)$$

Hence, the choice $v = D_\mu A^\mu$ is such that the stochastic evolution would drive the gauge field to zero, if there were no noise. Taking into account the fluctuations induced by the noise η , one sees that the gauge field A_μ tends to be confined in a region around $A_\mu = 0$. It is believed that this region might coincide with the one within the first Gribov horizon.

3. Stochastic quantization of two-dimensional gravity

3.1 THE BELTRAMI PARAMETRIZATION

The natural parametrization of the metric of a Riemann surface is in terms of the Beltrami differentials : $ds^2 = \exp(\varphi)(dz + \mu_z^z d\bar{z})(d\bar{z} + \mu_z^{\bar{z}} dz)$, or

$$g_{ab} = \exp(\varphi) \begin{pmatrix} \mu_z^z & \frac{1}{2}(1 + \mu_z^z \mu_z^{\bar{z}}) \\ \frac{1}{2}(1 + \mu_z^{\bar{z}} \mu_z^z) & \mu_z^{\bar{z}} \end{pmatrix}, \quad \sqrt{-g} = \frac{1}{2} \exp(\varphi) (1 - \mu_z^z \mu_z^{\bar{z}}) \quad (3.1a)$$

$$g^{ab} = \exp(-\varphi) \frac{4}{(1 - \mu_z^z \mu_z^{\bar{z}})^2} \begin{pmatrix} -\mu_z^z & \frac{1}{2}(1 + \mu_z^z \mu_z^{\bar{z}}) \\ \frac{1}{2}(1 + \mu_z^{\bar{z}} \mu_z^z) & -\mu_z^{\bar{z}} \end{pmatrix}. \quad (3.1b)$$

It will turn out to be even more convenient to parametrize g_{ab} in terms of μ_z^z , $\mu_z^{\bar{z}}$ and $\tilde{\varphi}$ where

$$\tilde{\varphi} = \varphi + \ln(1 - \mu_z^z \mu_z^{\bar{z}}) \quad \text{or} \quad \exp(\tilde{\varphi}) = 2\sqrt{-g} \quad (3.1b)$$

The action of the diffeomorphisms acting on the Beltrami differentials μ_z^z and $\mu_z^{\bar{z}}$ can be read from their BRS transformations¹⁶⁾ (s denotes the BRS operator)

$$s\mu_z^z = \partial_z c^z + c^z \partial_z \mu_z^z - \mu_z^z \partial_z c^z \equiv D_z c^z, \quad s c^z = c^z \partial_z c^z, \quad s\bar{c}_{z\bar{z}} = b_{z\bar{z}}, \quad s b_{z\bar{z}} = 0$$

$$s\varphi = \Omega + \partial_z c^z + \partial_z \bar{c}^z + \xi^z (\partial_z \varphi - \partial_z \mu_z^z) + \xi^{\bar{z}} (\partial_z \varphi - \partial_z \mu_z^{\bar{z}}) \quad (3.2)$$

$$\text{or} \quad s\tilde{\varphi} = \Omega + \partial_z \xi^z + \partial_z \bar{\xi}^z + \xi^z \partial_z \tilde{\varphi} + \xi^{\bar{z}} \partial_z \tilde{\varphi} \equiv \Omega + F(c^z, c^{\bar{z}}, \mu_z^z, \mu_z^{\bar{z}}, \tilde{\varphi}) \quad (3.2)$$

One also has the mirror equations for μ_z^z , c^z , $\bar{c}_{z\bar{z}}$, $b_{z\bar{z}}$ with z and $\bar{z}$ exchanged. The ghost fields c^z , $c^{\bar{z}}$ are related to the standard diffeomorphism ghosts ξ^z , $\xi^{\bar{z}}$ by

$$c^z = \xi^z + \mu_z^z \xi^{\bar{z}}, \quad c^{\bar{z}} = \xi^{\bar{z}} + \mu_z^{\bar{z}} \xi^z, \quad (3.3)$$

and Ω is the ghost of the Weyl transformations.

The string action also contains the space-time fields X transforming as

$$sX = \frac{1}{1 - \mu_z^z \mu_z^{\bar{z}}} (c^z (\partial_z - \mu_z^z \partial_z) + c^{\bar{z}} (\partial_z - \mu_z^{\bar{z}} \partial_z)) X. \quad (3.4)$$

The classical string action is BRS invariant and can be written as:

$$S_{cl} = \int \frac{d^2 z}{1 - \mu_z^z \mu_z^{\bar{z}}} (\partial_z - \mu_z^z \partial_z) X \cdot (\partial_z - \mu_z^z \partial_z) X, \quad sS_{cl} = 0. \quad (3.5)$$

The gauge-fixing action for the conformal gauge, in the standard Faddeev-Popov procedure, is

$$S_{gf} = \int d^2 z \mathcal{L}_{z\bar{z}}^{gf} = s \int d^2 z \left(\bar{c}_{z\bar{z}} (\mu_z^z - \mu_z^{(0)z}) + \bar{c}_{z\bar{z}} (\mu_z^{\bar{z}} - \mu_z^{(0)\bar{z}}) \right). \quad (3.6)$$

where $\mu_z^{(0)z}$ and $\mu_z^{(0)\bar{z}}$ are (classical) background fields (global zero-modes can be taken into account in eq. (3.6), see ref. 17). For the conformal gauge on the complex plane one has e.g. $\mu_z^{(0)z} = \mu_z^{(0)\bar{z}} = 0$. Using eq. (3.6) and integrating out the field $b_{z\bar{z}}$, $b_{\bar{z}\bar{z}}$, one finds

$$\begin{aligned} \frac{\delta S_{gf}}{\delta \mu_z^z} &= 2e^{-\tilde{\varphi}} \frac{\partial \mathcal{L}_{z\bar{z}}^{gf}}{\partial \mu_z^z} = 2e^{-\tilde{\varphi}} (-2\bar{c}_{z\bar{z}} \partial_z c^z - \partial_z \bar{c}_{z\bar{z}} c^z) \\ \frac{\delta S_{gf}}{\delta c^z} &= 2e^{-\tilde{\varphi}} \frac{\partial \mathcal{L}_{z\bar{z}}^{gf}}{\partial c^z} = 2e^{-\tilde{\varphi}} (\partial_z \bar{c}_{z\bar{z}} - 2\bar{c}_{z\bar{z}} \partial_z \mu_z^z - \mu_z^z \partial_z \bar{c}_{z\bar{z}}) \\ \frac{\delta S_{gf}}{\delta \bar{c}_{z\bar{z}}} &= 2e^{-\tilde{\varphi}} \frac{\partial \mathcal{L}_{z\bar{z}}^{gf}}{\partial \bar{c}_{z\bar{z}}} = 2e^{-\tilde{\varphi}} (\partial_z c^z + c^z \partial_z \mu_z^z - \mu_z^z \partial_z c^z) = 2e^{-\tilde{\varphi}} D_z c^z \end{aligned} \quad (3.7)$$

One also has the mirror equations with z and $\bar{z}$ interchanged. (We have applied the covariant functional differentiation rule $\frac{\delta \mu_z^z(z)}{\delta \mu_z^z(z')} = \frac{\delta(z-z')}{\sqrt{-g}}.$)

3.2 THE LAPLACE OPERATOR ON THE SPACE OF METRICS

In order to formulate the Fock-Planck equation for functionals of the metric g_{ab} , i.e. of $\mu_z^\pm, \mu_z^\pm, \varphi$ we have to construct the Laplace operator on the space of metrics. In general, for a manifold $\mathcal{M}$ with given metric G_{ij} , the Laplace operator on $\mathcal{M}$ is

$$\Delta = (\det G)^{-\frac{1}{2}} \sum_{i,j} \partial_i (\det G)^{\frac{1}{2}} G^{ij} \partial_j. \quad (3.8)$$

The metric $G_{ij}(z, z')$ on the space of metrics is read from the "invariant line element"

$$\begin{aligned} \|\delta g\|^2 &= \int d^2 z \sqrt{-g} \left(g^{ac} g^{bd} + \left(x - \frac{1}{2} \right) g^{ab} g^{cd} \right) \delta g_{ab} \delta g_{cd} \\ &= \int d^2 z \sqrt{-g(z)} \int d^2 z' \sqrt{-g(z')} G_{ij}(z, z') \delta g^i(z) \delta g^j(z') \end{aligned} \quad (3.9)$$

where

$$\delta g^i = (\delta \mu_z^\pm, \delta \mu_z^\pm, \delta \tilde{\varphi}). \quad (3.10)$$

Here x is some constant. $g^{ac} g^{bd} - \frac{1}{2} g^{ab} g^{cd}$ projects on the traceless part of δg_{ab} , and hence for $x = 0$, G_{ij} is singular, i.e. $\|\delta g\|^2$ is independent of $\delta \tilde{\varphi}$. In the following we will assume that $x \neq 0$.

Expressing all quantities in terms of $\mu_z^\pm, \mu_z^\pm, \tilde{\varphi}$, one can compute $G_{ij}(z, z')$. If we had parametrized δg in terms of $\mu_z^\pm, \mu_z^\pm$ and φ (instead of $\tilde{\varphi}$) G_{ij} would have a rather complicated structure. However, using $\tilde{\varphi}$ no cross-terms $\delta^2 / \delta \mu_z^\pm \delta \tilde{\varphi}$ or $\delta^2 / \delta \mu_z^\pm \delta \tilde{\varphi}$ are present. One has

$$G_{ij}(z, z') = \frac{4}{(1 - \mu_z^\pm \mu_z^\pm)^2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & x(1 - \mu_z^\pm \mu_z^\pm)^2 \end{pmatrix} \frac{\delta(z - z')}{\sqrt{-g(z)}}. \quad (3.11)$$

The inverse of G_{ij} is

$$G^{ij}(z, z') = \frac{(1 - \mu_z^\pm \mu_z^\pm)^2}{4} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & x^{-1}(1 - \mu_z^\pm \mu_z^\pm)^{-2} \end{pmatrix} \frac{\delta(z - z')}{\sqrt{-g(z)}}. \quad (3.12)$$

One clearly sees that G would be singular for $x = 0$. The Laplace operator is computed as

$$\Delta = (\det G)^{-\frac{1}{2}} \int d^2 z \sqrt{-g(z)} \frac{\delta}{\delta g^i(z)} \int d^2 z' \sqrt{-g(z')} (\det G)^{\frac{1}{2}} G^{ij}(z, z') \frac{\delta}{\delta g^j(z')}. \quad (3.13)$$

A priori, one expects undefined quantities proportional to $\delta(0)$ to appear. Remarkably, they all cancel and one ends up with

$$\Delta = \frac{1}{8} \int d^2 z \exp(\tilde{\varphi}) \left((1 - \mu_z^\pm \mu_z^\pm)^2 \left(\frac{\delta}{\delta \mu_z^\pm \delta \mu_z^\pm} + \frac{\delta}{\delta \mu_z^\pm \delta \mu_z^\pm} \right) + x^{-1} \frac{\delta^2}{\delta \tilde{\varphi}^2} \right). \quad (3.14)$$

Let us only note, that, for the bosonic string in a flat Minkowski background, the Laplace operator on the space of X^μ fields is:

$$\Delta_{(X^\mu)} = \int d^2 z \sqrt{-g(z)} \frac{\delta}{\delta X_\mu(z)} \frac{\delta}{\delta X^\mu(z)}. \quad (3.15)$$

This formula derives from the definition of the invariant line element $\|\delta X^\mu\|^2 = \int d^2 z \sqrt{-g} \delta X^\mu \delta X_\mu$.

3.3 THE FOCKER-PLANCK EQUATION

The stochastic evolution of the probability distribution P of two-dimensional gravity or of string theory is defined by the Focker-Planck equation. In the case of string theory one has

$$\begin{aligned} \frac{\partial}{\partial t} P &= (L^{(g)*} + L^{(X)*})P, \\ L^{(g)*} &= -\frac{1}{8} \int d^2 z e^{\tilde{\varphi}} \left((1 - \mu_z^z \mu_z^z)^2 \left(\frac{\delta}{\delta \mu_z^z} \left(\frac{\delta}{\delta \mu_z^z} + \frac{\delta S_{cl}}{\delta \mu_z^z} \right) + \frac{\delta}{\delta \mu_z^z} \left(\frac{\delta}{\delta \mu_z^z} + \frac{\delta S_{cl}}{\delta \mu_z^z} \right) \right) \right. \\ &\quad \left. + x^{-1} \frac{\delta}{\delta \tilde{\varphi}} \left(\frac{\delta}{\delta \tilde{\varphi}} + \frac{\delta S_{cl}}{\delta \tilde{\varphi}} \right) \right) \\ L^{(X)*} &= \frac{1}{2} \int d^2 z e^{\tilde{\varphi}} \frac{\delta}{\delta X_\mu} \left(\frac{\delta}{\delta X_\mu} + \frac{\delta S_{cl}}{\delta X_\mu} \right). \end{aligned} \quad (3.16)$$

The case of two-dimensional gravity is obtained, in a path-integral language, by integrating out the X^μ fields. The Focker-Planck equation then satisfied by P is the same as equation (3.16) but with $L^{(g)*}$ only.

As in Yang-Mills gauge theories, the gauge (diffeomorphism) invariance permits the modification of the evolution operator L^* into an operator $L_{v,W}^*$ where $v(z, \bar{z})$ is an arbitrarily chosen vector field and $W(z, \bar{z})$ is a function¹⁰⁾:

$$\begin{aligned} L_{v,W}^{(g)*} &= L^{(g)*} - \frac{1}{8} \int d^2 z e^{\tilde{\varphi}} \left((1 - \mu_z^z \mu_z^z)^2 \left(\frac{\delta}{\delta \mu_z^z} D_z v^z + \frac{\delta}{\delta \mu_z^z} D_z v^z \right) \right. \\ &\quad \left. + x^{-1} \frac{\delta}{\delta \tilde{\varphi}} (W + F(v^z, v^z, \mu_z^z, \mu_z^z, \varphi)) \right). \end{aligned} \quad (3.17)$$

$v^z, v^{\bar{z}}$ and W correspond to diffeomorphism and Weyl transformation parameters, respectively. $D_z v^z$ formally reproduces the action of a diffeomorphism on μ_z^z . D_z, D_z and F have been defined in eqs. (3.2). In the same way one has

$$L_{v,W}^{(X)*} = L^{(X)*} - \frac{1}{2} \int d^2 z e^{\tilde{\varphi}} \frac{\delta}{\delta X_\mu} (v^z (\partial_z - \mu_z^z \partial_z) + v^{\bar{z}} (\partial_{\bar{z}} - \mu_z^z \partial_z)) X. \quad (3.18)$$

3.4 EQUIVALENCE WITH THE STANDARD FADDEEV-POPOV GAUGE FIXING OF THE PATH INTEGRAL QUANTIZATION

We shall only consider two-dimensional gravity with an action $S(\varphi, \mu_z^z, \mu_z^z) \sim \int R \Delta^{-1} R$. (String theory can be considered as a particular case after integrating out the matter fields.) The Focker-Planck equation reads:

$$\frac{\partial}{\partial t} P = L_{v,W}^{(g)*} P. \quad (3.19)$$

We will show that there exists a choice of $v \equiv v(z; \mu_z^z, \mu_z^z, \varphi)$ and $W \equiv W(z; \mu_z^z, \mu_z^z, \varphi)$ such that the Faddeev-Popov distribution

$$P_{FP} = \int dc d\bar{c} \exp(-S_{tot}), \quad S_{tot} = S_{cl} + S_{gf} \quad (3.20)$$

is an equilibrium solution of the Focker-Planck equation (3.19), i.e. $L_{v,W}^{(g)*} P_{FP} = 0$.

Choosing $W = -F(v^z, v^z, \mu_z^z, \mu_z^z, \tilde{\varphi})$ one has

$$L_{v,W}^{(g)*} P_{FP} = L^{(g)*} P_{FP} - \frac{1}{8} \int d^2 z e^{\tilde{\varphi}} (1 - \mu_z^z \mu_z^z)^2 \left(\frac{\delta}{\delta \mu_z^z} D_z v^z + \frac{\delta}{\delta \mu_z^z} D_z v^z \right) P_{FP} \quad (3.21)$$

We choose for v

$$v^z = \frac{1}{P_{FP}} \int dc d\bar{c} d^2 z' c^z(z) \bar{c}_{zz}(z') \frac{\delta S_{gf}}{\delta \mu_z^z(z')} \exp(-S_{tot}) \quad (3.22)$$

One has a mirror equation for $v^{\bar{z}}$ with z and $\bar{z}$ interchanged. Using the last equation (3.7), we have

$$D_z v^z = \frac{1}{P_{FP}} \int dc d\bar{c} d^2 z' \bar{c}_{zz}(z') \frac{1}{2} e^{\tilde{\varphi}} \frac{\delta S_{gf}}{\delta \mu_z^z(z')} \frac{\delta}{\delta \mu_z^z(z')} \exp(-S_{tot}) \quad (3.23)$$

and after integrating by parts

$$\begin{aligned} D_z v^z &= \frac{1}{P_{FP}} \int dc d\bar{c} \frac{\delta S_{gf}}{\delta \mu_z^z(z)} \exp(-S_{tot}) \\ &= -\frac{1}{P_{FP}} \int dc d\bar{c} \left(\frac{\delta}{\delta \mu_z^z(z)} + \frac{\delta S_{cl}}{\delta \mu_z^z(z)} \right) \exp(-S_{tot}) \end{aligned} \quad (3.24)$$

Inserting this, and the analogue for $D_z v^z$, into (3.21) one finds

$$L_{v,W}^{(g)*} P_{FP} = -\frac{1}{8x} \int d^2 z e^{\tilde{\varphi}} \frac{\delta}{\delta \tilde{\varphi}} \left(\frac{\delta}{\delta \tilde{\varphi}} + \frac{\delta S_{cl}}{\delta \tilde{\varphi}} \right) \int dc d\tilde{c} \exp(-S_{tot}) = 0 \quad (3.25)$$

since $S_{tot} - S_{cl} = S_{gf}$ does not depend on $\tilde{\varphi}$. Q.E.D.

3.5 RESTRICTING THE MODULAR PARAMETER TO ONE FUNDAMENTAL DOMAIN

We will display a choice of v that tends to confine the metric within the analogue of the first Gribov horizon, i.e. one fundamental domain.

We define a norm for $\mu_z^z, \mu_{\bar{z}}^{\bar{z}}$ (Lyapunov function)

$$\|\mu\|^2 = \int d^2 z e^{\tilde{\varphi}} \mu_z^z \mu_{\bar{z}}^{\bar{z}} \quad (3.26)$$

Looking at the Focker-Planck equations (3.16) and (3.17), one sees that the equivalent Langevin equations for $\mu_z^z, \mu_{\bar{z}}^{\bar{z}}$ and $\tilde{\varphi}$ are:

$$\begin{aligned} \dot{\mu}_z^z &= -\frac{\delta S_{cl}}{\delta \mu_z^z} + \alpha D_{\bar{z}} v^z + b_z^z, \\ \dot{\mu}_{\bar{z}}^{\bar{z}} &= -\frac{\delta S_{cl}}{\delta \mu_{\bar{z}}^{\bar{z}}} + \alpha D_z v^{\bar{z}} + b_{\bar{z}}^{\bar{z}} \quad (3.27) \\ \dot{\tilde{\varphi}} &= -\frac{\delta S_{cl}}{\delta \tilde{\varphi}} + \alpha (\partial_z V^z + \partial_{\bar{z}} V^{\bar{z}} + V^z \partial_z \tilde{\varphi} + V^{\bar{z}} \partial_{\bar{z}} \tilde{\varphi}) + W + b_{\tilde{\varphi}}. \quad (3.28) \end{aligned}$$

We have introduced a real parameter α , and $b_z^z, b_{\bar{z}}^{\bar{z}}$ and $b_{\tilde{\varphi}}$ are the stochastic noises for $\mu_z^z, \mu_{\bar{z}}^{\bar{z}}$ and $\tilde{\varphi}$ respectively. The vector field V is related to the vector field v by

* Note that $\frac{\delta S_{cl}}{\delta \mu_z^z}$ (resp. $\frac{\delta S_{cl}}{\delta \mu_{\bar{z}}^{\bar{z}}}$) transform as $(\dots)_z^z$ (resp. $(\dots)_{\bar{z}}^{\bar{z}}$) tensors (cf. the remark after eq. (3.7)).

$v^z = V^z + \mu_z^z V^z, v^{\bar{z}} = V^{\bar{z}} + \mu_{\bar{z}}^{\bar{z}} V^{\bar{z}}$. In the limit $\alpha \rightarrow \infty$, we have

$$\begin{aligned} \frac{d}{dt} \|\mu\|^2 &= \alpha \int d^2 z (\partial_z (V^z e^{\tilde{\varphi}}) \mu_z^z \mu_{\bar{z}}^{\bar{z}} + e^{\tilde{\varphi}} D_{\bar{z}} v^z \mu_z^z + (z \leftrightarrow \bar{z})) \\ &= -\alpha \int d^2 z e^{\tilde{\varphi}} (V^z x_{\bar{z}} + V^{\bar{z}} x_z) \end{aligned}$$

$$x_z = \partial_{\bar{z}} (\mu_z^z \mu_{\bar{z}}^{\bar{z}}) + (1 - \mu_z^z \mu_{\bar{z}}^{\bar{z}}) \mu_z^z \partial_z \tilde{\varphi} + D_z \mu_z^z - \mu_z^z D_{\bar{z}} \mu_{\bar{z}}^{\bar{z}} \quad (3.29)$$

where D_z on μ_z^z and $D_{\bar{z}}$ on $\mu_{\bar{z}}^{\bar{z}}$ are defined by partial integration. Choosing

$$V^z = e^{-\tilde{\varphi}} x_{\bar{z}}, \quad V^{\bar{z}} = e^{-\tilde{\varphi}} x_z, \quad (3.30)$$

we obtain:

$$\frac{d}{dt} \|\mu\|^2 = -2\alpha \int d^2 z x_z x_{\bar{z}} \leq 0 \quad (3.31)$$

Hence, under stochastic evolution, taking into account the fluctuations induced by b_z^z and $b_{\bar{z}}^{\bar{z}}$, μ_z^z and $\mu_{\bar{z}}^{\bar{z}}$ will be confined in a domain around the origin $\mu_z^z = \mu_{\bar{z}}^{\bar{z}} = 0$. An interesting calculation would be to verify whether this domain is truly a fundamental domain.

Another gauge function than the one of the conformal gauge exists, namely one that is linear in the derivatives of the metric (see e.g. ref. 18). The corresponding gauge, called harmonic gauge, is the analogue of the Lorentz gauge in Yang-Mills theories. The gauge function $\mu_z^z - \mu_{\bar{z}}^{\bar{z}}$ in (3.6) is replaced by $D_z \mu_z^z$, and therefore instead of (3.6) one has

$$S_{gf} = s \int (\bar{c}_z D_z \mu_z^z + \bar{c}_{\bar{z}} D_{\bar{z}} \mu_{\bar{z}}^{\bar{z}}) \quad (3.32)$$

In this case, the Faddeev-Popov operator is (μ denotes μ_z^z and $\bar{\mu}$ denotes $\mu_{\bar{z}}^{\bar{z}}$)

$$FP = \left(\frac{\delta}{\delta \mu} (D_z \mu) D_z - \frac{\delta}{\delta \bar{\mu}} (D_{\bar{z}} \bar{\mu}) D_{\bar{z}} \right) \quad (3.33)$$

An easy calculation shows that (again in the $\alpha \rightarrow \infty$ limit)

$$\frac{d}{dt} \|D\mu\|^2 = \alpha \int d^2z (D_z \mu_z^z, D_z \mu_z^z) FP \left(\frac{v^z}{v^z} \right) \quad (3.34)$$

where

$$\|D\mu\|^2 = \int d^2z D_z \mu_z^z D_z \mu_z^z. \quad (3.35)$$

With the choice

$$v^z = e^{-\tilde{\varphi}} D_z \mu_z^z, \quad v^{\bar{z}} = e^{-\tilde{\varphi}} D_{\bar{z}} \mu_{\bar{z}}^{\bar{z}} \quad (3.36)$$

we find that $\frac{d}{dt} \|D\mu\|^2 \leq 0$ if the eigenvalues of FP are negative. Hence, the domain in the configuration space for μ_z^z and $\mu_{\bar{z}}^{\bar{z}}$, for which $D_z \mu_z^z = D_{\bar{z}} \mu_{\bar{z}}^{\bar{z}} = 0$ and the eigenvalues of FP are negative, is stable under stochastic evolution. It is therefore a good candidate for being the first fundamental domain in the harmonic gauge, defined by the gauge functions $D_z \mu_z^z$ and $D_{\bar{z}} \mu_{\bar{z}}^{\bar{z}}$. (See the works of Zwanziger in refs 2 and 3 where a detailed analysis is done for the corresponding Lorentz gauge of Yang-Mills theories.)

It is also possible to define a Lyapunov function for φ ,

$$\|\varphi\| = \int d^2z (\exp(\varphi) - \exp(\varphi_0)) , \quad (3.37)$$

where φ_0 is a given classical background such that $\exp(\varphi_0)$ transforms as a $(\dots)_{z\bar{z}}$ tensor. Using the Langevin equation (3.28) for $\tilde{\varphi}$, we deduce the one for φ . Knowing that, with the above choice of v^z and $v^{\bar{z}}$, μ_z^z and $\mu_{\bar{z}}^{\bar{z}}$ are confined in a small domain around 0, and we can make an expansion in powers of μ_z^z and $\mu_{\bar{z}}^{\bar{z}}$. Then $\varphi \simeq \tilde{\varphi} + \mu_z^z \mu_{\bar{z}}^{\bar{z}}$ and in the $\alpha \rightarrow \infty$ limit, we find

$$\begin{aligned} \frac{d}{dt} \|\varphi\| &= \alpha \int d^2z e^{\varphi} (\mu_z^z D_z v^z + \mu_{\bar{z}}^{\bar{z}} D_{\bar{z}} v^{\bar{z}}) \\ &\simeq -\alpha \int d^2z e^{\varphi} (x_{\bar{z}} v^z + x_z v^{\bar{z}}) \leq 0. \end{aligned} \quad (3.38)$$

(We integrated by parts and only kept terms that are first order in μ_z^z , $\mu_{\bar{z}}^{\bar{z}}$.) With

the same choice of v^z , $v^{\bar{z}}$ as above for μ_z^z and $\mu_{\bar{z}}^{\bar{z}}$, also φ is confined within a domain around the origin.

4. Relation with three-dimensional classical gravity

The stochastic equations (3.27) for μ_z^z and $\mu_{\bar{z}}^{\bar{z}}$ are (with $\alpha = 1$)

$$\begin{aligned} \partial_0 \mu_z^z &= \partial_{\bar{z}} v^z + v^z \partial_z \mu_z^z - \mu_z^z \partial_z v^z + B_z^z \\ \partial_0 \mu_{\bar{z}}^{\bar{z}} &= \partial_z v^{\bar{z}} + v^{\bar{z}} \partial_{\bar{z}} \mu_{\bar{z}}^{\bar{z}} - \mu_{\bar{z}}^{\bar{z}} \partial_{\bar{z}} v^{\bar{z}} + B_{\bar{z}}^{\bar{z}} \end{aligned} \quad (4.1)$$

We have defined $B_z^z = -\frac{\delta S_A}{\delta \mu_z^z} + b_z^z$ and $B_{\bar{z}}^{\bar{z}} = -\frac{\delta S_A}{\delta \mu_{\bar{z}}^{\bar{z}}} + b_{\bar{z}}^{\bar{z}}$. The Langevin equation for φ is easily deduced from (3.28):

$$\partial_0 \varphi = \partial_z v^z + \partial_{\bar{z}} v^{\bar{z}} + V^z (\partial_z \varphi - \partial_{\bar{z}} \mu_z^z) + V^{\bar{z}} (\partial_{\bar{z}} \varphi - \partial_z \mu_{\bar{z}}^{\bar{z}}) + W + B_\varphi \quad (4.2)$$

where $B_\varphi = -\frac{\delta S_A}{\delta \varphi} + b_\varphi$.

In the case of a Yang-Mills symmetry, the stochastic time can be unified with the ordinary coordinates in a gauge covariant way. One uses in this case the freedom in the Langevin equation as a gauge field component along stochastic time¹⁰⁾. For 2D gravity, we are going to interpret the stochastic time as a real time coordinate, which completes the complex coordinates z and $\bar{z}$. To get a covariant theory in the three-dimensional manifold with coordinates z , $\bar{z}$ and t , we must introduce a dreibein and a spin connection, relate their components to the quantities that we have used to describe the stochastic quantization of 2D gravity, and express all Langevin equations in terms of constraints on the curvatures of 3D space. As a result two-dimensional quantum gravity will be related to classical three-dimensional gravity.

We first notice that eqs (4.1) can be written in the following form¹⁰⁾:

$$\begin{aligned} dE^z - E^z \partial_z E^z &= B_z^z dt d\bar{z} \\ dE^{\bar{z}} - E^{\bar{z}} \partial_{\bar{z}} E^{\bar{z}} &= B_{\bar{z}}^{\bar{z}} dz dt \end{aligned} \quad (4.3)$$

In (4.3), we have defined the exterior derivative:

$$d = dz \partial_z + d\bar{z} \partial_{\bar{z}} + dt \partial_t \quad (4.4)$$

and the 1-forms :

$$\begin{aligned} E^z &= dz + \mu_z^z d\bar{z} + v^z dt \\ E^{\bar{z}} &= d\bar{z} + \mu_{\bar{z}}^{\bar{z}} dz + v^{\bar{z}} dt \end{aligned} \quad (4.5)$$

From (4.5) it is obvious that v^z and $v^{\bar{z}}$ should be written respectively as μ_0^z and $\mu_0^{\bar{z}}$.

Since v^z and its complex conjugate appear as differentials in the $0z$ and $0\bar{z}$ directions, it is clear that one must complete eq. (4.5) and introduce the following one-form:

$$E^0 = dt + e^\varphi (v^{\bar{z}} dz + v^z d\bar{z}) \quad (4.6)$$

The triplet of one-forms E^a , (the latin indices $a, b, \dots$ stand for $z, \bar{z}$ and 0), can be used to define a dreibein e^a . To restore the Weyl weights, we need to include the conformal weight φ for E^z and $E^{\bar{z}}$ and a new one (time dilatation weight) φ for E^0 . The dreibein e^a is defined as:

$$\begin{aligned} e^z &= e^{\varphi/2} E^z \\ e^{\bar{z}} &= e^{\varphi/2} E^{\bar{z}} \\ e^0 &= e^{\varphi_0/2} E^0 \end{aligned} \quad (4.7)$$

e^a can thus be written as follows:

$$e^a = \begin{pmatrix} e^{\varphi_0/2} & 0 & 0 \\ 0 & e^{\varphi/2} & 0 \\ 0 & 0 & e^{\varphi/2} \end{pmatrix} \begin{pmatrix} 1 & \mu_z^0 & \mu_{\bar{z}}^0 \\ \mu_0^z & 1 & \mu_z^{\bar{z}} \\ \mu_0^{\bar{z}} & \mu_{\bar{z}}^z & 1 \end{pmatrix} \begin{pmatrix} dt \\ dz \\ d\bar{z} \end{pmatrix} \quad (4.8)$$

where $\mu_0^z = v^z$, $\mu_0^{\bar{z}} = v^{\bar{z}}$, $\mu_z^0 = e^\varphi v^{\bar{z}}$, $\mu_{\bar{z}}^0 = e^\varphi v^z$.

As a whole e^a depends on six independent real functions as a 3D metric g_{ij} . We will show below that φ_0 can be determined, so *no* new degree of freedom is truly added to the degrees of freedom of the 2D model. (4.8) is a generalization of the parametrization of zweibeins using Beltrami differentials:

$$e^\varphi \begin{pmatrix} 1 & \mu_z^{\bar{z}} \\ \mu_{\bar{z}}^z & 1 \end{pmatrix} \begin{pmatrix} dz \\ d\bar{z} \end{pmatrix} \quad (4.9)$$

Eq.(4.3) is the signal that one can define a torsion which vanishes, where the spin connection is related to the variables B , i.e. the noise and the equations of motion of the 2D model. (See ref. 17, in which the BRST equations for the 2D diffeomorphisms are derived from analogous equations, linked to the Weyl-Poincaré algebra in two dimensions). To build such a torsion, we introduce a spin connection, which is made of three one-forms $\omega^a = \omega_b^a dx^b$ in the space with coordinates $(z, \bar{z}, 0)$. Since the Poincaré symmetry in three dimensions is a contraction of $SL(2, \mathbb{C})$ or $SU(2) \times SU(2)$, we impose that the six one-forms contained in the dreibein e^a and the spin connection ω^a are gauge fields either for the group $SL(2, \mathbb{C})$ or $SU(2) \times SU(2)$ or the 3D Poincaré group $ISO(3)$ itself. The difference between the three choices lies in the value of the cosmological constant (negative, positive or zero) when one expands the Chern-Simons action for these groups. More precisely, we define a connection A :^{*}

$$A = e^a N^a + \omega^a J^a \quad (4.10)$$

The torsions T^a and curvatures R^a are defined from the curvature of A :

$$F = dA + \frac{1}{2}[A, A] = T^a N^a + R^a J^a. \quad (4.11)$$

The generators N and T satisfy the following commutation relations:

$$[N^a, N^b] = \lambda \epsilon^{abc} J^c$$

* Our conventions are such that $e^a N^a = e^0 N^0 + e^z N^z + e^{\bar{z}} N^{\bar{z}}$, etc.

$$[J^a, J^b] = \epsilon^{abc} J^c$$

$$[N^a, J^b] = \epsilon^{abc} N^c. \quad (4.12)$$

with $\epsilon^{0z\bar{z}} = 1$ etc.[†] $\lambda = +1, -1, 0$ correspond to $SU(2) \times SU(2)$, $SL(2, \mathbb{C})$ and $ISO(3)$, respectively. The normalizations are $Tr J^a J^b = Tr N^a N^b = 0$ and $Tr J^a N^b = \delta^{ab}$ (with $\delta^{z\bar{z}} = \delta^{00} = 1$). Thinking of $SL(2, \mathbb{C})$ as the 4D Lorentz algebra, we can call the N and J boosts and rotations, respectively.

Combining (4.10) and (4.11), we obtain :

$$T^a = de^a + \epsilon^{abc} \omega^b e^c$$

$$R^a = d\omega^a + \frac{1}{2} \epsilon^{abc} \omega^b \omega^c + \frac{1}{2} \lambda \epsilon^{abc} e^b e^c. \quad (4.13)$$

The interesting observation is that the Langevin equations (4.3), i.e. (4.1), can be re-expressed as the vanishing condition for the curvature F , $F = 0$. There is of course more information in the equation $F = 0$, since this curvature is made of six three-forms, each having three projections along $dz, d\bar{z}$ and dt constrained by the Bianchi identities $dF = [F, A]$.

To study the consequences of $F = 0$ we consider separately $T^a = 0$ and $R^a = 0$.

The conditions of vanishing torsion $T^a = 0$ determine the nine components ω_b^a of the spin connection in terms of the dreibein components and its derivatives. Two of these equations are the stochastic equations (4.1). One finds that $B_z^{\bar{z}}$ and B_z^z in (4.1) can be identified with the spin-connection components $\omega_z^{\bar{z}}$ and ω_z^z . Another equation gives the stochastic equation (4.2) for φ , provided $\omega_z^{\bar{z}} - \omega_z^z = B_\varphi + \dots$. The six other equations are just definitions of some components of the spin connections as functions of the e^a 's.

[†] With this convention one has, e.g., for J^0 and $J^z \sim J^+$: $[J^0, J^z] = \epsilon^{0zz} J^z = +J^z$ as one should have for $SU(2)$.

The conditions of vanishing curvatures $R^a = 0$ give second order differential equations. Only six of them are independent because of the Bianchi identities $dF + [A, F] = 0$. Three of them permit the determination of the z and $\bar{z}$ derivatives of the physically spurious stochastic time rescaling factor φ_0 for any given choice of μ, v and φ . Combining two other equations with $T = 0$ gives two differential equations on the function W . If we choose $W = \partial_0 \varphi_0$ we get two more equations for φ_0 which allow us to completely fix the dependence on z and $\bar{z}$ of the three derivatives of φ_0 . The remaining equation gives a constraint on the rescaling factor φ which can be identified as the constraint that the Riemannian curvature of the 2D Riemann surface is equal to λ . The choices $SU(2) \times SU(2)$, $ISO(3)$ or $SL(2, \mathbb{C})$ are thus adapted to the cases where the 2D Riemann surface has positive, zero or negative curvature, i.e. its genus is $g = 0, g = 1$ or $g \geq 2$, respectively. What is important is that the conditions $R = 0$ do not provide unphysical restrictions on the physical variables of 2D gravity.

To understand this in detail, it is easier to work at first order in $v^z, \mu_z^z, \mu_z^{\bar{z}}$ and $v^{\bar{z}}$. From the nine equations stemming from $T^a = 0$, we find:

$$\omega_z^z = \partial_0 \mu_z^z - \partial_z v^z \quad (4.14a)$$

$$\omega_z^{\bar{z}} = \partial_z v^{\bar{z}} - \partial_0 \mu_z^{\bar{z}}$$

$$\omega_z^z = \partial_0 \varphi - \partial_z v^z \quad (4.14b)$$

$$\omega_z^{\bar{z}} = \partial_z v^z - \partial_0 \varphi$$

$$\omega_0^0 = \partial_z v^{\bar{z}} - \partial_z v^z$$

$$\omega_z^0 = \partial_z \mu_z^{\bar{z}} - \partial_z \varphi$$

$$\omega_z^{\bar{z}} = \partial_z \varphi - \partial_z \mu_z^z \quad (4.14c)$$

$$\omega_0^z = \partial_0 v^z - \partial_z \varphi_0$$

$$\omega_0^{\bar{z}} = \partial_z \varphi_0 - \partial_0 v^{\bar{z}}$$

We see that (4.14a) gives the stochastic equations (4.1) for $\mu_z^z, \mu_z^{\bar{z}}$ provided the components ω_z^z and $\omega_z^{\bar{z}}$ are identified with combinations of the stochastic noises

and the equations of motion of the 2D model, i.e. with the B_z^z and B_z^z . (4.14b) gives the stochastic equation for φ provided one has the relation $\omega_z^z - \omega_z^z = B_\varphi + \dots$

The 3×3 equations of zero curvature $R_\mu^\alpha = 0$ are related by 3 constraints $dT^a = \epsilon^{abc} R^b e^c$. 6 independent equations survive:

$$\partial_z \partial_z \varphi_0 = \lambda - \partial_0(W + B_\varphi)$$

$$\partial_z^2 \varphi_0 = \lambda \mu_z^z + \partial_0(\partial_z v^z - B_z^z) \quad (4.15a)$$

$$\partial_z^2 \varphi_0 = \lambda \mu_z^z + \partial_0(\partial_z v^z - B_z^z)$$

$$\begin{aligned} \partial_z(W + B_\varphi) &= \partial_z B_z^z - \partial_z \partial_z v^z + \lambda v^z \\ \partial_z(W + B_\varphi) &= \partial_z B_z^z - \partial_z \partial_z v^z + \lambda v^z \end{aligned} \quad (4.15b)$$

$$2\partial_z \partial_z \varphi - \partial_z^2 \mu_z^z - \partial_z^2 \mu_z^z = \lambda \quad (4.15c)$$

The equations (4.15a) determine the z and $\bar{z}$ derivatives of φ_0 up to a function of t . The equations (4.15b) give W (again up to a function of t). It seems natural to identify W with $\partial_0 \varphi_0$. Then equations (4.15a,b) determine all the derivatives of φ_0 up to a function of t . The l.h.s. of equation (4.15c) is nothing but the linearized curvature of the 2D Riemann surface. Hence this equation fixes the 2D curvature to the value of λ .

The equation $F = 0$ is the equation of motion of the Chern-Simons 3D action:

$$I_{CS} = \int_{\mathcal{M}_3} \text{tr} \left(AdA + \frac{2}{3} A^3 \right) \quad (4.16a)$$

Using our normalization convention for the traces of the generators, one finds

$$I_{CS} = 2 \int_{\mathcal{M}_3} \left(e^a (d\omega^a + \frac{1}{2} \epsilon^{abc} e^a \omega^b \omega^c) + \frac{1}{6} \lambda \epsilon^{abc} e^a e^b e^c \right) \quad (4.16b)$$

This action describes gravity in a Minkowskian 3D space. The last term, proportional to λ , can be interpreted as a cosmological term. If we had chosen a different

normalization of the generators, namely $\text{Tr} J^a J^b = \lambda \text{Tr} N^a N^b = \delta^{ab}$, $\text{Tr} J^a N^b = 0$ we would have obtained the gravitational Chern-Simons action

$$\int_{\mathcal{M}_3} \left(\lambda e^a (de^a + \epsilon^{abc} e^b \omega^c) + \omega^a d\omega^a + \frac{1}{3} \epsilon^{abc} \omega^a \omega^b \omega^c \right). \quad (4.16c)$$

Of course, both actions (4.16b) and (4.16c) give the same (classical) equations of motion, $T = R = 0$. In fact both actions can be related by a change of basis for the generators J and N .

We have thus displayed a relation between the classical solution of *three-dimensional* gravity with cosmological constant and the stochastic quantization of *two-dimensional* gravity. One can go further. To study the classical equations $F=0$, one can consider them as the classical limit of the quantum field theory defined from this Chern-Simons action. Clearly, it would be interesting to compare expectation values of Wilson loops in 3 dimensions, $\langle T \exp \oint A \rangle$, computed with the Chern-Simons action, with observables in two-dimensional conformal field theories. In the next section we will show that the theory in three dimensions is itself related to the four-dimensional topological gravity, defined as a gauge-fixed action of the Euler invariant $\int_{N_2 \times R \times R} \epsilon^{\mu\nu\rho\sigma} R_{\mu\nu}^{ab} R_{\rho\sigma}^{ab}$ by means of self-duality conditions.

5. Relation with four-dimensional topological gravity

To study the equations $F=0$, it is relevant to introduce the Chern-Simons action $I_{CS} = \int_{\mathcal{M}_3} (AdA + \frac{2}{3}A^3)$, and the corresponding quantum field theory. However, the Chern-Simons action is first order. Its Hamiltonian vanishes modulo gauge transformations and the ordinary path integral method is difficult to interpret.

It has been shown that stochastic quantization of the 3-D Chern-Simons action provides a supersymmetric functional integral representation with an action which is second order and of the type of a topological Yang-Mills theory (see refs 9 and 10 for the details).

We thus introduce a second stochastic time variable x^4 . Greek indices will stand for 4-D indices made of $z, \bar{z}, t$ and x^4 . We shall express the supersymmetric representation of the Langevin equation of the theory associated to I_{CS} , by means of an action defined over the 4-manifold $M_4 = N_2 \times R \times R$. The Chern-Simons action I_{CS} is gauge-invariant under the gauge transformations $\delta A_i = D_i \epsilon$, where D_i is the covariant derivative for $SL(2, \mathbb{C})$ $ISO(3)$ or $SU(2) \times SU(2)$. Therefore, in the Langevin equation for A , we have the freedom expressed in (2.7) or (3.27), that we denote A_4 instead of v :

$$\partial_4 A_i = -\frac{\delta I_{CS}}{\delta A_i} + D_i A_4 + b_i. \quad (5.1)$$

Defining $F_{4i} = \partial_4 A_i - \partial_i A_4 + [A_4, A_i]$, (5.1) reads:

$$F_{4i} = -c_i^{jk} F_{jk} + b_i \quad (5.2)$$

We define a one-form $A = A_\mu(z, \bar{z}, t, x^4) dx^\mu$ in the manifold $M_4 = N_2 \times R \times R$, with $A_\mu = (A_i, A_4)$. It makes sense to write the indices of A as A_μ^{ab} , with an antisymmetry on the indices a and b , $1 \leq a, b \leq 4$, and $A_\mu^{4i} = e_\mu^i$, $A_\mu^{ij} = \epsilon_\mu^{ij} \omega_\mu^k$. In the cases $\lambda = \pm 1$, $A_\mu dx^\mu$ can be interpreted as a spin-connection in M_4 , since

in these cases the groups $SL(2, \mathbb{C}) \simeq SO(3, 1)$ or $SU(2) \times SU(2) \simeq SO(4)$ are isomorphic to the symmetry group of the tangent spaces of M_4 .

Let $J_i(x, t)$ be the sources of the A_i 's. The stochastic partition function is:

$$Z[J] = \int [Db_i] \exp \int d^4x \text{Tr} \left(-\frac{1}{2} b_i^2 + J_i(x, t) A_i(x, t) \right), \quad (5.3)$$

where the $b_i(x, t)$'s satisfy (5.1).

Equation (5.2) can be inserted as a constraint in the generating functional of correlation functions. By eliminating the noises as functions of the A_i , and taking into account all relevant Jacobians, one obtains in the standard way a supersymmetric representation in four dimensions. The latter depends of course on the choice of A_4 . However, gauge-independent correlation functions are independent of A_4 . Thus we can further integrate functionally over arbitrary chosen configurations of A_4 . Our choice is that A_4 itself satisfies a Langevin equation:

$$\partial_4 A_4 = \mathcal{F} + b_4. \quad (5.4)$$

b_4 is a noise, and the functional $\mathcal{F}[A]$ is arbitrarily chosen, provided the stochastic process induced by the Langevin equations (5.1) and (5.4) is meaningful. ($\mathcal{F}[A] = \partial_i A^i$ is a natural choice to obtain Lorentz covariance in M_4).

Equations (5.2) and (5.4) show that the generating functional is obtained by a functional integral concentrated around the solutions of the (anti)self-duality equations $F_{4i} = -c_i^{jk} F_{jk}$ and the gauge condition $\mathcal{F}[A(x)] = 0$. Doing all relevant manipulations to eliminate the b 's in terms of the A 's, and choosing $\mathcal{F}[A] = g^{ij} \partial_i A_j$, one gets the following expression (see ref. 9):

$$Z[J] = \int [DA_\mu][D\Psi_\mu][D\chi_{\mu\nu}][Dc][D\bar{c}][D\bar{\eta}][D\Phi][D\bar{\Phi}] \exp \left(I_{GF} + \int d^4x \text{Tr} J_i A_i \right) \quad (5.5)$$

where:

$$\begin{aligned}
I_{GF} = \int d^4x \text{Tr} & \left(\frac{1}{2} (F_{\mu\nu} + \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma})^2 \right. \\
& + \frac{1}{2} (\partial_\mu A^\mu)^2 - \bar{\eta} D_\mu D^\mu c \\
& - \chi^{\mu\nu} D_{[\mu} \Psi_{\nu]} - \bar{c} \partial_\mu \Psi^\mu - \bar{\eta} D_\mu \Psi^\mu \\
& \left. + \bar{\Phi} (D_\mu D^\mu \Phi + [\Psi^\mu, \Psi_\mu + D_\mu c] + D_\mu [\Psi^\mu, c]) \right). \tag{5.6}
\end{aligned}$$

It is understood that a metric $g_{\mu\nu}$ has been introduced to define $\partial_\mu A^\mu$ as well as $F_{\mu\nu} + \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$ and its square in I_{GF} .

In (5.6), one has fermionic and bosonic ghosts fields. All of them take their values in the same Lie algebra as A . As shown in refs 9 and 10, they come from the exponentiation of Jacobians which arise when one eliminates the noises as functions of the A 's. The anticommuting one-form ghosts Ψ_μ are the stochastic ghosts for the A_μ . The self-dual two-form $\chi_{\mu\nu}$ and $\bar{c}$ are the corresponding antighosts. $\bar{c}$ and $\bar{\eta}$ play the role of ordinary Faddeev-Popov ghosts. Φ and $\bar{\Phi}$ are commuting fields: they are used to exponentiate the determinant of a map involving the anticommuting ghost $\bar{c}$. The gradings are defined by attributing ghost numbers to all fields: 0 for A_μ and b_μ ; 1 for c and Ψ_μ ; -1 for $\bar{c}$, $\bar{\eta}$ and $\chi_{\mu\nu}$; 2 for Φ and -2 for $\bar{\Phi}$, and the grading of any given form or operator is defined as the sum modulo two of the ghost number and form degree.

Since $\frac{1}{2} \text{Tr} (F_{\mu\nu} + \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma})^2 = F_{\mu\nu}^{ab} F^{ab}{}_{\mu\nu} + \epsilon_{\mu\nu\rho\sigma} F^{ab}{}_{\rho\sigma} F^{ab}{}_{\mu\nu}$, we recognize in (5.6) the action of topological 4D gravity¹⁴⁾, based on the BRST invariant gauge fixing of the Euler number $\int_{M_4} d^4x \epsilon_{\mu\nu\rho\sigma} F^{ab}{}_{\rho\sigma} F^{ab}{}_{\mu\nu}$ by means of self-duality equations¹⁵⁾. $\frac{1}{4} F_{\mu\nu}^{ab} F^{ab}{}_{\mu\nu}$ is the Weyl action for 4D gravity.

To recognize that I_{GF} is topological, one can introduce Lagrange multiplier

fields $\lambda_{\mu\nu}$ and λ . In this way, I_{GF} can be written as follows:

$$\begin{aligned}
I_{GF} = \int d^4x & \left(\frac{1}{2} \lambda_{\mu\nu} \lambda^{\mu\nu} + \lambda^{\mu\nu} (F_{\mu\nu} + \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}) \right. \\
& + \frac{1}{2} \lambda^2 + \lambda (\partial_\mu A^\mu) - \bar{\eta} D_\mu D^\mu c \\
& - \chi^{\mu\nu} D_{[\mu} \Psi_{\nu]} - \bar{c} \partial_\mu \Psi^\mu - \bar{\eta} D_\mu \Psi^\mu \\
& \left. + \bar{\Phi} (D_\mu D^\mu \Phi + [\Psi^\mu, \Psi_\mu + D_\mu c] + D_\mu [\Psi^\mu, c]) \right) \tag{5.7}
\end{aligned}$$

By functional integration, I_{GF} generates perturbatively expectation values which are independent of the choice of metric in M_4 . This property can be deduced from the fact that I_{GF} is a BRST exact action:

$$\begin{aligned}
I_{GF} = \int d^4x \text{Tr} \, s_{top} & \left(\chi^{\mu\nu} (F_{\mu\nu} + \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} + \frac{1}{2} \lambda_{\mu\nu}) + \bar{c} (\partial_\mu A^\mu + \frac{1}{2} \lambda) \right. \\
& \left. - \bar{\Phi} (D_\mu \Psi^\mu + D_\mu D^\mu c) \right) \tag{5.8}
\end{aligned}$$

where the topological BRST operator s_{top} is:

$$\begin{aligned}
s_{top} A_\mu &= \Psi_\mu \\
s_{top} c &= \bar{\Phi} \\
s_{top} \Psi_\mu &= 0 \\
s_{top} \bar{\Phi} &= 0 \tag{5.9a}
\end{aligned}$$

$$\begin{aligned}
s_{top} \bar{c} &= \lambda & s_{top} \lambda &= 0 \\
s_{top} \chi_{\mu\nu} &= \lambda_{\mu\nu} & s_{top} \lambda_{\mu\nu} &= 0 \\
s_{top} \bar{\Phi} &= \bar{\eta} & s_{top} \bar{\eta} &= 0. \tag{5.9b}
\end{aligned}$$

The geometrical BRST operator of reference 15 is identical to s_{top} in (5.9), up to the following redefinitions of the ghosts, $\Psi_\mu \rightarrow \Psi_\mu + D_\mu c$, $\Phi \rightarrow \Phi - \frac{1}{2} [c, c]$. s_{top} satisfies of course the nilpotent property $s_{top}^2 = 0$, and commutes with the energy momentum tensor since the action is s_{top} -exact. This explains formally the metric

independence of observables. Metric independence in the whole M_4 seems thus related to Polyakov summation over 2D metrics for N and to the inobservability of the stochastic times. Moreover, from what we have done, the gravity in two dimensions appears as an effective theory, obtained by the dimensional reduction of topological 4D gravity defined over a manifold $M_4 = N \times R \times R$ which has two trivial directions along x^4 and t that we interpret as two stochastic times.

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