Inherent limitations in the effective-beam technique for algorithmic solutions to radiative corrections

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Abstract

The algebraic structure of the amplitude describing the process $e^+e^- \rightarrow V + \gamma$, $V + \gamma\gamma$ is studied, where V stands for a (possibly off-shell) photon or Z boson which can decay into any final state. A simple algebraic rule is given for determining whether such an amplitude can be expressed in terms of the lowest-order, non-radiative amplitude multiplied by factors that take into account the emission of the bremsstrahlung photons. This rule is shown to be satisfied for all single-bremsstrahlung helicity amplitudes but not for some of the double-bremsstrahlung helicity amps. It follows that there are structure-function based algorithms which give the exact result for all possible distributions in single-photon bremsstrahlung, but that no such algorithms are possible for double-photon bremsstrahlung. An argument is given for the physics behind this formal result. The accuracy of approaches that rely on structure functions for more than one bremsstrahlung photon is discussed. It is shown that the ultimate accuracy possible when one restricts oneself to purely effective-beam techniques cannot be better than next-to-leading log, even if a K factor good to all orders is included.

1 Introduction

The advent of LEP⁺ has led to an increasing need for efficient high-precision Monte Carlo simulations of electroweak processes at high energies [1]. In particular the problem of accurately describing the emission of multiple photons from the incoming e^+e^- beams has been attracting much attention, since the resulting radiative corrections are quite large and have a complicated dynamical behaviour due to the resonant structure of the Z exchange. One possible approach to the problem of constructing reliable Monte Carlo programs that implement these effects is what can be described as the *algorithmic approach*. The idea behind this approach is that the cross section for a process in which one or more photons are radiated is described by the cross section for the same process *without* photon radiation, which is then ‘dressed up’ with expressions representing the emission of the photons by the incoming e^+e^- beams¹. The best-known examples of this kind are the so-called *structure function* methods: in these, the cross section is typically written as something like

$$\frac{d\sigma^{\text{obs}}(s)}{d\Phi_{\text{obs}}} = K \int dx_1 dx_2 D_+(x_1) D_-(x_2) \frac{d\sigma_0^{\text{CM}}(s')}{d\Phi^{\text{CM}}}, \quad (1)$$

where the $x_{1,2}$ are the bremsstrahlung energy and angular variables, and $D_{\pm}$ are structure functions describing the emission of bremsstrahlung from the positron and electron beams, respectively. The actual cross section is denoted by σ , and the non-radiative cross section by σ_0 . A so-called K -factor is included in Eq.(1); shortly we shall see why it always has to occur in algorithmic approaches. The quantities s and s' denote the total invariant mass of the system before and after the radiation of the bremsstrahlung, respectively. The variables Φ stand for any set of phase-space variables describing the final state, for instance the scattering angle of a produced muon pair. The superscript ‘lab’ indicates that the variables Φ have to be evaluated in the laboratory frame, and likewise ‘CM’ means that the variables are to be evaluated in the centre-of-mass frame of all final state particles apart from the bremsstrahlung photons. In other words, in the lab frame we have

$$\vec{p}_+ + \vec{p}_- = \vec{0}, \quad (2)$$

while (if n photons are radiated) we have in the CM frame

$$\vec{p}_+ + \vec{p}_- - \vec{k}_1 - \vec{k}_2 \dots - \vec{k}_n = \vec{0}. \quad (3)$$

Here, p_+ and p_- denote the positron and electron momenta, respectively, and k_i is the momentum of photon i . The expression in Eq.(1) should be understood only as a generic one: there are many versions, some of which combine D_+ and D_- into a single structure function [2,3,4] while others use one structure function for each radiated photon [5,6]. Also exponentiated and non-exponentiated versions occur. All these variations have in common that they lead immediately to a simple layout of a Monte Carlo algorithm to generate events:

1. First the photon variables are generated using appropriate mappings etc. The resonant structure of the Z exchange makes in particular the generation of the photon energies a quite non-trivial task, with which we shall not concern ourselves here. One thing, however, has to be stressed here. Once we adopt the structure function $D(x)$ as a recipe for generating variables x , it has to be normalized to unity, $\int D(x)dx = 1$. It

¹ Throughout this paper, we shall only discuss processes like $e^+e^- \rightarrow \mu^+\mu^-$ for which initial state radiation is a well-defined concept. An even more problematic case like $e^+e^- \rightarrow e^+e^-$ is not considered.

should not come as a surprise that in that case the convolution of Eq.(1) will not necessarily have the correct normalization. Hence the occurrence of the K factor, already well known from QCD studies of the Drell-Yan process in hadron collisions.

2. Secondly, the CM frame is constructed, and the so-called *effective energy*, namely s' , is computed.
3. Finally, the variables Φ^{CM} are generated, according to the non-radiative distribution $d\sigma_0$. The resulting four-momenta are then boosted back to the lab frame

This type of algorithm has the advantage of simplicity: the ‘QED’ effects (the multiple bremsstrahlung) are nicely separated from the actual ‘hard’ scattering described by $d\sigma_0$, into which one can insert such ‘purely weak’ corrections as are felt to be appropriate. In fact approaches like these have already been applied for many years in the context of perturbative QCD predictions, for instance for hadron-hadron collisions, with satisfactory results. In the context of precision tests of the Standard Model of electroweak interactions, however, the demands on the precision of such methods are much more stringent than in QCD, and it remains to be seen whether the strategy embodied in Eq.(1) can give results that are precise to the desired level of a few tenths of a per cent.

For the integrated cross section σ it has been shown that (1) holds to arbitrary precision provided the structure function(s) D are chosen correctly [7,8]. The various algorithmic results for the Z line shape agree to very high accuracy with the more straightforward computation of [2]. This is in fact no surprise since the integrated non-radiative cross section $\sigma_0(s')$ is a function of one variable only, and it is of course always possible to factorize such a simple scalar expression. The more subtle problems appear when we want to apply (1) to multidifferential distributions, which is unavoidable if an event generator is called for. In that case the above-mentioned agreement just tells us that one can get the *overall normalization* correct, but says nothing about for instance the angular distribution of the muon pair.

To illustrate this let us consider more closely steps (2) and (3) of the algorithm outlined above. We have to generate, say, a muon scattering direction. To do this we need to have a ‘beam direction’ with respect to which we can orient the muon direction. Naively this would be the direction of the beams, evaluated in the CM frame. But if the bremsstrahlung photons have any finite amount of transverse momentum the two beam directions are no longer back-to-back in the CM frame, and the orientation of the muons becomes ambiguous. At this point, some authors choose to pick one of the two beam directions at random, with equal probability [5,9], while others just take the z -axis, assuming the beams are along the z -axis in the lab frame [3]. In many cases these choices practically coincide, since the transverse momentum of the bremsstrahlung tends to be small – but for precision simulation it is unacceptable to have a possibly small but generally *unknown* bias in the distributions due to this effect. In fact for the case of *angle* bremsstrahlung it has been shown in [10] how to deal with this problem: one has to pick either one of the beams, with a *well-defined* probability depending (as we shall see) on the helicity of the bremsstrahlung photon. A similar result has also been derived for final-state radiation [11].

The aim of this paper is to study to whether these results can be extended to the case of multiple bremsstrahlung. As we shall see, for double bremsstrahlung with equal photon helicities we can in fact find a rule to choose an effective beam direction similar to the above-mentioned one. However, if the two photons have opposite helicity it turns out to be *impossible* to define an effective beam direction such that Eq.(1) holds exactly. This can

be shown by applying a simple algebraic criterion. It follows that algorithmic approaches are bound to be non-exact as far as differential cross sections are concerned, which in turn implies that there are sets of experimental cuts under which the total cross section will not be evaluated correctly. We shall then try to quantify the magnitude of these deviations from exactness, and present a few physical arguments to underpin the *no-go* theorem. Before finishing, we would like to stress that the amount of non-exactness in the effective-bram method is not very big, mainly because the QED coupling constant is small. However we are at present in a situation where the field of electroweak-correction algorithms is slowly being put on a solid computational basis, and it is important to have a clear understanding of what is, and what is not, formally possible.

2 Non-radiative and radiative amplitudes

In this section we shall derive reasonably compact forms for the amplitude describing production of an (off-shell) boson V in e^+e^- collisions, with up to 2 photons. Since we do not consider the way in which this boson decays into a final state (say, a muon pair) we keep a free Lorentz index μ in the amplitude, to which any decay current can in principle be connected, thus allowing for, say, production of a Z -Higgs boson pair, heavy quarks, radiative final states such as $\mu^+\mu^-\gamma$, and so on. Neglecting the electron mass as compared to s , the amplitude for the non radiative process can be written as

$$M_0^\mu = i u_\lambda(p_+) (v + a\gamma^5) \gamma^\mu u_\lambda(p_-), \quad (1)$$

where v and a denote the vector and axial-vector couplings of V to the electrons (for $V = \gamma$, we have $a = 0$ and $v = Q_e$, the electron charge). The electron and positron helicity is denoted by λ_e and defined by

$$u_\lambda(p) u_\lambda(p) = \frac{1}{2} (1 + \lambda_e \gamma^5) \not{p} \quad (2)$$

for any massless momentum p and helicity $\lambda_e = \pm 1$. Note that for massless particles we use the same convention for both fermions and antifermions: in this convention, helicity is conserved along all fermion lines. We can rewrite the non-radiative amplitude as

$$M_0^\mu = i (v - \lambda_e a) J_0^\mu(p_+, p_-), \quad (3)$$

where

$$J_0^\mu(p_+, p_-) = u_\lambda(p_+) \gamma^\mu u_\lambda(p_-). \quad (4)$$

The current J_0^μ contains the information on the spin structure of the initial state, and *all* differential cross sections are completely determined by its product with the final-state current.

In what follows we shall make use of the spinor techniques described in detail in [12]. These are based on a particular convention for the phase difference between positive- and negative-helicity spinors such that the *spinor product* becomes antisymmetric, i.e. for any two massless $p_{1,2}$ and $\lambda = \pm$ we have

$$s_\lambda(p_1, p_2) = u_\lambda(p_1) u_{-\lambda}(p_2) = -u_\lambda(p_2) u_{-\lambda}(p_1), \quad (5)$$

which immediately leads to the following useful relations:

$$\begin{aligned} u_\lambda(p_1) \gamma^\mu u_\lambda(p_2) &= u_{-\lambda}(p_2) \gamma^\mu u_{-\lambda}(p_1), \\ (u_\lambda(p_1) \gamma^\mu u_\lambda(p_2)) \gamma_\mu &= 2u_\lambda(p_2) u_\lambda(p_1) + 2u_{-\lambda}(p_1) u_{-\lambda}(p_2). \end{aligned} \quad (6)$$

Together with the property that

$$|s_\lambda(p_1, p_2)|^2 = 2 p_1 \cdot p_2 \quad (7)$$

these suffice to derive all our further results.

We now turn to the radiative amplitudes. The most convenient form for the polarization vectors ϵ of a bremsstrahlung photon with momentum k , helicity λ and Lorentz index μ turns out to be

$$\epsilon_\lambda^\mu(k) = \frac{\bar{u}_\lambda(k) \gamma^\mu u_\lambda(q)}{\lambda \sqrt{2} s_{-\lambda}(k, q)}, \quad (8)$$

where q is any massless four-vector not collinear with k . Different choices for q correspond to different gauge choices for ϵ , and we may exploit this freedom to choose q in a way which depends on the photon helicity λ as well as the electron helicity λ_e :

$$\begin{aligned} q &= p_- \quad \text{if } \lambda = \lambda_e, \\ q &= p_+ \quad \text{if } \lambda = -\lambda_e. \end{aligned} \quad (9)$$

The photon polarization vectors ϵ will enter in the e^+e^- fermion line as f . If we are only interested in bremsstrahlung from this fermion line, the overall complex phase of ϵ is unimportant. In that case, we can effectively use

$$\begin{aligned} f_{\lambda_e} &= N(k) \not{p}_+ \not{p}_-, \\ f_{-\lambda_e} &= N(k) \not{p}_+ \not{p}_-, \end{aligned} \quad (10)$$

with

$$|N(k)|^{-2} = 4(p_+ \cdot p_-)(p_+ \cdot k)(p_- \cdot k) = 2 s(p_+ \cdot k)(p_- \cdot k) \quad (11)$$

This choice for ϵ is in fact the one first described by the CALKUL collaboration [13]: it allows for very simple evaluation of Abelian amplitudes of the type considered.

We can now write down the amplitude M_1^μ for single bremsstrahlung. Neglecting an irrelevant overall complex phase we have for the two possible helicities of the photon:

$$\begin{aligned} M_1^\mu(\lambda = \lambda_e) &= N(k) Q_e (v - \lambda_e a) J_1^\mu(+), \\ M_1^\mu(\lambda = -\lambda_e) &= N(k) Q_e (v - \lambda_e a) J_1^\mu(-), \end{aligned} \quad (12)$$

where J_1^μ , the single-bremsstrahlung analogue of J_0^μ , is given by

$$\begin{aligned} J_1^\mu(+) &= \bar{u}_{\lambda_e}(p_+) \not{p}_- (\not{p}_+ - \not{k}) \gamma^\mu u_\lambda(p_-), \\ J_1^\mu(-) &= \bar{u}_{\lambda_e}(p_+) \gamma^\mu (\not{p}_- - \not{k}) \not{p}_+ u_\lambda(p_-). \end{aligned} \quad (13)$$

Again, the current J_1^μ completely determines all distributions of the final-state particles (e.g. the muon pair).

The amplitude M_2^μ for double bremsstrahlung with photon momenta $k_{1,2}$ and helicities $\lambda_{1,2}$ reads

$$\begin{aligned} M_2^\mu(\lambda_1 = \lambda_e, \lambda_2 = \lambda_e) &= N(k_1) N(k_2) Q_e^2 (v - \lambda_e a) J_2^\mu(+, +), \\ M_2^\mu(\lambda_1 = \lambda_e, \lambda_2 = -\lambda_e) &= N(k_1) N(k_2) Q_e^2 (v - \lambda_e a) J_2^\mu(+, -), \\ M_2^\mu(\lambda_1 = -\lambda_e, \lambda_2 = \lambda_e) &= N(k_1) N(k_2) Q_e^2 (v - \lambda_e a) J_2^\mu(-, +), \\ M_2^\mu(\lambda_1 = -\lambda_e, \lambda_2 = -\lambda_e) &= N(k_1) N(k_2) Q_e^2 (v - \lambda_e a) J_2^\mu(-, -), \end{aligned} \quad (14)$$

where for equal photon helicities we have

$$J_2^{\mu}(+, +) = s u_{\lambda}(p_+) \not{p}_+ (\not{p}_+ - \not{p}_1 - \not{p}_2) \gamma^{\mu} u_{\lambda}(p_-), \quad (18)$$

$$J_2^{\mu}(-, -) = s u_{\lambda}(p_+) \gamma^{\mu} (\not{p}_- - \not{p}_1 - \not{p}_2) \not{p}_+ u_{\lambda}(p_-), \quad (19)$$

while for unequal photon helicities

$$J_2^{\mu}(+, -) = \frac{s}{\Delta_+} H_1^{\mu} + \frac{s}{\Delta_-} H_2^{\mu} + H_3^{\mu},$$

$$H_1^{\mu} = u_{\lambda}(p_+) \not{p}_+ \not{p}_- (\not{p}_+ - \not{p}_1 - \not{p}_2) \gamma^{\mu} u_{\lambda}(p_-),$$

$$H_2^{\mu} = u_{\lambda}(p_+) \gamma^{\mu} (\not{p}_- - \not{p}_1 - \not{p}_2) \not{p}_+ \not{p}_- u_{\lambda}(p_-),$$

$$H_3^{\mu} = u_{\lambda}(p_+) \not{p}_+ (\not{p}_+ - \not{p}_1) \gamma^{\mu} (\not{p}_- - \not{p}_2) \not{p}_+ u_{\lambda}(p_-),$$

$$\Delta_{\pm} = (p_{\pm} - k_1 - k_2)^2, \quad (20)$$

$$J_2^{\mu}(-, +) = [J_2^{\mu}(+, -)]_{k_1 \leftrightarrow k_2}. \quad (21)$$

Having written the amplitudes for single and double bremsstrahlung in as compact a form as possible, we shall now turn our attention to the problem of finding effective beam directions as discussed in the previous section.

3 Effective beam directions: a no-go theorem

Once the phase space variables for one or more bremsstrahlung photons have been generated in a Monte Carlo procedure, the CM frame is fixed and in principle Φ , the rest of the final state variables, can be generated as well: all relevant distributions are given by contracting the radiative amplitudes $\mathcal{M}_{1,2,\dots}^{\mu}$ with the appropriate current describing the production of the final state. In the algorithmic approach, however, one prefers to generate the variables Φ using the same prescription as holds for the lowest-order cross section. Let us now concentrate on single bremsstrahlung. Since at lowest order the information on all final state distributions is contained in the vector $\mathcal{M}_0^{\mu}$ the relation between the radiative and non radiative cross sections implied by Eq.(1) can be satisfied if and only if there are some massless four-vectors $q_{1,2}$ such that

$$\mathcal{M}_1^{\mu} = C_1 \bar{u}_{\lambda}(q_1) \gamma^{\mu} u_{\lambda}(q_2) \quad (22)$$

for some complex constant C_1 which may depend on the photon momenta and helicities but may *not* depend on any other final-state momentum or helicity: this is obvious, since in the algorithmic approach the determination of the effective beams has to be performed *before* any final state momentum is generated. If furthermore we have

$$p_+^{\mu} + p_-^{\mu} - k^{\mu} = q_1^{\mu} + q_2^{\mu} \quad (23)$$

(which can always be arranged by rescaling of the q 's) it is seen that in the radiative amplitude the vectors q_1 and q_2 play exactly the same role as do the vectors p_+ and p_- , respectively, in the non-radiative one. In particular they are back-to-back in the CM frame, and together make up the correct CM energy s' . The conclusion is that *if* effective beam momenta $q_{1,2}$ can be found, *then* the algorithmic approach will give the correct answer for *every possible* distribution of the final state particles. A similar result holds of course also for $\mathcal{M}_{2,3,\dots}^{\mu}$.

It should be noted that Eq.(22) is non-trivial since it is a complex vector equation due to the occurrence of the index μ . If, in contrast, one is only interested in, say, $|\mathcal{M}|^2$, the

requirement Eq.(22) reduces to just having two momenta with total invariant mass s' , which is trivial — this is precisely the case for the *total cross section*, for which a factorization like Eq.(1) is expected to hold to great precision. It has been shown in [10] how to find effective beam momenta $q_{1,2}$ for the case of single bremsstrahlung: in this note we try to generalize and extend this result.

It turns out that there is a quite simple criterion by which one can check whether a solution for $q_{1,2}$ is in fact possible. It is based on the following theorem: let J^{μ} be a complex four-vector, and Q^{μ} be a real timelike four-vector, i.e. $Q^2 > 0$. Furthermore, let $J \cdot Q = 0$. We can then view J^{μ} as a conserved current, with total inflowing momentum Q^{μ} . Then there are two massless four-vectors $q_{1,2}$ and a complex number C such that

$$\begin{aligned} J^{\mu} &= C \bar{u}_+(q_1) \gamma^{\mu} u_+(q_2), \\ Q^{\mu} &= q_1^{\mu} + q_2^{\mu}, \end{aligned} \quad (24)$$

if and only if

$$J \cdot J = 0. \quad (25)$$

Note that here we have $J^{\mu} J_{\mu}$ and not $J^{\mu} (J^{\mu})^*$ which occurs in the total cross section. Furthermore, by Eq.(9) the opposite helicity of the two spinors can simply be arranged by interchanging q_1 and q_2 . One part of the above statement is proved by noting that

$$\begin{aligned} \{C \bar{u}_+(s_1) \gamma^{\mu} u_+(q_2)\} \{C \bar{u}_+(s_1) \gamma_{\mu} u_+(q_2)\} &= \\ 2C^2 \bar{u}_+(q_1) [u_+(q_2) \bar{u}_+(q_1) + u_-(q_1) \bar{u}_-(q_2)] u_+(q_2) &= 0, \end{aligned} \quad (26)$$

where one of the terms vanishes because of the masslessness of $q_{1,2}$ and the other because of helicity conservation. In the above we have used the second part of Eq.(9). The more familiar 'absolute value squared' gives

$$\begin{aligned} \{C \bar{u}_+(s_1) \gamma^{\mu} u_+(q_2)\} \{C \bar{u}_+(s_1) \gamma_{\mu} u_+(q_2)\}^* &= \\ |C|^2 \bar{u}_+(q_1) \gamma^{\mu} u_+(q_2) \bar{u}_+(q_2) \gamma_{\mu} u_+(q_1) &= \\ |C|^2 \bar{u}_+(q_1) \gamma^{\mu} \not{q}_2 \gamma_{\mu} u_+(q_1) &= \\ -2|C|^2 \text{Tr} \left(\frac{1}{2}(1 + \gamma^5) \not{q}_1 \not{q}_2 \right) &= -2|C|^2 Q^2. \end{aligned} \quad (27)$$

The other part is best proved by giving the explicit construction of $q_{1,2}$ and C . To this end, we compute

$$J^{\mu} = (J^{\mu})^* \quad \text{and} \quad R^{\mu} = \frac{i}{J \cdot J} \epsilon^{\mu\nu\rho\sigma} Q_{\nu} J_{\rho} \bar{J}_{\sigma}, \quad (28)$$

where $\epsilon^{\mu\nu\rho\sigma}$ denotes the Levi-Civita tensor. It is easily checked that $R \cdot J = 0$ and $R \cdot R = -Q^2$. The four vectors Q , R , J and $\bar{J}$ form a basis of the complex four-dimensional space, and we have

$$\begin{aligned} \frac{1}{Q^2} (Q^{\mu} Q^{\nu} - R^{\mu} R^{\nu}) + \frac{1}{J \cdot J} (J^{\mu} \bar{J}^{\nu} + J^{\nu} \bar{J}^{\mu}) &= g^{\mu\nu}, \\ i \epsilon_{\mu\nu\rho\sigma} Q^{\mu} R^{\nu} J^{\rho} \bar{J}^{\sigma} &= Q^2 (J \cdot J). \end{aligned} \quad (29)$$

We can now compute

$$\begin{aligned} \mathcal{Q}_1^{\mu} &= (Q^{\mu} + R^{\mu})/2, \\ \mathcal{Q}_2^{\mu} &= (Q^{\mu} - R^{\mu})/2, \\ |C| &= \sqrt{-(J \cdot \bar{J})/(2Q^2)}. \end{aligned} \quad (30)$$

■ should be stressed that all this can be done on the basis of just the components of J^μ in the form of complex numbers, without reference to any physics involved. The complex phase of ϵ^+ is not fixed yet since the complex phase of the spinors $u(q_{1,2})$ has not been fixed. An explicit construction of these spinors along the lines of [12] would also give the phase: at present, however, it is sufficient to note that indeed we can write

$$\begin{aligned} & \{u_+(q_1)\gamma^\mu u_+(q_2)\} \{u_+(q_1)\gamma^\nu u_+(q_2)\}^* \\ &= \frac{1}{2} \text{Tr} \left((1 + \gamma^5) \not{p}_1 \not{p}_2 \gamma^\mu \not{p}_1 \gamma^\nu \right) \\ &= 2(q_1^\mu q_2^\nu + q_1^\nu q_2^\mu - q_1^\alpha q_2^\beta g^{\alpha\beta} - i\epsilon^{\mu\nu\alpha\beta} q_1^\alpha q_2^\beta) \\ &= Q^\mu Q^\nu - R^\mu R^\nu - Q^2 g^{\mu\nu} + i\epsilon^{\mu\nu\alpha\beta} Q_\alpha R_\beta \\ &= \frac{2Q^2}{J \cdot J} J^\mu J^\nu \\ &= J^\mu J^\nu / |C|^2, \end{aligned} \quad (31)$$

where we have used the first part of Eq.(29) to rewrite the $g^{\mu\nu}$ term, and the second one to rewrite the $\epsilon^{\mu\nu\alpha\beta}$ term. This finishes the proof of our little theorem.

Returning to physics, we now have at our disposal a simple criterion to check whether any amplitude M^μ can be reduced to an effective 'Born' amplitude with given helicities: its square has to vanish, i.e. $M^\mu M_\mu = 0$. If the square of a given amplitude is non-zero, it is impossible to write it in the form of Eq.(22).

Let us now apply this to the radiative amplitudes of the previous section. It turns out to be handy to apply the second of the Eqs.(9) in the following form:

$$u_\lambda(p_-)u_\lambda(p_+) = \frac{1}{2} (1 + \lambda_c \gamma^5) \gamma^\alpha \{u_\lambda(p_+) \gamma_\alpha u_\lambda(p_-)\}. \quad (32)$$

Using this relation we can work out the squares of J^μ :

$$\begin{aligned} J_1(+)^{\mu} J_1(+)^{\mu} &= u_\lambda(p_+) \not{p}_- (\not{p}_+ - \not{E}) \gamma^\mu \gamma^\nu \not{p}_- (\not{p}_+ - \not{E}) \gamma_\mu u_\lambda(p_-) \\ &\quad \times \bar{u}_\lambda(p_+) \gamma_\nu u_\lambda(p_-) \\ &= -2\bar{u}_\lambda(p_+) \not{p}_- (\not{p}_+ - \not{E}) (\not{p}_+ - \not{E}) \not{p}_- \gamma^\nu u_\lambda(p_-) \\ &\quad \times u_\lambda(p_+) \gamma_\nu u_\lambda(p_-) \\ &= 0, \end{aligned} \quad (33)$$

since $\not{E}^2 = Q^2$ for any q^μ . Similarly it is proved that $J_2(-)^\mu J_2(-)_\mu = 0$. It follows that for single bremsstrahlung the algorithmic approach will give the exact result provided we use the correct effective beams $q_{1,2}^\mu$. These have already been derived in [10]; however it is instructive to see how they can be computed with the help of the constructive recipe given above. Furthermore, in this way the intimate relation between the effective beam choice and the photon helicity is made clear. First of all, we have

$$\begin{aligned} J_1(+)^{\mu} J_1(+)^{\mu} &= u_\lambda(p_+) \not{p}_- (\not{p}_+ - \not{E}) \gamma^\mu \not{p}_- \gamma_\mu (\not{p}_+ - \not{E}) \not{p}_- u_\lambda(p_-) \\ &= -8s(p_- \cdot Q)^2, \end{aligned} \quad (34)$$

where $Q^\mu = p_+^\mu + p_-^\mu - k^\mu$. We now compute R^μ according to its definition given above. Using the rules (9) to write

$$\begin{aligned} \gamma_\mu J_1^\mu(+)^{\mu} &= 2 \left(u_\lambda(p_-) u_\lambda(p_+) \not{p}_- \not{Q} + \not{Q} \not{p}_- u_\lambda(p_+) u_\lambda(p_-) \right), \\ \gamma_\mu J_1^\mu(+)^{\mu} &= 2 \left(u_\lambda(p_-) u_\lambda(p_+) \not{p}_- \not{Q} + \not{Q} \not{p}_- u_\lambda(p_+) u_\lambda(p_-) \right). \end{aligned} \quad (35)$$

we have

$$\begin{aligned} i\epsilon^{\mu\nu\rho\sigma} Q^\nu J_1^\rho(+)^{\mu} J_1^\sigma(+)^{\mu} &= \frac{1}{4} \text{Tr} \left(\gamma^5 \gamma^\mu \not{Q} \gamma^\nu \not{p}_- \gamma^\rho \not{p}_- \gamma^\sigma \right) \\ &= \frac{1}{4} \text{Tr} \left(\gamma^5 \gamma^\mu \not{Q} \left[u_\lambda(p_-) u_\lambda(p_+) \not{p}_- \not{Q} u_\lambda(p_-) u_\lambda(p_+) \not{p}_- \not{Q} + \right. \right. \\ &\quad \left. \left. \not{Q} \not{p}_- u_\lambda(p_+) u_\lambda(p_-) \not{Q} \not{p}_- u_\lambda(p_+) u_\lambda(p_-) \right] \right) \\ &= -8s(p_- \cdot Q)^2 Q^\mu + 8s(p_- \cdot Q) p_-^\mu. \end{aligned} \quad (36)$$

This leads immediately to explicit formulae for R^μ , $|C|$ and the effective beam momenta:

$$\begin{aligned} R^\mu &= Q^\mu - \frac{Q^2}{p_- \cdot Q} p_-^\mu, \\ q_1^\mu &= Q^\mu - \frac{Q^2}{2p_- \cdot Q} p_-^\mu, \\ q_2^\mu &= \frac{Q^2}{2p_- \cdot Q} p_-^\mu, \\ |C| &= 2(p_- \cdot Q) \sqrt{s/s'}. \end{aligned} \quad (37)$$

The final result is that the single-bremsstrahlung current can, up to an overall phase, be written as:

$$\begin{aligned} J_1(+)^{\mu} &= 2(p_- \cdot Q) \sqrt{s/s'} \bar{u}_\lambda \left(Q - \frac{Q^2}{2p_- \cdot Q} p_- \right) \gamma^\mu u_\lambda \left(\frac{Q^2}{2p_- \cdot Q} p_- \right), \\ J_1(-)^{\mu} &= 2(p_- \cdot Q) \sqrt{s/s'} u_\lambda \left(\frac{Q^2}{2p_+ \cdot Q} p_+ \right) \gamma^\mu u_\lambda \left(Q - \frac{Q^2}{2p_+ \cdot Q} p_+ \right), \end{aligned} \quad (38)$$

where the result for the other helicity case is derived in precisely the same manner. To summarize: the algorithmic approach can in principle give the exact result for all distributions in single bremsstrahlung. To this end effective beams have to be chosen carefully in a way which depends on the bremsstrahlung helicity. As can be seen, the probabilities for having a + or - helicity are in a ratio of $(p_- \cdot Q)^2$ to $(p_+ \cdot Q)^2$, respectively. The effective beams coincide with the direction of p_-^μ for helicity + and with that of p_+^μ for helicity - in the CM frame. The results are precisely the ones derived in [10]. It also follows that picking the effective beams at random, with a 50%-50% probability, will never generate the single-bremsstrahlung matrix element correctly.

We now turn to the case of double bremsstrahlung. The total momentum flow is now given by: $Q^\mu = p_+^\mu + p_-^\mu - k_1^\mu - k_2^\mu$. If the two photon helicities are equal, the currents J_2^μ are essentially the same as J_1^μ , except for the replacement $k \rightarrow k_1 + k_2$. Since in the derivation of the effective beams we have nowhere used that $k \cdot k = 0$, we can immediately write these J_2^μ using effective beams:

$$\begin{aligned} J_2(+, +)^{\mu} &= 2s(p_- \cdot Q) \sqrt{s/s'} \bar{u}_\lambda \left(Q - \frac{Q^2}{2p_- \cdot Q} p_- \right) \gamma^\mu u_\lambda \left(\frac{Q^2}{2p_- \cdot Q} p_- \right), \\ J_2(-, -)^{\mu} &= 2s(p_- \cdot Q) \sqrt{s/s'} u_\lambda \left(\frac{Q^2}{2p_+ \cdot Q} p_+ \right) \gamma^\mu u_\lambda \left(Q - \frac{Q^2}{2p_+ \cdot Q} p_+ \right), \end{aligned} \quad (39)$$

²In fact it has been proved by the CALKUL collaboration [11] that this simple form persists for an arbitrary number of bremsstrahlung photons if they all have the same helicity. This result has also been extended to the case of QCD in [15].

and we conclude that also for double bremsstrahlung the algorithmic approach will work if the photon helicities are equal.

The case of opposite photon helicities is more complicated. The three subcurrents $H_{1,2,3}^\mu$ have indeed zero square: $H_1 \cdot H_1 = H_2 \cdot H_2 = H_3 \cdot H_3 = 0$. However the products of different H 's do not vanish:

$$\begin{aligned}
H_1 \cdot H_2 &= u_{\lambda_1}(p_+) \not{p}_+ \not{p}_- \not{p}_2 (\not{p}_+ - \not{p}_1) \gamma^\mu u_{\lambda_2}(p_-) \\
&\quad \times u_{\lambda_3}(p_+) \gamma_\mu (\not{p}_- - \not{p}_2) \not{p}_1 \not{p}_+ \not{p}_2 u_{\lambda_4}(p_-) \\
&= u_{\lambda_1}(p_+) \not{p}_1 \not{p}_- \not{p}_2 (\not{p}_+ - \not{p}_1) \gamma^\mu \gamma^\alpha \\
&\quad \times \gamma_\mu (\not{p}_- - \not{p}_2) \not{p}_1 \not{p}_+ \not{p}_2 u_{\lambda_2}(p_-) \\
&\quad \times \frac{1}{2} u_{\lambda_3}(p_+) \gamma_\alpha u_{\lambda_4}(p_-) \\
&= -2 u_{\lambda_1}(p_+) \not{p}_1 \not{p}_- \not{p}_2 (\not{p}_+ - \not{p}_1) u_{-\lambda_2}(p_+) \\
&\quad \times u_{-\lambda_3}(p_-) (\not{p}_- - \not{p}_2) \not{p}_1 \not{p}_+ \not{p}_2 u_{\lambda_4}(p_-) \\
&= 16(p_+ \cdot k_1)(p_- \cdot k_2)(k_1 \cdot k_2) \\
&\quad \times \{u_{\lambda_1}(p_+) \not{p}_1 u_{\lambda_2}(p_-)\} \{u_{\lambda_3}(p_+) \not{p}_2 u_{\lambda_4}(p_-)\} .
\end{aligned} \tag{10}$$

A similar derivation yields

$$\begin{aligned}
H_1 \cdot H_3 &= H_2 \cdot H_3 \\
&= -8s(p_+ \cdot k_1)(p_- \cdot k_2) \\
&\quad \times \{u_{\lambda_1}(p_+) \not{p}_1 u_{\lambda_2}(p_-)\} \{u_{\lambda_3}(p_+) \not{p}_2 u_{\lambda_4}(p_-)\} .
\end{aligned} \tag{11}$$

Combining all this we see that

$$\begin{aligned}
J_2^\mu(+, -) J_{2\mu}(+, -) &= 16s^2 \frac{(p_+ \cdot k_1)(p_- \cdot k_2)(s - s')}{\Delta_+ \Delta_-} \\
&\quad \times \{u_{\lambda_1}(p_+) \not{p}_1 u_{\lambda_2}(p_-)\} \{u_{\lambda_3}(p_+) \not{p}_2 u_{\lambda_4}(p_-)\} .
\end{aligned} \tag{12}$$

We conclude that in general the algorithmic approach will *not* work in the case of double bremsstrahlung: it is impossible to find effective beams $q_{1,2}^\mu$ so that the angular distributions of the produced final state can be generated according to the non-radiative expression.

4 Physical considerations

In the above section we have studied under which circumstances effective beams can be constructed, and seen that this is not possible for more than single bremsstrahlung, even in the simplest case of initial state radiation only. A simple physical picture helps us to understand this, as follows. For massless fermions, helicity must be conserved at 1- or 4-type vertices. Now consider the Feynman diagram in which the e^- emits a photon before collision. After having emitted the photon, its original pure-helicity state is transformed into a superposition of + and - helicity states. When this e^- meets the e^+ line at the vertex where the Z is produced, the e^+ helicity will automatically pick out the correct state of the e^- to interact with; the other state does not couple. Therefore, in this simple picture only one pure helicity state of the e^- comes into play. In fact, in the gauge we have chosen to work in, the other Feynman diagram in which the photon is emitted by the e^+ vanishes. The same goes through when the e^- emits two photons with equal helicities: only 2 out of the 6 diagrams survive in our gauge, and the Z production can be interpreted as due to the

scattering of pure states. Hence it is reasonable that in that case we can in fact find effective beams. If the photons have opposite helicities diagrams survive in which one of them comes from the e^+ and the other one from the e^- (corresponding to the quantity H_3 in Eq.(21)). Now, the Z production vertex sees two superpositions of (+) and (-) states colliding, and out of the 4 combinations (+,+), (+,-), (-,+), and (-,-) the first two will survive: this will lead to a complex linear combination of the two non-radiative amplitudes for $\lambda_e = +$ and $\lambda_e = -$, which has no counterpart in the non-radiative cross section. Indeed, we can extend Eq.(24): if $J \cdot J$ is nonzero, we can write

$$J^\mu = C_1 u_+(q_1) \gamma^\mu u_+(q_2) + C_2 \bar{u}_-(q_1) \gamma^\mu u_-(q_2) , \tag{13}$$

where $C_{1,2}$ and $q_{1,2}^\mu$ can be computed by a procedure similar to that described in section 3. This can be interpreted as the original non-radiative amplitude, now for the case of generally (not longitudinally) polarized beams. However, one should realize that in such a rewriting the combinations $(v \pm a)$ would not turn out correctly. Therefore, an effective-beam approach could in principle work for double bremsstrahlung if it is based on the generally polarized expression for the non-radiative amplitude, provided that $a = 0$, which makes this possibility irrelevant in practice.

Let us now turn to the magnitude of the deviation from exactness. In Eq.(12) it is seen that $J_2 \cdot J_2$ does vanish whenever at least one of the photons becomes soft, or collinear with the beams. These are precisely the phase space regions that contribute to the large logarithms in the cross section: deviations, therefore will be expected at the order α^2 without large logarithms, or (if we continue to three or more photons) at order α^3 with one large logarithm, and so on. In general, only the leading and next-to-leading logarithms will be produced correctly in the result for the total cross section. Hence for line-shape measurements, and the like, overall precision is not noticeably affected; but for more exclusive measurements, and especially in distribution for muon pairs with high p_T , deviations are to be expected.

Another important point that should be stressed here is that one should take care to choose the effective beams correctly in the single-bremsstrahlung case: indeed, most of the recent structure-function based Monte Carlos ([6,5,9]) do not assign effective beams correctly even in this simple case, and therefore their predictions can only be trusted up to terms of order α with no logs, i.e. they correctly predict the leading terms but not the non-leading terms. It is one of the aims of this note to point out such inherent subtleties in the algorithmic approach, and stress the fact that where high precision is aimed for, a careful examination of what distributions are actually generated by a given algorithm, as well as formal limitations to any such procedure, is imperative.

References

- [1] 'Monte Carlos for electroweak physics', R. Kleiss et al., in 'Z physics at LEP I', Vol.3, G. Altarelli, R. Kleiss and C. Verzegnassi, eds., CERN Yellow Report 89-08 (1989).
- [2] F.A. Berends, G. Burgers and W. van Neerven, Nucl.Phys. **B297**(1988)118; E **B304**(1988)921.
- [3] S. Jadach and B. Ward, Univ. of Tennessee and Jagiellonian University preprints UTHEP-89-0763 / TPJU-19/89 (1989).

- [4] 'The Z line shape', F.A. Berends et al., Z physics at LEP I, Vol.1, G. Altarelli, R. Kleiss and C. Verzegnassi, eds., CERN Yellow Report 89-08 (1989).
- [5] G. Bonvicini and L. Trentadue, Nucl. Phys. **B323**(1989)253; G. Bonvicini, Univ. of Michigan preprint UALHE 89/35 (1989).
- [6] J.-E. Campagne and R. Zitoun, Z.f.Phys. **C43**(1989)169;
- [7] O. Nicrosini and L. Trentadue, Z.f.Phys. **C39**(1988)179.
- [8] J.-E. Campagne, Ph.D. thesis, Univ. Paris VII, L'PHE (1989).
- [9] H. Dahnou et al., Z.f.Phys **C44**(1989)139, and Monte Carlo program UNIBIB (unpublished).
- [10] F.A. Berends and R. Kleiss, Nucl.Phys. **B178**(1981)111; **B260**(1985)32.
- [11] R. Kleiss, Phys.Lett. **B180**(1986)100.
- [12] R. Kleiss, Nucl.Phys. **B241**(1984)100; R. Kleiss and W.J. Stirling, Nucl.Phys. **B262**(1985)235.
- [13] The CALKUL collaboration, F.A. Berends et al., Phys.Lett. **103B**(1981)121; Nucl.Phys. **B206**(1982)61.
- [14] The CALKUL collaboration, F.A. Berends et al., Nucl.Phys. **B264**(1986)213.
- [15] F.A. Berends and W. Giele, Nucl.Phys. **B306**(1988)759.