CERN-TH.5534/89

**MONTE-CARLO SIMULATIONS ON AND OF
TWO-DIMENSIONAL DYNAMICAL LATTICES.
A THEORETICAL OVERVIEW.**

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Abstract

A short review is presented on different projects for regularization of two dimensional surfaces with lattices. The problems with the emergence of acceptable continuum limits are briefly referred, but the emphasis is put on the attempts to saturate the functional measure in the integral over the intrinsic metrics.

CERN TH.5534/89
September 1989

(Expanded version)

March 1990

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1 Introduction

The interest in random surfaces in high-energy physics has changed periodically over the past few decades, where the concept was tentatively applied over different areas of particle physics and, more recently, on low-dimensional quantum gravity and in cosmology.

Compared with these trends of fashion, random surfaces have enjoyed a more stable success in condensed matter physics, where they became a normal tool to attack problems in the area of phase transitions, biological membranes, vortex lines in superconductors, etc.

One of the reasons for this fact (i.e., its partial acceptance in high-energy and gravity physics) may come from the different uses the two fields make of random surfaces. In fact, high energy physics and gravity demand the understanding of the continuous random surfaces' geometry basically on all distance scales. This requires regularization and mastering the renormalization of different surface models. That, in turn, implies the use of a theory of random surfaces which, as opposed to the random walk case, does not exist yet.

Condensed matter physics, on the other hand, is generally equipped with natural short distance cut offs which not only provide a natural framework for regularization but moreover render the renormalization problem more amenable, if it is present at all. The need for a real theory of random surfaces is, strictly speaking, not urgently felt here and progress can be made, on a more pragmatic basis, by exploring advantages of particular phenomenological models proposed to deal with specific type of problems.

A drawback of this situation is that not only the models proposed are generally not obtained from first principles (they are phenomenological) but also that, once under study, small differences of detail may lead to drastically different types of behaviour for the surfaces they describe. In other words, there is no room for universality.

Universality is generally a characteristic of statistical systems undergoing critical phase transitions, which is only possible if they are infinitely large. Of course this situation hardly arises in condensed matter physics, with discrete systems. But it arises when dealing with discrete approximations to finite sized continuous systems, as those representing (super) strings and space time manifolds.

By showing that many details of a model become irrelevant, universality reduces the variety of possible behaviours for continuous systems. On the positive side, it decreases the degree of arbitrariness of a model by showing its equivalence with an entire class of other models sharing the same continuum properties. And since the (first) steps in the search for adequate models are always plagued by some degree of uncertainty, it is not only reassuring but also practically necessary to know that an a priori reasonably looking model has a finite chance of describing the correct continuum properties of the random surface, even when it looks somewhat arbitrary.

Here we will succinctly describe the main conceptual steps made in the direction of constructing discrete models of random surfaces and understanding their continuum (i.e., critical) properties using, for that, facts which may, one day, belong to the theory of random surfaces.

By now there is a large amount of work done on the uses of lattice methods to study two dimensional surfaces in high energy physics and quantum gravity¹. We did not intend to present an exhaustive review of that work and certainly many relevant contributions had to be left aside. Instead, and given the large variety of methods followed – ranging from numerical Monte Carlo (MC) simulations [4,5,6] to proving theorems [7,8,9,10] [11,12,13] – we decided to emphasize some of the main trends and problems common to practically all of them, which are the attempts to progressively obtain more complete integration measures for the intrinsic metric in lattice regularized surfaces and the crisis (normally associated with bad continuum limits) pointing to the insufficiency of each method. In this way, many details of the analysis near the continuum were left aside, leaving the problems typically related to the dynamics of the regularization itself as the ultimate focus of interest.

Now we will briefly indicate the contents of this paper:

In Section 2 we introduce the basic concepts common to the study of bosonic strings and two dimensional quantum gravity in the continuum, which are the Polyakov action functional, its classical symmetries and the possible additional terms allowed by Poincaré invariance and renormalizability, and finally its classical form, obtained when the world sheet metric

¹ Among the available review articles, we found the subject of random surfaces treated in great detail in [1,2,3].

is not quantized.

In section 3 we take this classical form of the action (Nambu-Goto action) and look at its historical importance in describing the QCD colour flux tube on space time hypercubic lattices. The question of the existence of confinement deconfinement phase transition is referred together with some of its characteristics. We stress particularly the importance of the fluctuations in the topology (or in the connectivity) of the string world sheet for counting the number (entropy) of the corresponding random surface configurations on the lattice. We discuss why analytical and computational limitations impose a freeze on such topological fluctuations even though this truncates the picture of fully interacting random surfaces.

In section 4 we accept this truncation on the assumption that the continuum limit properties of the lattice systems will not depend crucially on global aspects, such as the topology, and concentrate on the spherical topology. We study the critical exponents and the Hausdorff dimension of the surfaces and show why the Nambu-Goto action is basically a bad choice. It leads to polymerization of the surface in the continuum, equivalent to a non interacting model.

Section 5 presents some of the attempts to overcome the polymerization problem by using intrinsic (on the surface) triangulated lattices as an alternative to space time ones, and how such efforts were frustrated by the formation of “spiky” configurations near the critical point. As we indicate in the sequence, the cure to that problem came from un-freezing the surface’s intrinsic metric degrees of freedom. We explain how that is done on the triangulated lattices by summing over classes of isomorphic triangulations, and why one might hope that the usual principle of reparametrization invariance should emerge from these summations. The section ends with a resumé of analytical computations of the surfaces’ susceptibility, at lattice criticality and directly in the continuum (using conformal techniques), and showing a remarkable, yet intriguing agreement between the two sets of results (with space time dimensionality $D \leq 1$), suggesting that fundamental properties of the continuum random surfaces are caught by the dynamically triangulated lattices at their critical points.

Apart from pure gravity ($D = 0$), most of the systems which random surfaces try to simulate (e.g. strings) are naturally embedded in spaces with $D \geq 1$. Here however, all attempts to perform exact analytic cal-

culations on lattices have been frustrated so far (except at $D = 1$ and in the limiting case $D = +\infty$). The need for Monte Carlo (MC) simulations and the ways to implement them are explained in section 6. Once again, a more complete integration measure is required in the path integral which appears, in the lattice case, as an extra sum over the number of lattice sites (grand canonical ensemble). The Besides remaining so far out of reach for analytical studies directly in the continuum, the sector $D > 1$ is also very hard to penetrate by numerical methods. Here, the existence and characteristics of continuum limits for lattice systems is a very actively studied question today.

Finally, in section 7 we go one step forward in the direction of “completing” the measure over metrics, by considering topological fluctuations. We indicate the urgency of this question in the area of euclidean quantum gravity (also in higher dimensions) and we shortly present a personal point of view according to which the need to sum over topologies might be related to the implementation of a larger reparametrization symmetry on the surfaces. We end this section by discussing the dependence of the critical susceptibility on the surface’s topology and its embedding dimensionality as well and briefly referring to very recent treatments of the old question on topological sums.

2 Embedding in Space-Time: Bosonic Strings

As is well known, Feynman’s concept of the Sum over histories, which is fundamental in the study of quantum mechanical point particles and quantum field theories was generalized to the case of two dimensional systems by Polyakov [14]. Such systems can be of quite different types, ranging from biological membranes to interfaces between distinct chemical or physical phases, to string world sheets up to rather idealized models such as two dimensional gravity.

The interest in these systems comes either from their intrinsic geometric properties or from the extrinsic properties they acquire when adequately embedded in various types of spaces (matter fields or the space time itself).

The (closed) bosonic string represents a typical example where the intrinsic variables $g_{\mu\nu}(\xi)$ (metric) interact with the extrinsic parameters

$X^\mu(\xi)$ (space time coordinates) in a well defined way, described by the Polyakov action [14, 15]

$$S[g, X] = \beta \int_{\mathcal{M}} d^2\xi \sqrt{g} g^{\alpha\beta}(\xi) h_{\mu\nu}(X) \partial_\alpha X^\mu(\xi) \partial_\beta X^\nu(\xi) \quad (1)$$

where β is a parameter.

Poincaré invariance and power counting renormalizability allow two more terms to be added to this action namely, the term with the cosmological constant (λ),

$$\lambda \int_{\mathcal{M}} d^2\xi \sqrt{g} \quad (2)$$

and the scalar curvature term (i.e., the topological term in two dimensions)

$$\frac{1}{2\pi} \int_{\mathcal{M}} d^2\xi \sqrt{g} R \quad (3)$$

The action (1) is invariant under reparametrizations

$$\delta X^\mu = \epsilon^\alpha \partial_\alpha X^\mu \quad (4)$$

$$\delta g^{\alpha\beta} = \epsilon^\gamma \partial_\gamma g^{\alpha\beta} - \partial_\alpha \epsilon^\gamma g^{\gamma\beta} - \partial_\beta \epsilon^\gamma g^{\alpha\gamma} \quad (5)$$

$$\delta(\sqrt{g}) = \partial_\alpha (\epsilon^\alpha \sqrt{g}) \quad (6)$$

and Weyl re scalings

$$\delta g^{\alpha\beta} = \Lambda(g) g^{\alpha\beta} \quad (7)$$

From the point of view of the classical equations of motion, the topological term is irrelevant since it is the integral of a total derivative. The cosmological term, on the other hand, is forbidden and the intrinsic metric freezes around the induced metric

$$(g_{\alpha\beta})_{\text{classical}} \propto \partial_\alpha X^\mu \partial_\beta X_\mu \quad (8)$$

With these classical metric configurations, the action (1) reduces to the Nambu Goto action [16, 17], which is proportional to the area of the internal parameter space $\mathcal{M}$

$$S[g_{\text{dynamical}}, X] \propto \int_{\mathcal{M}} d^2\xi \sqrt{g_{\text{dynamical}}} \equiv \text{Area}(\mathcal{M}) \quad (9)$$

Before we consider the Nambu Goto action and its historical importance, we should come back to the problem of extending the path integral to two dimensional objects. As is well known, the Polyakov path integral represents a sum over an ensemble of all possible configurations and is described by the partition function

$$Z = \int [dg] \int [dX] e^{-S[g,X]} \quad (10)$$

where $[dg]$ and $[dX]$ denote the measures for the dynamical variables g and X , and $\mathcal{M}$ denotes the manifold of space time coordinates and other fields which may be introduced in string theory. In pure gravity, the integral over $\mathcal{M}$ is absent and only the metric is quantized.

The adequate definition of the measure $[dg]$ remains an open mathematical problem and depends on the regularization scheme used². As we will refer in paragraph 5, recent numerical results [19,20,21] using lattice regularizations indicate that the physics near and at criticality (i.e., at the continuum) may generally depend on the choice of such measure. On the other hand, the class of metrics over which one integrates depends on various physical criteria and constitutes the main question discussed throughout this work.

Thus the integral (10) can be written only formally. In a first attempt to give meaning to it, one may use a discretized space time lattice and freeze the metric $g_{\mu\nu}$ at its classical configuration $(g_{\mu\nu})_{\text{classical}}$. In this way, the integral (10) reduces to a sum over configurations of a statistical mechanical system. Such a system should exhibit critical behaviour for some values of its parameters (couplings or temperatures), where the correlation length of the dynamical variables becomes much larger than the lattice step length, thus mimicking the continuum limit.

Besides giving meaning to the formal expression (10), the use of lattices should have still another advantage, namely it should allow the study of typical non perturbative effects through the use of numerical simulations, as

²For a comparative study of different proposals for $[dg]$ see [18] and references therein.

the example of lattice quantum field theory would suggest. However, as we will see in the next paragraphs, such prospects for non perturbative analysis do not seem to be easily implementable in the case of two dimensional lattice objects.

3 Nambu-Goto Action

As the intrinsic metric is now considered a classical quantity, there will be no independent intrinsic degrees of freedom on the world sheet spanned by the closed string (no gravity), and the whole string dynamics is dictated by the embedding fields X^μ (space time coordinates). Actually, reducing now to the area of the string world sheet, the action contains only the minimal information about the surface geometry.

Under these simplified assumptions, several calculations were carried out on hypercubic lattices in the early eighties [22,23,24,25], trying to simulate the open string of gluons [26,27,28,29] between two static quarks. One of the main motivations was to check if the non perturbative phenomenon of quark confinement could be seen on the lattice [30,31,32,33]. The distinctive signal for confinement would be the transition between a phase where the world sheet spanned by the colour flux tube, would fluctuate in shape and area more or less wildly, to a phase where the string excitations would lead almost exclusively to small disturbances of its world sheet around a tidy and stretched surface [34,35].

The selection of one phase or the other is determined by the value of the inverse of the string tension, β , in front of the action (1).

Since the Nambu Goto action depends only on the area, it is the interplay between the action and the exponent of the number of configurations with a fixed area A (i.e., the entropy of these configurations) that determines the phase, as usual.

Hence, it is crucial to estimate the dependence of the entropy on the area. Such results are well known [36,37,38], and they can be enunciated as follows:

If one does not restrict the topology of the surfaces but instead allows for all topologies compatible with the lattice sizes, the number of surfaces with given area A grows like

$$n(A) \sim A! \quad (11)$$

If, on the other hand, the topology is restricted to a fixed genus h , that number grows like

$$n_h(A) \sim A^{b_h} C^A \quad (12)$$

where C is a cut off dependent parameter, independent of h , and b_h is cut off independent but generally varies with h . The exponent b_h plays an important role in the continuum theory since it is related to critical exponents and the Hausdorff dimension of the random surfaces (see later) ³. In the spherical topology ($h = 0$), b_0 can be calculated perturbatively [38] and one gets:

$$b_0 \sim \frac{D-25}{6} \quad (13)$$

We see that, in the case (12), the leading term of the entropy is additive in A (for large A) as opposed to (11) where the leading term grows like $A \ln A$. This means that, whereas by summing over all topologies the most probable configurations have an arbitrarily large area and the action can never compete with the entropy, the action can do so in the second case, of additive entropy, large areas being preferred only if $\beta < \beta_c \equiv \frac{1}{2} \ln C$. If $\beta > \beta_c$ the most common configurations will have the minimal area (compatible with the distance between the static quarks), given by

$$\langle A \rangle \simeq \frac{b_h}{\beta - \ln C} \quad (14)$$

In this case it is reasonable to choose the physical vacuum configuration corresponding to a string stretched between the static sources. Excitations will take place roughly as small amplitude undulations around these configurations and the piling up of such modes will eventually lead to a transition into a deconfining phase where, although still intact, the string becomes extremely de-localized since its tension is not able to damp out the fluctuations of large amplitude.

³In paragraphs 5 and 6 we will see that b_h relates to the critical index γ by $b_h \simeq \gamma - 3$.

It is important to stress that, in the previous scheme, the string remains intact in the deconfined phase, although it is essentially “unobservable”. The reason is again that the counting of the number of configurations on the lattice has, in this case, to take care of only one fixed topology (or connectivity). In the case of a meson for instance, the transition would lead to a free quark antiquark pair. However we know that, much before this happens, the original meson has split into two or more mesons which, in the old days of strong interaction physics, would have been interpreted as the breaking of the original (stretched) string into bits of stretched strings. That is, the string should be allowed to break before its “ends” become free. This means that the connectivity of the world sheet spanned by the string should be allowed to change or, in the case of the closed string, its topology should be allowed to fluctuate (see figs. 1 and 2).

From the domain of strong interactions, the string concept entered elementary particle physics [39] and gravitation [40]. In the framework of the new string physics, the fluctuations in connectivity and topology are a natural consequence of the string self-interactions. In spite of this, in the analytical studies and numerical simulations of the string on lattices, from the first days until very recently ⁴, the constraints of fixed connectivity and world sheet topology have been systematically imposed, although recent studies allow for toroidal and eventually even higher (but fixed) topologies [11,42]. It is in this sense that the access to non perturbative, fully interacting results still remains limited, as was mentioned before.

In terms of the string dynamics, once one abandons the spherical topology (corresponding to the free string case), it is not clear what dynamical truncations the constraint of constant topology will correspond to unless, of course, one argues that the object of study is now a special statistical system whose boundary conditions have been chosen and fixed in peculiar ways.

Certainly there were, and there still are, reasons to impose such global constraints, however unrealistic they may be. The most obvious one is that the numerical simulations become often exceedingly complex once the constraints are relaxed. The second, and most important reason, stems

⁴See the end of paragraph 7 for a short comment on recent works where the problem of summing over topologies is re-considered.

from the fact that, to a large extent, the continuum limit properties of these statistical systems have not been tamed yet.

Hence, if any pathological behaviour is found near the continuum limit (typically short distance effects), it will show the need for an improvement of the model but it will not generally stem from any global, topological aspects of the configurations considered unless, of course, these topological effects are, after all, not so global, which would be the case if the system were not exactly critical or if configurations with arbitrarily large topologies somehow became dominant.

As a third reason, we might refer that planar surfaces are the only ones which contribute significantly in the large N limit, to the high temperature expansion of $SU(N)$ pure lattice gauge theories, whose formulation in terms of string type degrees of freedom was extensively studied in the past [13, 14].

Now and for a while we will stick to configurations with fixed, spherical topology and consider some issues related to the continuum limit.

4 Critical behaviour of the (Planar) Random Surfaces

In the study of random surfaces, spanned by bosonic strings, one constructs the n loop Green functions [15]

$$G_n(\gamma_1, \dots, \gamma_n) = \sum_{S(\gamma_1, \dots, \gamma_n)} e^{-\beta A} \quad (\beta \geq \beta_c) \quad (15)$$

where A is the area of the surface (proportional to the number of plaquettes forming it in the hypercubic lattice) and the sum extends over all the random surfaces, $S(\gamma_1, \dots, \gamma_n)$, having n fixed boundaries (denoted by $\gamma_1, \dots, \gamma_n$), with positions kept fixed ⁵ and the rest of the surface remaining free. In particular, these n loops can be the boundaries of n plaquettes (see fig. 3). In that case, the compact random surfaces are recovered by “filling in” the interior of the fixed plaquettes, which does not affect the

⁵If these boundaries were associated with world lines of quarks (or anti quarks) this would mean that they were static sources, i.e., non dynamical.

dynamics. It is important to notice that the previous sum converges only if $\beta \geq \beta_c$, where β_c is the inverse of the critical temperature.

Let us consider only the cases where the γ loops represent simple plaquette boundaries.

Physical quantities like the mass, string tension and susceptibility can be defined in terms of the Green functions in the limit when some typical distances become very large [15]. Being functions of β , these quantities will be singular or vanish at the critical value β_c , depending on the values of the corresponding critical exponents. The *mass gap* is defined as

$$m(\beta) = - \lim_{|a| \rightarrow \infty} \frac{\ln G_\beta(\gamma_0, \gamma_a)}{|a|} \sim (\beta - \beta_c)^\nu \quad (16)$$

where $|a|$ represents the minimal number of lattice links (lattice distance) between the plaquettes γ_0 and γ_a . ν is the critical exponent for the mass-gap.

The definition of the *string tension* requires that we consider again a random surface, this time with a large boundary, γ_{LM} (and not simply a plaquette), having for simplicity a rectangular shape and dimensions L and M :

$$\tau(\beta) = - \lim_{L, M \rightarrow \infty} \frac{\ln G_\beta(\gamma_{LM})}{LM} \quad (17)$$

The corresponding critical exponent, μ , is defined with respect to the derivative of $\tau(\beta)$:

$$\frac{d\tau(\beta)}{d\beta} \sim (\beta - \beta_c)^{\mu-1} \quad (18)$$

Finally, the *susceptibility* χ , defined by

$$\chi(\beta) = \sum_{P \in Z^D} G_\beta(\partial P_0, \partial P) \sim (\beta - \beta_c)^{-\gamma} \quad (19)$$

involves a summation over all possible positions of one of the boundaries, (∂P) , on the hypercubic lattice Z^D . In particular, the boundaries ∂P_0 and ∂P may be those of simple plaquettes, as before.

Near critical points the statistical systems generally obey scaling relations. They are simple relations between the critical exponents. In the present case they are also valid and read ⁶

$$\begin{aligned} \gamma &= \nu(2-\eta) & (Fisher's relation) \\ \nu &= \frac{1}{\delta} \end{aligned} \quad (20)$$

An important geometrical (and physical) characteristic of the surface is the *Hausdorff dimension* [17,18], δ . We recall its general definition:

A D_0 dimensional object (with D_0 a positive integer), embedded in a D dimensional space (with $D \geq D_0$), is said to have Hausdorff dimension δ if the minimal number of D dimensional spheres, each with radius R , needed to cover it completely, grows like $R^{-\delta}$ when $R \rightarrow 0$.

If the object is smooth, δ coincides with D_0 . Generally however, δ can be non integer and larger than D_0 . Furthermore δ does not depend on D and is another intrinsic characteristic of the object. δ can be calculated in different ways. For a random surface on a lattice we may consider the following construction [15]: Consider an ensemble of surfaces with fixed area A , containing a given arbitrary lattice site $\mathcal{P}$ as well as a given arbitrary plaquette P_0 which has $\mathcal{P}$ as a vertex (see fig. 1.). For each such surface one constructs a two dimensional sphere with radius R , centered at $\mathcal{P}$, and measures how much of the random surface falls inside that sphere. Let S_R be the area trapped inside the sphere. Finally let $u_0(1)$ count the number of surfaces with area A satisfying the conditions indicated above. The Hausdorff dimension can then be determined as follows:

$$\delta = \lim_{R \rightarrow \infty} \lim_{A \rightarrow \infty} \frac{1}{A} \sum_{\substack{P_0 \in A \\ S \in \mathcal{P}}} \frac{\ln S_R}{\ln R} \quad (21)$$

An additional critical index, η , is often used in statistical mechanics. It is the *anomalous dimension* of the plaquette-plaquette correlation function and is defined by considering each individual term in the sum giving the susceptibility (two loop correlation function)

⁶For a review of critical phenomena in field theory, see ref. [16].

$$(\tau_P(\partial P_0, \partial P) \sim |X_P - X_{P_0}|^{-(D-2+\eta)} \quad (22)$$

where $|X_P - X_{P_0}|$ is the distance between ∂P_0 and ∂P , and D is the lattice (or space time) dimensionality.

Using the Nambu-Goto action to generate lattice configurations on hypercubic lattices with $D \geq 2$, and the assumption that $\gamma > 0$ ⁷, the values obtained for the critical exponents are the following [15,48,49,50]:

$$\begin{aligned} \nu &= \frac{1}{4} \\ \mu &= \gamma = \frac{1}{2} \\ \eta &= 0 \\ \delta &= 4 \end{aligned} \quad (23)$$

These values are disappointing in the sense that they all coincide with the values one would have obtained had one used simple mean field theory for the surface on the lattice [45]. However, it is well known that mean field theories are essentially free theories.

Hence, even though one started with the non trivial Nambu-Goto action, this non triviality seems to be completely washed out as one approaches the continuum limit.

There is an additional effect which indicates that this model should be substituted by a more elaborated one. It is associated with the fact that the string tension $\tau(\beta)$ given by (17) does not vanish at the critical point (instead, its derivative with respect to β diverges at β_c) but stays finite and positive

$$\lim_{\beta \searrow \beta_c} \tau(\beta) > 0 \quad (24)$$

Now, on the lattice, as in the continuum, one can always re-express the correlation functions in terms of modified variables. Thus, consider

⁷I.e., that $\chi(\beta)$ diverges at β_c . As we will see later on, this is a natural assumption in view of the fact that $\chi(\beta)$ gives a measure of the average number of lattice sites near the critical point, a quantity which necessarily diverges if there is a continuum limit.

the set $\{b_i\}$ of lattice links and let X_i collectively denote the space time coordinates of their middle points, measured in lattice units

$$b_i \equiv b(X_i) \quad (25)$$

Define then new coordinates Y_i by the relations

$$X_i = \theta Y_i \quad (26)$$

where $\theta \geq 1$ is a scale parameter. The quantities Y_i represent continuous coordinates in the limit $\theta \rightarrow \infty$ (continuum limit).

The vanishing of $m(\beta)$ at $\beta = \beta_c$ and the fact that $m(\beta)$ is the inverse of the correlation length means that, as $\theta \rightarrow \infty$

$$\beta \equiv \beta(\theta) \rightarrow \beta_c \quad (27)$$

By using the continuum type variables Y_i and simple dimensional arguments, one can define continuum Green functions G_θ from the lattice Green functions G_θ' , when $\theta \rightarrow \infty$, as

$$G_\theta(Y_1, \dots, Y_n) \equiv [\alpha(\theta)]^n G_{\theta\theta'}(b(\theta)Y_1, \dots, b(\theta)Y_n) \rightarrow G(Y_1, \dots, Y_n) \quad (28)$$

where $G(Y_1, \dots, Y_n)$ are the physical Green functions. Demanding, in particular, that the two loop function $G(Y_1, Y_2)$ does not vanish (otherwise $m(\beta)$ would probably diverge since it depends on the logarithm of $G_\theta(X_1, X_2)$) and using (22), we obtain the explicit form of the scaling factor in (28)

$$\alpha(\theta)^2 = \theta^{D-2+\eta} \quad (29)$$

The dependence of β on θ is dictated by demanding that

$$\theta m(\beta(\theta)) \rightarrow m^* > 0 \quad \theta \rightarrow \infty \quad (30)$$

with m^* being the physical mass gap.

Analogously, the physical string tension τ^* is given by

$$\tau^* = \lim_{\theta \rightarrow \infty} \theta^2 \tau(\beta(\theta)) \quad (31)$$

and since $\tau(\beta(\theta))$ does not vanish at β_c , τ^* is finite.

This unacceptable conclusion is related to the fact that, as β gets near to β_c , the surfaces, originally compact and reasonably "smooth", will become preferentially distorted in such a way that, at $\beta = \beta_c$, their area will vanish. On the other hand, their Hausdorff dimension being 1 (i.e., larger than 2) means that, whatever shape the surfaces acquire, they will fill a higher dimensional volume in the continuum limit. The only way they can have zero surface but non zero volume is by collapsing into fractal structures, which are the continuum limit of branched polymers on the lattice [15,51].

5 Generalized Actions and Triangulations

The suggestion that this unphysical behaviour was an artifact of the lattice regularization motivated further studies using a different type of regularization for the area action. The suspicion was that metastable states, associated with the finite space time lattice step (see fig. 5.a), once created were difficult to "kill" near the continuum limit [52]. The idea was then to start working with a continuum space time and, simultaneously, use a lattice, with all its advantages, by triangulating the random surfaces intrinsically. The lattice is then formed by the set of sites on the triangulated surface and the links joining them.

In this way fine changes in the surface shape and area can easily be made (since now the space time coordinates of each point on the surface are represented by a set of D continuous fields) (see fig. 5.b). Another advantage of this proposal had to do with the space time dimensionality D . Whereas with the hypercubic lattice, each time a new simulation with different D has to be made one must reconstruct the entire lattice, using triangulated surfaces the same can be done simply by adding (or subtracting) the same number of fields at each lattice site.

Theoretical analysis [53,54] and numerical simulations [52] were done with intrinsic lattices of the simplest type, namely regular honeycomb structures on spherical and toroidal surfaces. However, they again revealed the

formation of fractal structures, this time with infinite Hausdorff dimension ($\delta = \infty$). This prevents the existence of a continuum limit, as was rigorously shown in [54].

As are now well understood, such results should be expected. The simplest way to understand them is in terms of the entropy of the surfaces [55]. The action, being essentially their area, will not discriminate between surfaces with identical areas but different shapes. However, on the lattice, the number of smooth or approximately smooth (in the lattice sense) surfaces is much smaller than the number of those which have “spikes” (see fig. 6.) or other defects. The reason is that each elementary spike contributes to the entropy with an amount proportional to the number of positions it can occupy on the lattice. When the number of such elementary spikes increases beyond the point where excluded volume problems start being felt, some of them may coalesce to form a larger one⁸.

At large distances this results in a large surface “floppyness”. Ultimately, the easy formation of composite spikes with arbitrarily large sizes will be responsible for the appearance of fractal trees or polymers, in the continuum limit and this effect might be the discretized version of the well known tachyonic instability in bosonic strings.

A way to reduce this floppyness is to choose a more discriminative action, namely one that depends on detailed geometric characteristics of the surface and not simply on its area.

Indeed, generalized actions satisfying this requirement were studied in [54,56,57]. Basically they depend on the space time coordinates of the surface sites or reflect constraints imposed on the configurations by forbidding special orientations of each plaquette with respect to its neighbours on the surface [58].

Their eventual success in reducing the surface polymerization is achieved by making use of the already existing space time degrees of freedom and directly increasing the surface rigidity. They are called **extrinsic** actions. Different forms of extrinsic actions have been studied, some of them originally motivated from outside the area of high energy physics [59] but with possible relevance to it [60,61].

⁸ Actually, since all surfaces with identical area come out with identical weights, it is not even necessary to reach a high density of elementary spikes before composite spikes start forming.

Another class of generalized actions depend on the **intrinsic** metric of the surface, $g_{\mu\nu}$, and possibly on functions of it such as the powers of the scalar curvature. However, to be effective in reducing the polymerization, the external space time coordinates must somehow be coupled to $g_{\mu\nu}$. In the continuum, the Polyakov action (1) is a typical example of this.

In models equipped with extrinsic actions, the extrinsic curvature terms may arise from integrating fermionic degrees of freedom originally present [62] as, e.g., in the superstring. The possible success of extrinsic actions in eliminating the lattice polymerization should then be understood as consequence of the fact that the superstring is free of tachyons. This idea remains so far unchecked due to the difficulty of constructing lattice discretizations of supersymmetric theories. Nevertheless, some interesting results exist already. Even though they are preliminary and much remains to be understood, they show strong model dependence and, in some cases, seem to be somehow contradictory and plagued by serious violations of universality near the (expected) critical points. Moreover, in these models the discretized action is purely bosonic from the start.

We will not review here the work done on models with extrinsic actions, but refer the reader to recent literature on the subject [63,64,65,66,67]. Instead, we will consider a different point of view that is, roughly speaking, it is not excess of floppyness, but rather lack of it, which is responsible for the lattice polymerization of the random surfaces at the critical point, as was already suspected when the hypercubic lattices were replaced by intrinsic surface triangulations.

The question now is, where to look for additional degrees of freedom to “unfreeze” in order to get more floppyness?

The answer can be found in the intrinsic metric tensor components $g_{\mu\nu}$, frozen at their classical values to yield the Nambu-Goto action from the Polyakov one.

If this point of view turns out to be justified, this will mean that the persistence of trivial critical behaviour might come from the fact that the fixed lattice of intrinsic surface coordinates (honeycomb in the case studied), which represents the parameter space, describes properly only small fluctuations around the flat internal metric $g_{\mu\nu} \propto \delta_{\mu\nu}$.

Let us comment a bit more on this point. Although one can simulate a surface fluctuating in space time by using a statistical ensemble of fields

(the space time coordinates) defined on a fixed lattice in particular of the honeycomb type the lattice itself, as seen by an observer living on it and who ignores those fields, is rigid. For that observer, the distance between any two lattice sites as measured say, by the minimal number of links joining those sites, remains constant.

Therefore, to switch on the dynamical fluctuations of $g_{\mu\nu}$, as required by the Polyakov path integral quantization (10), these distances must be allowed to fluctuate.

The rationale for this hope lies in the fact that a large, approximate symmetry emerges as a result of summing over all intrinsic lattice configurations.

In the continuum limit this approximate symmetry becomes exact and expresses the principle of local coordinate invariance on the random surface [68,69]. Each random triangulation is to be interpreted as an arbitrary (discrete) coordinate system. Freezing $g_{\mu\nu}$, as with the Nambu Goto action, is equivalent to choosing a particular class of triangulations (isometrically related) and prevents the continuous symmetry to manifest itself. Furthermore, it seems natural to let the system evolve towards a set of "most probable" intrinsic geometries and only then consider additional dynamics (embeddings) on it. If this sequence is not followed, unexpected results may arise.

There are essentially two methods to accomplish this purpose. The first one is by keeping the lattice structure fixed but allow for changes of the link lengths, which become the dynamical variables (subject, of course, to several constraints). In the second method one keeps those lengths fixed (and say, identical) but allows the lattice structure to change that is, one devises algorithms which artificially remove some links and create other links on the lattice. After computing the variation of the action, the change is accepted or rejected by using the well known Metropolis algorithm [21].

Certainly such procedures must be subjected to constraints, to guarantee that each new configuration of sites and links still corresponds to an acceptable triangulation of the surface.

We should remark, in passing, that in the continuum limit, these two procedures should yield identical results [1] since the choice of triangulations with equilateral triangles is equivalent, in the continuum, to selecting local coordinates in which the intrinsic metric tensor is

$$g = e^{2\phi}\eta \quad (32)$$

where η is the diagonal metric $\eta_{\alpha\beta} = \delta_{\alpha\beta}$, and $e^{2\phi}$ is the conformal factor. The integration over this last factor [11] should then result from the sum over all classes of isomorphic triangulations^a.

When space time is no longer the embedding space (as in string theory) but becomes the internal parameter space itself (in this case the surface), the first method referred to above, constitutes the basis for lattice discretizations of space time and numerical simulations of general relativity, called Regge calculus [70]. Given its generality, this method has been applied recently to study quantum gravity [18,71,72,73,74] [75,76,77,78].

Here we will only consider results related to the second method.

A simple example of the type of algorithm cited above, called a FLIP, [21] changes one link into another one each time it is applied (see fig. 7.a). When iterated sufficiently many times, it may change the intrinsic lattice by creating defects (see fig. 8.) and/or by affecting those already present. In this sense, the intrinsic lattice itself becomes dynamical.

Taking again the example of the FLIP, and considering it as a sort of building block for more complicated algorithms, a few remarks are in order:

- i) By iterating sufficiently many times the FLIP algorithm, the distance between any pair of sites may eventually be altered, as the simplest example of fig. 9. shows. In this way the metric is changed.
- ii) To saturate the space of all internal metrics in the Polyakov path integral (10), one should sum over all possible triangulations of the surface.
- iii) By the Euler relation for the lattice,

$$\chi_{Euler} = 2 - 2h = N_0 - N_1 + N_2 \quad (33)$$

where

$$\begin{aligned} \chi_{Euler} &= \text{Euler characteristic of the lattice polyhedron} \\ h &= \text{Topology (no. of Handles)} \\ N_0 &= \text{no. of Sites} \end{aligned}$$

^a i.e., different triangulations in which the distances between identical pairs of sites are the same.

$$\begin{aligned} N_1 &= \text{no. of Links} \\ N_2 &= \text{no. of Triangles} \end{aligned}$$

we see that the FLJP algorithm preserves the topology ($\delta\chi_{Euler} = 0$), since all geometric elements are conserved in number ($\delta N_0 = \delta N_1 = \delta N_2 = 0$). The ensemble generated in this way is thus a microcanonical ensemble [20,21,79] [80,81].

iv) The number q_i of nearest neighbours of (or links emanating from) a given site i may thus vary between three (we do not consider configurations with tadpoles or more than one link joining each pair of nearest neighbours (see fig. 10.)) and arbitrarily large numbers. On the type of lattices that we will consider from now on, all links have identical lengths, and q_i measures the deficit angle on the site i which relates to the scalar curvature R_i , at that site, by

$$R_i = \pi \frac{6 - q_i}{q_i} = \pi \frac{2 - \sigma_i}{\sigma_i} \quad (31)$$

with σ_i being the area of the cell dual to site i (fig. 11.). Using now the continuum to discrete prescription

$$\begin{aligned} R(\xi) &\longrightarrow R_i \\ \sqrt{g(\xi)} &\longrightarrow \sigma_i \\ \frac{1}{2} \int_{\mathcal{M}} d^2\xi &\longrightarrow \sum_i \end{aligned} \quad (35)$$

the Gauss-Bonnet formula for a surface without boundaries

$$2\pi\chi_{Euler} = \frac{1}{2} \int_{\mathcal{M}} d^2\xi \sqrt{g(\xi)} R(\xi) \quad (36)$$

reduces, on the lattice, to the Euler formula

$$2\pi\chi_{Euler} = \sum_i \sigma_i R_i = 2\pi(N_0 - N_1 + N_2) \quad (37)$$

After the basic issues about internal metric fluctuations and the associated sum over triangulations have been stated, the next step is to actually

check whether over- or under-floppyness (rigidity) is the cause for the observed pathological behaviour near the continuum limit.

To do that, the dynamical lattice is re-embedded back into space-time and the partition function

$$Z(\beta) = \sum_T \rho(T) \Gamma(T) e^{-\beta N_2(T)} \quad (38)$$

may in principle be computed.

Here T denotes each triangulation and $\rho(T)$ is the weight attributed to it. In principle $\rho(T)$ is related to the inverse of the volume of the group of symmetry of T (which should become the group of diffeomorphisms in the continuum)¹⁰ and to the distribution of the coordination numbers (q_i) of the sites on the intrinsic lattice [19,21]. Contrasting with the case of random walks [54] however, $\rho(T)$ cannot generally be determined for random surfaces and no universality arguments seem to be valid in this case either. In fact, as we will see in the next paragraph, the dependence of the numerical results near criticality, on different choices of $\rho(T)$, seems to be evident in many cases. We will choose $\rho(T) = 1$ unless indicated otherwise¹¹.

Dealing with the quantization of our system, we were now allowed to include the “cosmological” term $\beta N_2(T)$, where β is a bare string tension and $N_2(T)$ is proportional to the intrinsic area of the surface that is, to the number of simplicial plaquettes (=triangles) in the configuration T (β may alternatively be interpreted as the bare cosmological constant of the space-time surface).

Notice the absence of the topological term, linear in the (intrinsic) scalar curvature $R(\xi)$.

The problem of selecting the discretized action $S[\{X_i\}]$ is analogous to the question of how to choose $\rho(T)$: In the one-dimensional case (random walk) there is an enormous freedom in the choice of the action since one can

¹⁰In the continuum, it would contain the gauge fixing terms and the integration over ghosts.

¹¹By requiring that all triangulations generated should have a simplicial plaquette fixed in the internal parameter space [19], the problem of determining the symmetry-weight factors in $\rho(T)$ is avoided, all isomorphic classes of triangulations contributing with identical weights.

prove a central limit theorem for the random variables $\{X_i\}$, which remains valid under rather general conditions. This is a fixed point theorem for the action and yields a gaussian form leading immediately to the free particle propagator. However, no such theorem has been proved so far in the case of random surfaces. Lacking such result, it is natural and convenient to select, from the start, the quadratic expression

$$S[\{X_i\}] = \sum_{\langle ij \rangle} \frac{(X_i - X_j)^2}{(a_0)^2} \quad (39)$$

as the discretized form of the Polyakov action (1) for the bosonic string [51, 79, 82]. The sum extends over all pairs of sites, i and j , which are nearest neighbours, with euclidean space time coordinates collectively denoted by X_i and X_j respectively. a_0 is the lattice step (or link intrinsic length), assumed to be uniform, which acts as ultra violet cut off.

Finally the factor $V(T)$ takes care of the integration over the D space time coordinates in the path integral:

$$V(T) = \int \prod_{i=1}^{N_0-1} \frac{d^D X_i}{\pi^{\frac{D}{2}}} e^{-S[\{X_i\}]} \quad (40)$$

The space time translational mode is removed by integrating only over $N_0 - 1$ sets of D coordinates.

Alternatively, (38) (40) can also describe two dimensional gravity coupled to a massless free scalar field X with D components.

Notice that the gaussian integral over the continuous variables X_i can formally be solved. The resulting determinant depends of course on T as well as on the D coordinates which were not integrated in (40). The final expression may then be analytically continued to arbitrary values of D .

The results obtained on the model just described can be summarized as follows:

The case with $D = 0$ is two dimensional pure gravity and is the simplest (no X_i integrals); all one has to do is to count planar triangulations which are dual to Feynman graphs of ϕ^3 theory [83, 84].

The critical exponent γ , computed analytically [79] is $\gamma = -\frac{1}{2}$.

For the unphysical dimension $D = -2$ the model can also be solved exactly [79] in particular, by mapping it to a $D = 0$ supersymmetric field

theory and hence reduced to pure combinatorics again [85]. One obtains in this case [79] $\gamma = -1$.

For $D = 1$ the gaussian model cannot be solved analytically. However, changing the form of the action as

$$\sum_{\langle ij \rangle} \frac{(X_i - X_j)^2}{a_0^2} \rightarrow \sum_{\langle ij \rangle} \frac{|X_i - X_j|}{a_0} \quad (41)$$

one can extract exact results for the critical exponents [86] which, by universality, should be identical with those obtained by using the quadratic action (39). One gets $\gamma = 0$.

Analogously, the case with $D = \frac{1}{2}$ requires the use of a modified discrete action. The same sort of argument gives $\gamma = -\frac{1}{3}$.

In the limit $D \rightarrow -\infty$, the model can be solved exactly and it corresponds to the one loop effective action for triangulated Polyakov surfaces, as computed in [38].

In this limit the dominant configurations tend to be the most regular triangulations which can cover the surface [21, 87, 19], in the sense that the (intrinsic) curvature is nearly constant on the surfaces.

The asymptotic dependence of γ on D , based on the one loop correction around the saddle point configurations of constant curvature, is [38]

$$\gamma(D) = \frac{(D-7)}{6} \quad (D \rightarrow -\infty) \quad (42)$$

and does not change when higher loop corrections are taken into account [88] i.e., the last expression is perturbatively exact in the limit $D \rightarrow -\infty$.

In the limit $D \rightarrow +\infty$ the model is also exactly solvable and its results coincide with the predictions of mean field theory [15], as already pointed out in (23).

The interest in the values of γ just listed, calculated at physically uninteresting dimensions D , comes from the fact that being exact, they can be compared with an expression for γ in terms of D , recently obtained [89] in the framework of conformal field theory for strings, in the range $D \leq 1$

$$\gamma_{conf}(D) = \frac{D-1-\sqrt{(D-1)(D-25)}}{12} \quad (43)$$

The exact values of γ , calculated on triangulated lattices, fit the exact formula above, valid in the continuum, and this should be an indication that the model of dynamically triangulated random surfaces (DTRS) considered is able to represent (some) properties of the continuum limit at least when $D \leq 1$.

The perturbatively exact result (12) is actually the limit of (43) when $D \rightarrow -\infty$. For finite D , it underestimates γ .

Nevertheless, $\gamma_{conf}(D)$ remains negative for $D < 1$ and this makes obscure any possible connection between the lattice configurations at critical points and the continuum surfaces. When analytically continued beyond $D = 1$, $\gamma_{conf}(D)$ goes through complex values and only for $D \geq 25$ will $\gamma_{conf}(D)$ recover real (and positive) values.

The critical exponent γ was however assumed to be positive in the case of random surfaces in hypercubic space time lattices, with the Nambu-Goto action. Such assumption is natural since the susceptibility should diverge near β_c , if there is any continuum limit. However, the results presented were obtained only for physical dimensions $D > 1$ and cannot be compared with the numerical values of γ just reported. Moreover, the values of γ for the hypercubic lattices were all consistent with $\gamma = \frac{1}{2}$, fact which has lead to the conjecture [87] that, if $\gamma > 0$ then $\gamma = \frac{1}{2}$ in those models. Now, if universality holds, their (correct) continuum limit should equally well be obtainable from DTRS's (with identical values of D). Hence they should have the same γ 's when $D > 1$.

This, together with the fact that γ complex is not physically acceptable, can be used to form a phase diagram [19] in which the dimension $D = 1$ separates two phases.

In the first phase, $D < 1$ and $\gamma(D)$ is correctly described by (43). The limit (12) falls inside this phase and results from small deformations around constant curvature surfaces.

On the other hand, (nearly) constant curvature surfaces (with given topology) are represented by a small class of triangulations (most of them) isometrically related and thus not distinct from each other. In other words, the limit $D \rightarrow \infty$ severely restricts the type of metric fluctuations in the path integral¹². This is thus a weak coupling phase for gravity.

¹²In the same sense that, using a regular triangulation of the torus means complete

In the second phase, $D > 1$ and γ settles at its mean field value, $\gamma = \frac{1}{2}$, for all values of D . This is a strong coupling phase for gravity. If the hypothesis holds, then one is not allowed to use (43) in the strong coupling phase even at $D \leq 25$, where γ becomes positive.

To resume the situation, exact lattice calculations made on the model just described lead to conclusions which completely agree with the corresponding results obtained directly from the continuum theory, suggesting that the triangulated random surfaces described are acceptable regularizations of the continuum model when $D \leq 1$. However, the negative values of γ make this connection obscure and intriguing.

For $D > 1$, numerical simulations are mandatory and the situation is even much less clear, as we will see now.

6 The Grand-Canonical Ensemble

So far we have considered only numerical simulations in which the number of sites, N_0 , was kept fixed (microcanonical case). This means that we are not yet summing over all possible (and physically admissible) triangulations of the parameter space. That will require at least an extra sum over the number of sites N_0 [5, 20, 21] [51, 80, 81].

In that case the full partition function, Z , will be a linear combination of the partition functions $Z(N_0)$, computed with N_0 fixed, as given by (38) but without the cosmological term ($\beta = 0$):

$$Z = \sum_{N_0} z^{N_0} Z(N_0) \quad (44)$$

where z represents a fugacity of the number of sites N_0 and $Z(N_0)$ can be expressed as [21]¹³

$$Z(N_0) \sim C t e^{-N_0^{2-3} z^{-N_0}} \quad (45)$$

when $N_0 \rightarrow \infty$. γ is the critical exponent for the susceptibility $\chi(z)$ and, on the lattice, is generally an *a priori* unknown function of D i.e., the number of fields defined at each lattice site:

¹³In the same sense that, using a regular triangulation of the torus means complete absence of metric fluctuations i.e., total decoupling of gravity, as already stressed. See also eqs. (12) and (13).

$$\begin{aligned}
\chi &\sim Cte (\mu - \mu_c)^{-\gamma} \\
\gamma &= \gamma(D) \\
\mu &\equiv -\ln z = 2\beta
\end{aligned} \tag{16}$$

Moreover one can prove [87] that

$$\chi(z) \sim < N_0 >, \tag{17}$$

where $< N_0 >$ denotes the average number of sites computed with fixed fugacity z .

From the point of view of numerical simulations, the advantage of generating grand canonical ensembles has been known for some time [58]. In fact, as we already pointed out, this is the most natural thing to do if one looks for properties near the continuum. The number of geometrical components of each lattice configuration is allowed to vary.

The first attempts to generate grand canonical ensembles [54,58] were limited by the fact that the surfaces were embedded in hypercubic lattices. More recently, the same idea was implemented on intrinsically triangulated surfaces [79,82] and ensembles described by (41) were obtained.

The new ingredients in the MC algorithms used for varying N_0 are the SPLIT and JOIN operations [79] shown in figs. 7.b and 7.c.

This method allowed one to go beyond the exact analytical calculations (which, so far, have been possible only at $D = -\infty, -2, 0, \frac{1}{2}, 1$ and $+\infty$, as we referred in the last paragraph) and to determine the critical exponent γ [11].

The exponential γ could be calculated from the large volume limit (15) of microcanonical ensembles. However, it is well known that the overall normalization constant Cte in (15) cannot be determined. This means that the coefficients of Z^N in that expansion cannot be obtained from independent microcanonical ensembles.

To solve this problem, three methods have been devised:

In the first method [90] one finds identities relating the ratios $r_N \equiv \frac{Z(N-1)}{Z(N)}$ to some averages with respect to a distribution of Feynman graphs

¹¹The Hausdorff dimension and the mass gap can be extracted using the microcanonical ensemble. They are related by (20).

with fixed number (N) of loop propagators, which are dual to triangulations with fixed number (N) of vertices. This method consists basically of a MC calculation of the perturbative expansion of any physical quantity, order by order.

Within the numerical uncertainties, the values of γ , computed for $D = 1, 2, 3$ and 4 were all compatible with $\gamma = 0$ (which, at $D = 1$ confirms the exact result (13)).

Strong coupling expansion methods can also be used to compute γ [82,83]. In this case, the determinants for closed surfaces of increasing sizes are estimated. In principle they bring the advantage of automatically providing the weight factor $\rho(T)$. This method would give the critical point β_c as well as γ itself if one were able to handle arbitrarily large surfaces. In practice, however, the limitation on the lattice sizes which can be generated in a computer's memory imposes a limit on the precision of γ obtained in this way. We will not review this method.

As the third method, we have the grand canonical ensemble to get γ for arbitrary values of D [19,91]. In particular, the dependence of γ on D (for $D \geq 0$) was studied numerically in [19] over a wide range of values of D . The authors concluded that the form of $\gamma(D)$ depended on the particular measure $\rho(T)$ chosen. Although this effect was shown not to be an artifact of the numerical approximation, it may indicate either a lack of universality or that the lattices used were still too small to show the relevant asymptotic behaviour of the physical quantities studied.

The conclusion was that, in terms of D and the measure $\rho(T)$ (parameterized by a quantity α as $\rho_\alpha(T) = \prod_{i \in T} q_i^\alpha$), one can distinguish two phases, with $\gamma < 0$ and $\gamma > 0$, separated by a boundary curve with $\gamma = 0$. Inside the phase $\gamma > 0$, γ settles near to $\frac{1}{2}$ and the configurations are dominated by polymer like surfaces and become trivial in the continuum, as already indicated before. It was argued that only on the curve $\gamma = 0$ a non trivial continuum limit might be possible and, if such was the case, it could be found only in the interval

$$D'_0 \leq D \leq D_0 \tag{18}$$

Hence D'_0 and D_0 act respectively as lower and upper critical dimensions for the model. The estimates [87,19], although lacking precision, indicated

that $D'_0 \approx 2 - 4$ and $D_0 \approx 20 - 30$.

Thus one may say that an indirect but important consequence of summing over a more complete set of (discretized) metrics – when going from micro to grand canonical ensembles – is the conclusion that continuous euclidean space-time may be approximated by fields living on discrete triangulated random surfaces (in the way described above) only when its dimensionality falls in the interval (18). However, the errors affecting the available MC data still remain too large to allow for more precise conclusions concerning the universality with respect to $\rho(T)$, or the values of γ .

From the point of view of the continuous symmetries, the Weyl rescaling symmetry (7) should emerge from the summation over N_0 , since the metric can be related to the density of vertices [92] of the Feynman graphs dual to the discretized random surface.

7 Varying Topologies

The attempts to summing over more and more complete sets of configurations in the path integral leads naturally to the next (and last ?) step, namely, the sum over topologies. However, as we said before, the partition functions which include this sum are not well defined.

By analogy with the grand canonical ensemble, we can formally introduce a “chemical potential” ω for the topology, and write the partition function as

$$Z = \sum_h e^{-\omega h} Z(h) \quad (49)$$

where h is the surface genus and $z(h)$ represents the sum over all surfaces with given genus h .

In pure gravity ($D = 0$) the term with the chemical potential ω is nothing but the Einstein action (in two dimensions), and the coefficients $Z(h)$ represent a measure arising possibly from intrinsic metric degrees of freedom.

It was in the area of euclidean quantum gravity and cosmology that the need for a sum over topologies (this line of 4 dimensional random manifolds) was first recognized.

Such (formal) sum appears in the definition of the wave function of the Universe. This is defined as a configuration in the space of **three-geometries** ($g(3)$) and its wave function reads [98,93,91]

$$\Psi[g(3)] = \sum_{g(4)} e^{-S_E[g(4)]} \quad (50)$$

where S_E is the euclidean action and the sum extends over all **four-geometries** (or metrics) $g(4)$, having a single boundary on which the induced metric is $g(3)$.

As with the case of random surfaces, S_E can be discretized using Regge calculus’ type of lattices and (50) can be evaluated [95,96].

In (50) a summation over topologies is implicit.

Concerning the analogy with the grand canonical ensemble, there is a significant distinction between varying the number of sites of a lattice and varying its topology: The first process can be achieved through local changes whereas the second one is generally non local. This fact suggests that going from one topology to another might resemble the process of tunneling between classically allowed configurations through interpolating, but classically forbidden ones.

If this is the correct picture, those interpolating, genuinely quantum configurations may well fail to represent triangulations of regular, smooth manifolds. Within such “foamy” (or “unruly”) [97] configurations, some lattice sites will have more than one set of nearest neighbours in such a way that no site in one set will be nearest neighbour to any other site in a different set¹⁵. If the naive continuum limit is taken, such sites (with two or more sets of nearest neighbours) will not have well defined tangential spaces, which is a necessary condition for the validity of the equivalence principle of general relativity. In other words, the quantum configurations will often violate the equivalence principle.

If the action in (50) is the Einstein action

$$S_E = - \int d^2 \xi \sqrt{g(\xi)} R(\xi) \quad (51)$$

¹⁵ Additionally, one should demand that the structure remains strongly connected i.e., that any two plaquettes either share a common link (contiguous plaquettes) or else they can be joined by a row of contiguous plaquettes.

it is easy to prove that smooth configurations¹⁶ (in particular the sphere) will be singled out as those which minimize S_E , provided that the configurations considered in the sum (50) are restricted to a certain class [97] inside which χ_{Enter} (given by (37)) remains a metric invariant (i.e., local lattice deformations do not affect χ_{Enter}).

This class includes (but is larger than) the triangulations of smooth manifolds. Its configurations are either smooth manifolds or pseudo manifolds that is, the unruly structures described above, which may be obtained from smooth manifolds by making self intersections on it, at separated pairs of sites.

This generalization raises an interesting question related to the parametrization (choice of local coordinates) on such pseudo manifolds: As we said, at the self intersection sites no coordinate frames can be chosen consistent with the two dimensionality of the structure. However, the pseudo manifolds may be interpreted as perfectly smooth (abstract) manifolds on which general coordinate patches are superimposed in such a way that at least one pair of points (sites on the lattice) are identified. From this point of view, the inclusion of pseudo manifolds in the sum over triangulations is simply a consequence of generalized reparametrization freedom (GRF) [98], where distinct sites on an abstract smooth manifold (representing a sort of covering space) may acquire identical coordinates. Of course, the probability for such coincidences will vanish on continuous manifolds, if the coordinates (and therefore the metric) are randomly chosen. However, on a (finite) lattice that probability will be finite and self intersections will effectively be generated.

From the point of view of lattice algorithms, the identification of pairs of sites which are not nearest neighbours is a generalization of the JOIN operation in grand canonical ensemble simulations, and it does not keep χ_{Enter} invariant. Obviously, the extension of the path integral to this larger class of pseudo manifolds raises the question of whether even larger classes of triangulations should be considered as a next step. Allowing for GRF is the simplest method for generating a much larger class of parametrized lattices.

The difficulty with the proposal above is that, although still well defined

¹⁶From now on we will use the term "smooth" as the opposite of "unruly" or "foamy".

for each configuration, the quantity χ_{Enter} is no longer able to select the smooth triangulated manifolds as extrema of the Einstein action. In fact, χ_{Enter} can become arbitrarily positive if the configuration is sufficiently foamy. This means that either one has to find a more sophisticated physical mechanism which prefers smooth manifolds or else somehow argue that what looks like being smooth manifolds is nothing but the limit of very large but foamy configurations. In this case, the structure would appear rather irregular at short distances.

Although being also speculative, related ideas were proposed a long time ago, suggesting that, at distances of the order of the Planck length, the space time metric should undergo enormous fluctuations away from the flat metric, some of them being enough to modify its topology [99]. Even though the topological structures formed in this way ("wormholes") should typically be very small, the space time always retained its smooth manifold character and no interpolating configurations were considered. However, as we discussed before, local changes in the metric structure are generally not capable of producing such topological variations. More recently, related [94,100] and more radical ideas [101,102,103] have been put forward, some of which possibly having cosmological implications [104,105,106,107] [109,110,111,112].

On the positive side, we should mention that the GRF scheme presented above provides a natural mechanism for generating interpolating configurations between topologically distinct smooth manifolds [113]. The set of such smooth manifolds (with any topology) is however very small when compared with the total number of accessible configurations. This reflects the fact that, starting from any smooth triangulation, most of the local lattice changes (JOIN and generalized JOIN operations) will lead to foamy structures, and to get back to a smooth triangulation (with the same or different topology), a non local (or, at least, quasi local) change is necessary, and this is very unlikely to occur.

Now let us come back to 2 dimensional triangulated random surfaces and shortly refer to the results so far obtained on the dependence of their critical behaviour on the topology.

To begin with, the susceptibility exponent γ was shown to depend linearly on χ_{Enter} [11]

$$\gamma = 2 + \frac{D-19}{12} \chi_{Euler} \quad (52)$$

As in the case of spherical topology ($\chi_{Euler} = 2$), this result becomes exact when $D \rightarrow -\infty$ (weak coupling limit of gravity). Non perturbative effects will become important for larger values of D and may completely alter the critical behaviour of higher genus random surfaces as compared with (52). Such effects are however very difficult to estimate, except for the special dimensions $D = 0$ and $D = -2$, where γ was computed exactly:

$$\gamma = 2 + \frac{5}{4} \chi_{Euler} \quad (D = 0) \quad (53)$$

$$\gamma = 2 + \frac{3}{2} \chi_{Euler} \quad (D = -2) \quad (54)$$

Following the conformal techniques of ref. [89], the result (43) was generalized [114] to take into account the dependence on χ_{Euler}

$$\gamma_{conf}(D; \chi_{Euler}) = 2 + \frac{\chi_{Euler}}{24} ((D-25) - \sqrt{(D-1)(D-25)}) \quad (55)$$

which coincides with (13) for $\chi_{Euler} = 2$ and reduces to the exact random lattice results (53) and (54) when $D = 0, -2$. In the limit $D \rightarrow -\infty$ it recovers the perturbative result (52).

Finally, we should say that, very recently, renewed attention was paid to the problem of summing over topologies in 2 dimensional random surfaces [115, 116, 117, 118]. Although we will not review here the exciting possibilities opened by these developments, we will just say that they were obtained using the well known correspondence [81] between discretized random surfaces and the expansion of the free energy of single $M \times M$ matrix field theory in terms of ϕ^4 Feynman diagrams with well defined topologies. The question of the divergence of the partition function for the surfaces, when the topologies are summed, is avoided by renormalizing the new "string tension" $e^{-\frac{1}{2}}$ (see (19)) first, and then taking the large M limit (matrix size) adequately.

In this way, a differential equation was derived for the free energy of the matrix model (which counts the number of ϕ^4 graphs with given topology),

whose solutions reveal in general the presence of essential singularities in the effective coupling. These arise typically from non perturbative effects.

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Figure Captions

Fig.1

World sheet spanned by a closed string between its creation from the vacuum and its annihilation;

- a) For a free string the world sheet has no holes;
- b) When the string self interactions are switched on, the string may break into smaller closed strings. In a vacuum to vacuum amplitude diagram, these smaller strings rejoin and make higher topologies.

The cross sections drawn represent possible “instantaneous” configurations of the world sheet.

Fig.2

Processes of the type described in fig. 1., but taking place on a cubic euclidean space time lattice.

Fig.3

A random surface $S(\gamma_1, \dots, \gamma_n)$ may fluctuate between its n boundaries $\gamma_1, \dots, \gamma_n$, which are kept fixed. $\gamma_1, \dots, \gamma_n$ are loops surrounding holes cut out on the surface or frozen parts of it, which are assumed not to contribute to the dynamics.

Fig.4

A surface S with fixed area A , contains the point P as a site of its lattice (not drawn). P_0 is a plaquette on S having P as a vertex. S is intersected by a sphere (dotted) with radius R , centered on P , which encloses an area S_R of S (grey) inside it. The number of lattice surfaces satisfying the conditions above is denoted by $n_0(A)$.

Fig.5

- a) The lattice step of an (hyper)cubic space time lattice determines the smallest possible changes between distinct configurations consisting of creation or annihilation of a fraction of the (hyper)surface of an elementary (hyper)cube.
- b) An intrinsically triangulated surface can suffer infinitesimal deformations when embedded e.g. in space time, keeping however its intrinsic structure of regular honeycomb $2 - D$ lattice.

Fig.6

Simple “spikes” and more complicated structures may, under adequate conditions, proliferate on surface lattices, tending to reduce the area of those surfaces (subtended by fixed boundary loops) to their minimal values compatible with the boundaries and making them occupy finite volumes that is, degenerating the surfaces into fractal structures.

Fig.7

The three basic operations used for modifying the lattice configurations. **SPLIT** and **JOIN** are specific to simulations in the grand canonical ensemble, since they change the number of sites on the lattice.

Fig.8

The repeated use of the **FLIP** operation generally produces defects on a lattice, corresponding to sites with various numbers of nearest neighbours. They are interpreted as points of the random surface where the original (scalar) curvature suffered a variation.

Fig.9

Changes in the lattice configuration generally alter the distance (= minimal number of lattice steps) between two fixed sites. The minimal path between two different sites is in general not uniquely defined, even with trivial lattice topologies, except when the sites are nearest neighbours.

Fig.10

Configurations which are not considered:

- a) Tadpole at a site (1);
- b) Two different links joining A and B .

Fig.11

The Wigner-Seitz cell (dotted) dual to the site i , with area σ_i .

Fig. 3

Fig. 4

Fig. 1

Fig. 2

Fig. 5. a)

Fig. 5. b)

Fig.6

Fig.7

Fig.9

Fig.8

Fig.10

Fig.11