EXCEPTIONAL MODULAR INVARIANTS OF $N=2$ TENSOR PRODUCTS

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ABSTRACT

Some expected relations among modular invariants of $N = 2$ minimal models, namely $(4)_D \approx (1) \times (1)$, $(10)_E \approx (1) \times (2)$ and $(28)_E \approx (1) \times (3)$, are shown to hold exactly for the entire massless spectrum of string theories built out of these minimal models, not just for $(2,2)$ but also for $(2,1)$ and $(1,1)$ theories. Some new exceptional invariants of $N = 2$ tensor products are also discussed. Unlike $(10)_E$ and $(28)_E$, these new invariants yield entirely new spectra.

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Simple currents [1] provide a natural definition of "exceptional" modular invariant partition functions for any conformal field theory. The set of simple currents of the theory organizes all primary fields into orbits. The simple currents generate modular invariant partition functions that do not mix different orbits. Conversely, it is quite likely (but has not yet been proved) that any modular invariant that does not mix different orbits can be obtained by means of simple currents. It is therefore natural (and in agreement with known cases) to call any invariant with matrix elements between different orbits exceptional (this definition still includes some trivial cases, such as automorphisms generated by charge conjugation).

A rather intensively studied class of partition functions is that of tensor products of $N = 2$ minimal models [2]. Most work in this field has concentrated on four-dimensional string theories of type $(2, 2)$, for which nearly all massless spectra that can be obtained with simple currents have now been computed. The most extensive lists are [3] and [4]. The former covers all $(2, 2)$ and $(2, 1)$ models and lists their complete massless spectra (including in particular gauge singlets), while the latter concentrates on $(2, 2)$ models and contains the complete massless spectra only for a small subset. There are also some differences in the completeness of these lists: in [3] an arbitrary number of simple currents is considered, but these currents were randomly generated and not constructed systematically, while in [4] only D -invariants with at most one additional twist were considered. In practice these differences are not very relevant. Multiple twists turn out to be important mainly for tensor products with many identical factors (at least as far as $(2, 2)$ models are concerned). On the other hand the statistical sample studied in [3] can easily be made large enough to include all spectra listed in [4], and does indeed include all of them now.

For most practical purposes the study of $(2, 2)$ compactifications that can be obtained with $N = 2$ minimal models is now complete, but nevertheless some questions remain. The first question we will address in this paper is which overlaps exist between different spectra, modulo simple currents. The second question is whether the presently known set of modular invariant partition functions of $N = 2$ tensor products is really complete.

We begin with some definitions. We denote $c = 1$ circle compactifications by U_n , where n is the number of primary fields (n is always even). The primary fields are labelled by $q = 0, \dots, n-1$, and their conformal weights are equal to $\frac{q^2}{2n} \bmod 1$. (Here q is defined modulo n , and should be mapped to the range $-\frac{n}{2} < q \leq \frac{n}{2}$ to get the exact conformal

weight including the integer part.) The $SU(2)$ level k Kac-Moody algebra is denoted as $SU(2)_k$, and its primary fields are labelled by integers $l = 0, \dots, k$, where l is twice the spin of the ground state representation. Note that U_2 is equal to $SU(2)_1$. The $N = 2$ minimal models will be denoted as (k) , and are best thought of as cosets:

$$(k) = \frac{SU(2)_k \times U_4}{U_{2k+4}} \Big|_J,$$

where J is the identification current [5]. We will label the fields of these theories as $(l, s; q)$, where the semicolon separates the indices corresponding to the numerator and the denominator of the coset. The identification current is characterized by the labels $(l, 2; k+2)$, and enforces the field identification [6-8][5] implied by the selection rules of the embedding of U_{2k+4} in $SU(2)_k \times U_4$. It may be thought of as any other integer spin current that is put into the chiral algebra of the model, with one important difference: whereas one normally adds the characters that are related by the current to get the characters of the new theory, here one should average them, i.e. add them and divide the result by two (there are no identification fixed points for these minimal models, unlike the more general cosets discussed in [9]). The identification current projects out half the fields (those with odd values of $l+q+s$), and combines the remaining ones into pairs with identical characters.

The simple currents of these models are $(0, s; q)$, where we have used the field identification to set the first label to zero. They generate a center $Z_{4(k+2)}$ for k odd and $Z_{2(k+2)} \times Z_2$ for k even. The orbits are distinguished by the $SU(2)_k$ labels $l = 0, \dots, [k/2]$, so that the orbits of an $N = 2$ tensor product with p factors are labelled by $(l_1, \dots, l_p)$. We expect that for each minimal model the simple currents generate all invariants that are diagonal in $(l_1, \dots, l_p)$. Invariants that are not diagonal in l can be built out of E invariants of $SU(2)_k$, $k = 10, 16$ or 28 . It has been conjectured [2] that this exhausts the list of modular invariant partition functions, but this is not correct, as we will show below.

The simple current procedure does not make any fundamental distinction between D -invariants and any other simple current invariant. The D -invariants are generated by the simple current $(0, 2; k+2)$ which is identified with the $SU(2)$ simple current $(k, 0; 0)$. Indeed, from the results of [4], in which this distinction is made, it is clear that there are large overlaps between twisted versions of combinations of D and A invariants. On the other

hand, when the E -invariants of one or more of the $SU(2)$'s is used, one obtains spectra that are really different from the corresponding A and D invariants.

Nevertheless two of the E -invariants do not really yield anything new. There is substantial evidence in favor of the identifications $(28)_E = (1) \times (3)$, $(10)_E = (1) \times (2)$, as well as $(4)_D = (1) \times (1)$. These relations are suggested on the one hand by the classification of the renormalization group fixed points of Landau-Ginzburg superpotentials [10,11] and on the other hand by the fact that the tensor products on the right-hand side have a central charge below the accumulation point $c = 3$ of the $N = 2$ discrete series. Unfortunately the first argument does not imply an exact relation, but only one up to continuous deformations (which might in principle affect the number of scalars), whereas the second does not specify exactly which modular invariants of (28), (10) or (4) should match the right-hand side. Indeed, there appears to be some confusion in the literature regarding the exactness of these relations, and there are even two independent computations [12], [4] that are in agreement with each other, and that claim to find violations of the exact LG-relations for the number of scalars in $T_2 \times K_3$ compactifications. On the other hand a third computation [3] finds full agreement with the LG-relations, even for $(2,1)$ compactifications. Although the importance of this discrepancy is rather minor in itself, it might point to more serious difficulties with the procedures used in some of these papers, and deserves therefore some attention.

The first step in the proof that one of the LG-relations is exact is the observation that there are two conformal embeddings of $SU(2)$'s in G_2 , namely $SU(2)_{28} \subset G_2$ and $SU(2)_3 \times SU(2)_1 \subset G_2$. This fact has been observed before [13]. Although it certainly goes in the direction of proving $(28)_E = (1) \times (3)$, there are several remaining problems. The first of these problems is the fact that the $U(1)$ factors in the coset theories defining (28), (3) and (1) are totally different. The second problem is that the adjoint representation of G_2 decomposes to $SU(2)_3 \times SU(2)_1$ as $(2,0) + (0,2) + (3,1)$, and hence odd and even l 's are mixed ($SU(2)$ representations of spin j are labelled here by $l = 2j$). Thus this embedding leads to violations of the selection rule $l + q + s = 0 \pmod{2}$, and therefore it does not yield a valid modular invariant partition function of $(3) \times (1)$. Finally, this observation seems to be useless for the other two relations, since for example $SO(5) \supset SU(2)_{10}$ does not contain $SU(2)_2 \times SU(2)_1$.

The crucial point that solves all of these problems is the fact that cosets involving $U(1)$'s

are often related. A simple example is the relation

$$\frac{U_2 U_4}{U_6} \Big|_J \equiv U_{12}, \quad (1)$$

where J is the identification current $(1,2;3)$. To prove this relation it suffices to observe that the left-hand side is a $c = 1$ theory with simple fusion rules for all fields, and hence can only be a U_n conformal field theory, where n is determined by counting the number of primary fields.

In a similar way, the relation $(28)_E = (3) \times (1)$ is due to the following relation among coset conformal field theories:

$$\frac{SU(2)_3 \times U_2 \times U_4}{U_{60}} \Big|_{J_1, J_2} = \frac{SU(2)_3 \times U_4 \times U_2 \times U_4}{U_{10} \times U_6} \Big|_{K_1, K_2, K_3},$$

where

$$\begin{aligned} J_1 &= (3, 1, 0; 0) & K_1 &= (3, 2, 0, 0; 5, 0) \\ J_2 &= (0, 0, 2; 30) & K_2 &= (0, 0, 1, 2; 0, 3) \\ & & K_3 &= (0, 2, 0, 2; 0, 0). \end{aligned}$$

The current J_1 combines $SU(2)_3$ and $U_2 \equiv SU(2)_1$ to G_2 , which in its turn can be written in $SU(2)_{28}$ basis. The left-hand side is thus nothing but the E -invariant of (28). The current J_2 is the identification current of the coset. Note that it does not have components in G_2 since G_2 (or in other words the $(28)_E$ invariant) contains only even values of l . The right-hand side can be recognized as the tensor product $(3) \times (1)$, and K_1 and K_2 as the identification currents of the two factors. The current K_3 is the product of the supercurrents of the two factors, and is needed to make sure that the tensor product is supersymmetric. The effect of this current is to project out all mixed combinations of NS and R fields.

The fields on the left- and right-hand sides can be labelled respectively by integers $(l, m_1, m_2; m_3)$ and $(l, n_1, n_2, n_3; n_4, n_5)$. Because of the selection rules implied by the currents the combinations $l + m_1, m_2 + m_3, l + n_1 + n_4, n_2 + n_3 + n_5$ and $n_1 + n_3$ should be even. Furthermore we may select one representative from each orbit to label the primary fields. This allows us to restrict ourselves to the ranges $0 \leq l \leq 1, m_1 = 0, 1, m_2 = 0, 1,$

$0 \leq m_3 \leq 59$, $0 \leq n_1 \leq 3$, $n_2 = 0$, $n_3 = 0, 1$, $0 \leq n_4 \leq 9$ and $0 \leq n_5 \leq 5$. Thus the values of l, m_2, n_2 and n_3 have been restricted to half their natural range, and the currents may be used to generate the other half.

It is not hard to establish that the number of primary fields on the left- and right-hand side is the same, and also that the same conformal weights occur. The precise map that relates the two sets of primary fields is

$$\begin{aligned} n_1 &= 2m_1 + m_3 \bmod 4 \\ n_2 &= 0 \bmod 2 \\ n_3 &= m_2 \bmod 4 \\ n_4 &= 5m_1 + m_3 \bmod 10 \\ n_5 &= m_3 \bmod 6, \end{aligned}$$

with the same value of l on the left- and the right-hand side. Note that the selection rules for the labels n_i are implied by those of the labels m_i , and that each orbit representative occurs precisely once. It is easy to check that the conformal weights match at least up to integers. Furthermore, since the map is linear, the fusion rules are manifestly preserved, and the matrix elements of S are identical. (The U_n factors contribute a phase $e^{2\pi i \vec{v} \cdot \vec{w}}$ to S , where $\vec{v}$ and $\vec{w}$ are lattice vectors; U_n factors in the denominators contribute with opposite sign. One can use $\vec{v} \cdot \vec{w} = \frac{1}{2}((v+w)^2 - v^2 - w^2)$ to see that equality of the conformal weights modulo integers implies equality of the matrix elements of S .) The final result that is important in computations of massless spectra is the exact values of the conformal weights and the ground state degeneracies. These can most easily be computed from the known results for the $N = 2$ minimal models. On the left-hand side this is done by associating each primary field to a G_2 primary field, which in its turn belongs to one of the two terms in the $(28)_E$ invariant. The ground state conformal weight and degeneracy are then easily obtained from those of the four characters appearing in a term of the $(28)_E$ invariant. The details are not very illuminating, but the result is an exact agreement for all 120 ground states. Hence the complete set of massless spectra of string theories built with $(28)_E$ is identical to that of string theories built with $(1) \times (3)$. This is true not only for $(2, 2)$ compactifications, but also for $(2, 1)$, $(1, 2)$ and $(1, 1)$. Differences can only start occurring if one breaks the world-sheet supersymmetry in the left-moving (bosonic string) sector of the heterotic string

to obtain a $(2, 0)$ theory. In that case one may omit the current K_3 , and the relation is no longer valid.

One can still do better than this, and derive an exact relation between the characters. This requires a rotation that takes the lattice sum on the left-hand side to that on the right-hand side. This rotation can be written down easily for (1), but becomes rather complicated for the other relations. Since the results obtained so far already imply that the massless spectra must be identical, we have not pursued this further.

The other two expected relations can be proved in a similar way. Although $SO(5)$ (which contains $SU(2)_{10}$ as a conformal sub-algebra) does not contain $SU(2)_2 \times SU(2)_1$, it *does* contain $SU(2)_2 \times U_4$ (note that $SU(2)_2 = SO(3)_1$ and $U_4 = SO(2)_1$). Hence the problem reduces to showing the equivalence of the cosets

$$\frac{SU(2)_2 \times U_4 \times U_4}{U_{24}} \Big|_{J_1, J_2} = \frac{SU(2)_2 \times U_4 \times U_2 \times U_4}{U_6 \times U_6} \Big|_{K_1, K_2, K_3},$$

where

$$\begin{aligned} J_1 &= (2, 2, 0; 0) & K_1 &= (2, 2, 0, 0; 4, 0) \\ J_2 &= (0, 2, 2; 12) & K_2 &= (0, 0, 1, 2; 0, 3) \\ & & K_3 &= (0, 2, 0, 2; 0, 0). \end{aligned}$$

Here the currents play exactly the same rôle as above. We introduce labels n_i and m_i as before. The selection rules are exactly as before, with the exception of the one due to J_2 , which now becomes $m_1 + m_2 + m_3 = 0 \bmod 2$. There is a minor technical complication in comparison with the previous relation. Because the $SU(2)$ factor has now an even level, its simple current has a fixed point. This requires a somewhat more complicated choice of the parameter ranges. For even l there is no essential difference, and the ranges we choose are $l = 0$, $m_2 = 0, 1$, $n_2 = 0$ and $n_3 = 0, 1$, with all other parameters within their natural bounds. For odd l , we choose $l = 1$, $m_1 = 1$, $m_2 = 0, 1$, $n_2 = 0$ and $n_4 - n_1 = 1 \bmod 4$.

The map that relates the primary fields that are now uniquely identified is then

$$\begin{aligned} n_1 &= m_1 + m_3 \bmod 4 \\ n_2 &= 0 \bmod 2 \\ n_3 &= m_2 \bmod 4 \\ n_4 &= 2m_1 + m_3 + 4l \bmod 8 \\ n_5 &= m_3 + 3l \bmod 6, \end{aligned}$$

One may check that the left- and right-hand sides do obey their respective selection rules, that all primary fields are represented precisely once on the left-hand side, and that the conformal weights and ground state degeneracies match. Because of the presence of the l -dependent terms this is not strictly a linear map between the $U(1)'$ s, but nevertheless it is not hard to show that that the matrix elements of S between related fields are indeed the same.

To prove the last relation we use $SU(3)_1 \supset SU(2)_4$ and $SU(3)_1 \supset SU(2)_1 \times U_6$. The expected relation $(4)_D = (1) \times (1)$ is then equivalent to

$$\left. \frac{U_2 \times U_6 \times U_4}{U_{12}} \right|_{J_1, J_2} = \left. \frac{U_2 \times U_4 \times U_2 \times U_4}{U_6 \times U_6} \right|_{K_1, K_2, K_3},$$

where

$$\begin{aligned} J_1 &= (1, 3, 0; 0) & K_1 &= (1, 2, 0, 0; 3, 0) \\ J_2 &= (0, 0, 2; 6) & K_2 &= (0, 0, 1, 2; 0, 3) \\ & & K_3 &= (0, 2, 0, 2; 0, 0). \end{aligned}$$

The rest of the discussion is exactly as above. The map between the two theories is given by:

$$\begin{aligned} n_1 &= m_3 \bmod 4 \\ n_2 &= 0 \bmod 2 \\ n_3 &= m_2 \bmod 4 \\ n_4 &= 2m_1 + m_3 \bmod 6 \\ n_5 &= m_1 + m_3 \bmod 6, \end{aligned}$$

It does not follow from these identities that the lists of (2,2) and (2,1) models that one

obtains by applying simple currents of the original $N = 2$ models must be the same. For example, the simple currents of $(1)^9$ produce some models that do not appear on the list for $(1)^7(4)$ (in this case the D -invariant of (4) is known to lead to the same answer as the A -invariant [4]). The explanation is that the D -invariant of $SU(2)_4$ has a fixed point, and the fixed-point fields become simple currents after this fixed point has been resolved into two fields (corresponding to the (3) and $(\bar{3})$ of $SU(3)_1$). These simple currents are new, and not yet included in those of $(1)^7(4)$. In general, to exploit the simple current formalism fully one has to check all integral spin invariants with fixed points for the presence of such "secondary" simple currents, which may be used to build additional invariants. On the other hand, if one replaces $(4)_D$ by $(1) \times (1)$, the extra currents are simple currents from the start. Thus the list for $(1)^9$ contains, for example, torus compactification on the E_6 torus (an integer spin invariant of $SU(3)^3$), whereas the list for $(1)^7(4)$ does not. This problem does not arise for the other two identities: G_2 has no simple currents at all, and the one of $SO(5)$ does not come from a fixed point of $SU(2)_{10}$, but from a combination of two fields, one of which is the simple current of $SU(2)_{10}$. Thus one expects the lists of string models related by these identities to be exactly the same, and indeed they are [3].

This conclusion disagrees with the results of [12] and [4], who find differences in the number of scalars for $T_2 \times K_3$ compactifications. We suspect that these authors have overlooked a set of scalars whose left-moving vertex operator is one of the $U(1)$ generators of the $N = 2$ factors. Such scalars exist in $T_2 \times K_3$ compactifications, but not in (2,2) theories with $N = 1$ space-time supersymmetry. This would nicely explain these authors' observation that the number of scalars in $T_2 \times K_3$ compactifications increases by one when for example $(1) \times (2)$ is replaced by $(10)_E$.

It is an interesting question whether any other exact relations exist. To investigate this we have checked our list of (2,2) models (1145 in total) for multiple occurrences of spectra. This would not be possible without a knowledge of the number of massless E_6 singlets, since otherwise there are many degeneracies. With this extra information nearly all spectra coming from different tensor products are different, and apart from the expected degeneracies for reasons discussed above, only 44 spectra occur more than once. A rather intriguing fact is a pair-wise correlation among tensor products that are adjacent in the ordering of [12]. We found 20 pairs of tensor products that have one (2,2) spectrum in common. Usually there is also an exact match for one (sometimes two or three) of the (2,1) models. Some of these pairs can already be identified in the table of [12]. The fact that

they are adjacent is of course not essential in itself, but indicates that the real origin of this matching is the presence of common $N = 2$ factors in the tensor product. This is a general property seen in nearly all cases where the same spectrum is found in more than one tensor product. It would be interesting to know whether the corresponding string theories are really identical, or whether only the massless modes agree. In any case the pattern is too clear to be a mere coincidence.

Having eliminated two of the three known exceptional invariants, we will now discuss a few new ones. We will focus here on modular invariants of a tensor product of $SU(2)_k$ factors. Clearly such an invariant can be used to build an invariant of a tensor product of $N = 2$ minimal models if it does not mix odd and even values of l . It is not too surprising that such invariants exist: by examining the list of conformal embeddings [14] one finds not only the well-known E_6 and E_8 invariants [15], but also several conformal embeddings of more than one $SU(2)$ factor in some simple group. In many of these cases the extension of the chiral algebra involves just simple current, and hence the corresponding invariants are already included in the simple current procedure. Other cases have to be eliminated because they violate the selection rule. Finally we are left with the following three embeddings

$$\begin{aligned} SO(9) &\supset SU(2)_6 \times SU(2)_6 \\ Sp(6) &\supset SU(2)_3 \times SU(2)_6 \\ SO(12) &\supset SU(2)_3 \times SU(2)_3 \times SU(2)_6 \end{aligned}$$

These embeddings yield the following modular invariant combinations of $SU(2)$ -characters

$$\begin{aligned} &|(0,0) + (2,4) + (4,2) + (6,6)|^2 \\ &+ |(0,6) + (2,2) + (4,4) + (6,0)|^2 \\ &+ |(1,3) + (3,1) + (5,3) + (3,5)|^2 \end{aligned}$$

for $SU(2)_6 \times SU(2)_6$,

$$\begin{aligned} &|(0,0) + (2,4) + (0,8)|^2 \\ &+ |(0,4) + (2,2) + (2,6)|^2 \\ &+ |(1,2) + (1,6) + (3,4)|^2 \\ &+ |(1,4) + (3,0) + (3,8)|^2 \end{aligned}$$

for $SU(2)_3 \times SU(2)_6$ and

$$\begin{aligned} &|(0,0,0) + (0,2,4) + (0,0,8) + (2,0,4) + (2,2,2) + (2,2,6)|^2 \\ &+ |(0,1,4) + (0,3,0) + (0,3,8) + (2,1,2) + (2,1,6) + (2,3,4)|^2 \\ &+ |(1,0,4) + (3,0,0) + (3,0,8) + (1,2,2) + (1,2,6) + (3,2,4)|^2 \\ &+ |(1,1,2) + (1,1,6) + (1,3,4) + (3,1,4) + (3,3,0) + (3,3,8)|^2, \end{aligned}$$

for $SU(2)_3 \times SU(2)_3 \times SU(2)_6$.

For these three invariants we have computed the four-dimensional string spectra for all tensor products for which they occur, and modified them with all possible combinations of simple currents (for a more detailed explanation we refer to [3]). As in that paper, we have studied the effect of simple currents by taking a large and well-saturated random sample). The results are listed in table I. To save space we only list the spectrum that is obtained if no simple current modifications are included. The various theories obtained by means of simple currents from a given exceptional invariant show a pattern already noticed in [3]: the difference of the number of generations and anti-generations is quantized in terms of a certain integer Δ . The value of Δ is indicated in the table.

The general features of these new spectra are not different from those of the already known invariants. All combinations of numbers of generations and anti-generations have occurred before. However, if one also examines the number of gauge-singlet vector bosons and scalars one finds that for all genuine $(2,2)$ models (*i.e.* not $T_2 \times K_3$ or torus compactifications, which are not sufficiently distinct from each other) the detailed spectra are different from the ones obtained in [3]. Hence the new exceptional invariants define theories that are really new.

In a few cases in table I one can use a $(3,8)$ or $(6,6)$ invariant more than once. For example, in the tensor product $(3,8,8)$ one can multiply the matrices defining the exceptional invariant of factors 1 and 2 with those of 1 and 3. It turns out that this product yields a new matrix which at least superficially seems sensible. The same is true for two or more overlapping $(6,6)$ invariants, but in this case the product can also be obtained by tensoring a $(6,6)$ invariant with another (6) , and adding an integer spin simple current to the chiral algebra. Thus the table of simple current invariants for $(6,6',6'',6^*)$ is contained in that of $(6,6,6^*,6^*)$, and similarly $(6',6'',6^*,6^*)$ is contained in $(6',6',6^*,6^*)$. Remarkably,

the simple current invariants of $(6^*, 6^{**}, 6^*, 6^*)$ (which all have at least $N = 2$ space-time supersymmetry) are also contained in the list of invariants for $(2, 2, 2, 2, 2, 2)$. Finally, the analogous product of modular invariants for the tensor product $(3) \times (3) \times (8)$ yields an invariant in which the identity appears more than once, and which can not be normalized. The last two entries in table I are on a less firm basis than the others, since we have no proof that the corresponding conformal field theory exists. For the other theories in table I we can rely on the fact that these are related to conformal embeddings.

Of course these new spectra do not modify the picture substantially. Nevertheless, they demonstrate quite clearly that we do not yet know all partition functions for the $N = 2$ minimal model tensor products. The new invariants described here are all derived rather easily from conformal embeddings. But there is no reason to believe that this completes the classification, since modular invariants that are not related to conformal embeddings are already known in other conformal field theories [1] [16].

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basic model	Δ	# spectra	N_L	N_H	N_+	N_-	N_S	N_G
$(1, 1, 1, 2, 6^*, 6^*)$	0	4	2	2	20	0	150	18
$(1, 1, 1, 3, 3^*, 8^*)$	0	1	2	2	20	0	144	15
$(1, 1, 3^*, 8^*, 28)$	0	1	1	1	23	23	249	5
$(1, 1, 6^*, 6^*, 10)$	0	7	1	1	19	19	206	6
$(1, 2, 4, 6^*, 6^*)$	24	15	1	1	31	7	232	10
$(1, 4, 3, 3^*, 8^*)$	0	1	2	2	20	0	144	15
$(2, 2, 2, 6^*, 6^*)$	12	146	1	1	39	3	226	6
$(2, 2, 3, 3^*, 8^*)$	0	2	2	2	20	0	144	13
$(3^*, 4, 8^*, 28,)$	0	3	1	1	29	29	286	4
$(3, 6^*, 6^*, 18)$	0	7	1	1	23	23	245	5
$(3^*, 8^*, 8, 8)$	12	12	1	1	47	11	286	4
$(4, 6^*, 6^*, 10)$	12	28	1	1	27	15	225	5
$(6, 6, 6^*, 6^*)$	12	62	1	1	35	11	245	5
$(6^*, 6^*, 6^*, 6^*)$	0	29	1	1	15	15	215	11
$(1, 1, 1, 3^*, 3^*, 8^*)$	0	1	4	4	0	0	0	32
$(1, 4, 3^*, 3^*, 8^*)$	0	1	4	4	0	0	0	32
$(2, 2, 3^*, 3^*, 8^*)$	0	1	4	4	0	0	0	30
$(3^*, 8^*, 8^*, 8)$	12	7	1	1	19	7	198	4
$(3^{*-}, 8^-, 8^-, 8')$	6	5	1	1	13	7	189	4

Table 1. Four-dimensional string spectra obtained with the new exceptional invariants. Column 1 gives the factors in the tensor product. Factors combined by one of the exceptional invariants are indicated by a *, ′ or •. Column 3 gives the number of different massless spectra obtained by applying simple currents. Column 2 gives the largest common divisor Δ of $N_+ - N_-$ of all these spectra, where N_+ is the number of generations and N_- the number of anti-generations (the chirality is chosen so that $N_+ \geq N_-$). The last 6 columns give the spectrum obtained with no other simple currents than those required for space-time and world-sheet supersymmetry. Here N_R is the number of space-time gravitino's, N_L the analog for the left-movers (i.e. the gauge group G is $SO(10)$, E_6 , E_7 and E_8 for $N_L = 0, 1, 2$ and 4 respectively), N_S the number of G -singlet scalars and N_G the number of additional gauge bosons.