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MASS FORMULAE IN TODA FIELD THEORIES

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Abstract

The characteristic equations which give the mass spectrum of the affine Toda field Theories are analyzed. In order to determine the decoupling point of the finite renormalization due to the tadpoles we introduce the so-called *mass formulae*. We derive the recursive equations the *mass formulae* satisfy and we study their properties for all Lie algebras. For the ADE series the *mass formulae* turn out to be sequences of integers.

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1 Introduction

Since the seminal works of Zamolodchikov [1] the off-critical statistical systems have been object of intense analysis [1-22]. A particular role is played by a class of (classical) integrable field theories, namely the Affine Toda Field Theories (ATFT). Eguchi and Yang [2] and, independently, Hollwood and Mansfield [3] pointed out the relevance of these integrable field theories in the description of the off-critical systems obtained as perturbations of Conformal Field Theories (CFT). Many other contributions are due to other groups [5,6,7,8,11,12,14,16,17,18]. In particular, one of us computed in a series of works the factorizable S -matrices of those integrable theories relevant in the analysis of the statistical models away from criticality [1,5,6,7,8].

In this paper we concern ourselves with the mathematical analysis of the mass sp of the ATFT and some *mass formulae* which arise considering the perturbative analysis of these theories. In next section we briefly recall the basic features of these theories, referring to [7,8] for a detailed discussion of them and a more complete list of references on the subject.

The ATFT present two different behaviours depending if they make use of the simply laced algebras or of the non-simply laced ones. Both are integrable theories at the classical level [23,21,25,26]. The quantum version of these theories, on the contrary, seems to select the simply laced ATFT as the only consistent theories [7,8,11]. For instance, the simply-laced ATFT have an exact S -matrix which can be consistently checked against a perturbative analysis [5,7,8,11,12,11]. Remarkably, the ATFT constructed on the exceptional algebras E_n have the property that their minimal S -matrices describes the massive regime of particular deformations of the first minimal models in CFT: the Ising model in a magnetic field (at $T = T_c$) is related to E_6 , as shown initially in [1] and successively analyzed also in [8,11,14,17], the Ising model with annealed vacancies in his high temperature phase is related to E_7 [5,17], and the tricritical 3-state Potts models in his high-temperature disorder phase has a description in terms of E_8 [17,18]. This picture has been confirmed both by numerical simulations [19] and by the solution of the thermodynamic Bethe-ansatz equations which permit the analysis of the ultraviolet limit of these massive field theories [20,21,22]. As was referred to above, the perturbative analysis of the simply-laced ATFT shows some nice features of these theories. The classical mass ratios are preserved by the quantum fluctuations and they indeed correspond to the exact mass spectrum of the theories. Moreover, the highest order poles in the exact S -matrix find a natural explanation in terms of multi-scattering processes [5,7,11], along the way shown by Coleman and Thun for the Sine-Gordon model [27].

The quantum fluctuations of the non-simply laced ATFT, on the contrary, seem to spoil the

classical properties of the theories [7,11]. In these cases the classical mass spectrum is no longer robust under the loop corrections and moreover the conjectured exact S -matrices present mysterious higher order poles which do not find explanation in the perturbation theory. The only exception seems to be the $A_{2n}^{(2)}$ series. Partial understanding of these results may be found in the *folding* construction of the non-simply laced ATFT [7,11,21].

In this paper we analyze the characteristic equations which give the mass spectrum of the ATFT. For the classical algebras they can be expressed in terms of the Chebyshev polynomials. We obtain the mass ratios for all ATFT. Their values are collected in Table 2 and 3.

The perturbative evaluation of the self-energy diagrams of ATFT leads us to consider some special expressions constructed in terms of the exact mass eigenvalues (see formula (2.11)). We call them the *mass formulae*. For the simply-laced ATFT they turn out to be always integer sequences while for the non-simply laced ATFT only the twisted theories of the exceptional algebras G_2 and F_4 give rise to integer values but not the other non simply laced algebras. Besides the mathematical interest in analyzing these sequences, at least in the case of the A_n series, there is an interesting by-product and, at the same time, an understanding of the integerness nature of these numbers. In fact, for this series the *mass formulae* coincide with a formula, derived by Verlinde [28], which expresses the dimensions of the vector-bundle V_g of the minimal CFT series on a Riemann surface of genus g . *A priori* these sequences should have integer values but the mathematical proof has not been obtained so far. It is not yet clear if the sequences for the other ATFT might be relevant in physics or in number theory. In any case they encode some information on the Lie algebras.

The paper is organized as follows: in Section 2 we discuss the elementary definitions of the ATFT. We introduce the *mass formulae* (MF hereafter) in relation with the tadpole diagrams of the perturbation theory. In Section 3 we study the characteristic equations which give the classical mass spectrum and we express the MF in terms of the coefficients of the characteristic equations. We derive the recursive equation the MF satisfy. Section 4 is devoted to the systematic analysis of ATFT. We make use of the properties of the Chebyshev polynomials (see Appendix A) for studying the classical algebras. The ADE , the non simply-laced series and the exceptional non-simply laced G_2 and F_4 algebras are analyzed in detail. Our conclusions are in Section 5.

2 Mass formulae for the ATFT

The ATFT constructed on a Lie algebra $\mathcal{G}$ describe the interaction of n bosonic fields ϕ_i ($i = 1, 2, \dots, n$) where n is the rank of $\mathcal{G}$. The Lagrangian is given by

$$\mathcal{L} = \frac{1}{2} \sum_{i=1}^n (\partial_\mu \phi_i)^2 - \frac{m^2}{\beta^2} \sum_{i=0}^n q_i [\exp(\beta \alpha_i^T \phi^j) - 1] \quad (2.1)$$

m^2 sets the mass scale. α_i ($i = 1, 2, \dots, n$) are the simple roots¹ of $\mathcal{G}$ and α_0 is a linear combination of the simple roots with integer coefficient q_i :

$$\alpha_0 = - \sum_{i=1}^n q_i \alpha_i \quad (2.2)$$

In (2.1) we treat α_0 on the same footing as the other α_i , taking $q_0 = 1$. The set of integers $\{q_i\}$ is specific to each algebra (see the Tables 1). In the case of simply-laced ATFT α_0 is the highest root of the algebra. For the non-simply laced ATFT there are, on the contrary, two possibilities: α_0 can be either the highest long root or the highest short root. To distinguish them we keep the usual notation for the first case while we called the second one the *twisted* ATFT, denoting them by a tilde, $\tilde{\mathcal{G}}$. In all cases the system of the roots α_i ($i = 0, \dots, n$) forms the Dynkin diagram of an affine (twisted) Lie algebra [29,30]. The basic properties of the integer numbers $\{q_i\}$ are

$$\sum_{i=0}^n q_i \alpha_i = 0, \quad \sum_{i=0}^n q_i = h \quad (2.3)$$

For the untwisted models, h is equal to ψ , the Coxeter number of $\mathcal{G}$. In the twisted situation it will coincide with the dual Coxeter number $\tilde{\psi}$, either of $\mathcal{G}$ itself or of another non simply-laced Lie algebra [7].

The classical mass ratios (for the simply-laced they come out to be also the exact ones [7,11,12]) are given by the eigenvalues of the mass-matrix

$$\mathcal{M}_{ab} = m^2 \sum_{i=0}^n q_i \alpha_i^a \alpha_i^b \quad (2.4)$$

The characteristic equations for the different theories will be presented in Section 3. For the moment let assume that we are in the basis of the eigenvectors where the mass-matrix is diagonal $\mathcal{M}_{ij} = m_i^2 \delta_{ij}$. The mass spectrum for the ATFT can be found in the Tables 2 and 3. Notice that there is a mass degeneracy if the group of automorphisms of the non affine Dynkin diagram of the

¹For simply-laced $\mathcal{G}$ we use the normalization $|\alpha_i|^2 = 2$ whereas for the non simply laced ones, we normalized the longest roots to be $|\alpha_i|^2 = 2$

Lie algebra is non trivial. An interesting property of the mass spectrum is that the masses of the ATFT series and of the untwisted non-simply laced ATFT can be organized in a vector

$$\mathbf{m} = (m_1, m_2, \dots, m_n)$$

that is, an eigenvector of the incidence matrix of the algebra $\mathcal{G}$ [11,10]. Indeed, $\mathbf{m}$ is the Perron-Frobenius eigenvector of $\mathcal{G}$ and its components can be associated with the circles of the Dynkin diagram² [31].

The expansion in β of (2.1) gives the whole set of the interaction terms. The Feynman rules are easily obtained. The perturbative quantization of the ATFT faces the problem of the ultraviolet divergences. These are simply the tadpole diagrams. The other graphs with no self-contractions are finite. For instance, the graph in Figure 1 (which enters the expression of the self-energy of the field ϕ_i) gives for the mass shift of the simply-laced ATFT a nice expression [7,8,11]

$$\frac{\Delta m_i^2}{m_i^2} = \frac{\beta^2}{m_i^2} \cot\left(\frac{\pi}{h}\right) \quad (2.5)$$

A geometrical interpretation of this formula is that the mass shift is a universal quantity, equal to the area of a regular planar polyhedron with h equal sides (h is the Coxeter number of the algebra) and perimeter β .

About the tadpole diagrams, they can be consistently subtracted just renormalizing the overall mass scale [7]. In theories with different masses finite effects are also present. However they act uniformly on all the fields and do not spoil the renormalization property. To the lowest order in β the contributions to the self-energy of the field ϕ_i are those shown in Fig.2. Their sum turns out to be proportional to the square of the mass m_i^2 with the same constant of proportionality for all fields ϕ_i :

$$\frac{\Delta m_i^2}{m_i^2} = \beta^2 \sum_{k=1}^n N_{ik} \log \frac{m_k^2}{\mu^2} = \text{const} \quad (2.6)$$

In this expression μ is the scale of renormalization. Eq. (2.6) is a linear equation for $\log \mu^2$, therefore exists a value of μ that we can choose in order to discard the finite term of the tadpoles as well. To the order β^2 , the decoupling point of the tadpole effects is given by

$$\log \mu^2 = \frac{\sum_{k=1}^n N_{ik} \log m_k^2}{\sum_{k=1}^n N_{ik}} \quad (2.7)$$

²This property has been used in [8] to compute the decay processes in the Conformal Field Theories of the E_n series perturbed by two relevant fields.

Expressing N_k in terms of the coupling constants of the AFTT, (2.7) becomes

$$\log p^2 = \frac{\sum_{a=1}^n m_a^2 \log m_a^2}{\sum_{a=1}^n m_a^2} \quad (2.8)$$

The numerator can be written³

$$\sum_{a=1}^n m_a^2 \log m_a^2 = \frac{d}{dx} \sum_{a=1}^n (m_a^2)^x \Big|_{x=1} \quad (2.9)$$

Then to compute the decoupling point one may consider the function $F(x)$ given by

$$F(x) = \frac{\sum_{a=1}^n (m_a^2)^x}{\sum_{a=1}^n m_a^2} \quad (2.10)$$

In the following we study a slight modification of this expression. Namely, we consider the numerical values obtained by

$$\mathcal{S}_{g-1} = \frac{\sum_{i=1}^n (m_i^2)^{(1-g)}}{(\sum_{i=1}^n m_i^2)^{(1-g)}} \quad g = 0, 1, 2, \dots \quad (2.11)$$

Making use of the mass matrix (2.4), we can express (2.11) in the following way

$$\mathcal{S}_{g-1} = \frac{\text{Tr}(\mathcal{M})^{1-g}}{(\text{Tr} \mathcal{M})^{1-g}} \quad g = 0, 1, 2, \dots \quad (2.12)$$

$\mathcal{S}_g$ will be referred to as the *mass formulae*. In the next section we will prove that $\mathcal{S}_g$ assume only integer values for the simply-laced AFTT but not for the non-simply laced ones.

3 Recursive equations for $\mathcal{S}_k$

In this section we derive the recursive equations which $\mathcal{S}_g$ satisfy. We need to discuss preliminary the characteristic equations associated to the mass matrix (2.4). A more detailed discussion will be given later on.

The classical mass spectrum of any AFTT is given by the roots of the characteristic equation

$$\|\mathcal{M} - x \cdot 1\| = 0 \quad (3.1)$$

The r.h.s. of (3.1) is a polynomial of order n whose general form is

$$P(x) = x^n - p_1 x^{n-1} - p_2 x^{n-2} - \dots - p_n \quad (3.2)$$

p_1 is simply the trace of $\mathcal{M}$ and for the ADE cases is twice the Coxeter number. The other coefficients p_k can be expressed in terms of the trace of higher powers of $\mathcal{M}$ as well. Their expression is

$$k p_k = a_k - p_1 a_{k-1} - \dots - p_{k-1} a_1 \quad (3.3)$$

³One of us (G.M.) thanks J.L. Cardy for this simple but useful remark.

where

$$\begin{aligned} a_1 &= \text{Tr} \mathcal{M} = \sum_i m_i^2 \\ a_2 &= \text{Tr} \mathcal{M}^2 = \sum_i m_i^4 \\ &\vdots \\ a_n &= \text{Tr} \mathcal{M}^n = \sum_i m_i^{2n} \end{aligned} \quad (3.4)$$

The validity of (3.3) can be shown by induction.

To evaluate $\mathcal{S}_k$ we need the expression of the traces of the inverse powers of $\mathcal{M}$. To achieve this goal we use the operatorial equation associated to (3.1) and (3.2), i.e.

$$\mathcal{M}^n - p_1 \mathcal{M}^{n-1} - p_2 \mathcal{M}^{n-2} - \dots - p_{n-1} \mathcal{M} - p_n 1 = 0 \quad (3.5)$$

Denoting $\mathcal{M}^{-1} = C$ and multiplying (3.5) by C^n we obtain

$$C^n - c_1 C^{n-1} - c_2 C^{n-2} - \dots - c_{n-1} C - c_n 1 = 0 \quad (3.6)$$

where c_i can be computed by the p_i 's

$$c_1 = -\frac{p_{n-1}}{p_n}, \quad c_2 = -\frac{p_{n-2}}{p_n}, \quad \dots, \quad c_n = -\frac{1}{p_n} \quad (3.7)$$

Equivalently, c_i are the coefficients of the characteristic equation for the inverse matrix C and can be determined by using (3.3) and (3.1) (this time for the inverse matrix). They are expressed in terms of the trace of higher powers of the inverse matrix, i.e. $t_k = \text{Tr} C^k$.

Since $\text{Tr} \mathcal{M} = p_1$, we may express $\mathcal{S}_k$ (for $k \leq n$) in the following way

$$\begin{aligned} \mathcal{S}_k &= [\text{Tr} \mathcal{M}]^k [\text{Tr} C^k] = p_1^k t_k = \\ &= (p_1 c_1) p_1^{k-1} t_{k-1} + (p_1^2 c_2) p_1^{k-2} t_{k-2} + \dots + (p_1^{k-1} c_{k-1}) p_1 t_1 + k p_1^k c_k \end{aligned} \quad (3.8)$$

A similar formula can be derived for $k > n$. Defining

$$r_1 = p_1 c_1, \quad r_2 = p_1^2 c_2, \quad \dots, \quad r_k = p_1^k c_k \quad (3.9)$$

we arrive at the following recursive equations for $\mathcal{S}_k$

$$\begin{aligned} \mathcal{S}_k &= r_1 \mathcal{S}_{k-1} + r_2 \mathcal{S}_{k-2} + \dots + r_{k-1} \mathcal{S}_1 + k r_k \quad k \leq n \\ \mathcal{S}_k &= r_1 \mathcal{S}_{k-1} + r_2 \mathcal{S}_{k-2} + \dots + r_n \mathcal{S}_{k-n} \quad k > n \end{aligned} \quad (3.10)$$

Notice that $\mathcal{S}_k$ depend only on the values of r_i since $\mathcal{S}_1 = r_1$, $\mathcal{S}_2 = r_1^2 + 2r_2$ etc. In the next section we analyze the characteristic equations of AFTT and show that for the ADE series the coefficients r_i take integer values (then all the $\mathcal{S}_k$ are integer numbers) whereas for the non-simply laced AFTT this is not generally the case.

4 Characteristic equations of ATFT

Firstly we show that the traces on the higher power of $\mathcal{M}$ may be equivalently computed in terms of another matrix directly related to the Dynkin diagrams. Let us define the matrix $\mathcal{N}$ with elements

$$\mathcal{N}_{ij} = (g_i \alpha_i, \alpha_j) = \sum_{k=1}^n q_k \alpha_i^k \alpha_j^k \quad (1.1)$$

It is easy to prove that

$$\text{Tr } \mathcal{M}^s = \text{Tr } \mathcal{N}^s, \quad s = 1, 2, \dots, n \quad (1.2)$$

Therefore the characteristic equation of $\mathcal{M}$ should coincide with that of $\mathcal{N}$. However $\mathcal{N}$ is a $(n+1) \times (n+1)$ matrix but n_0 is a linear combination of the other simple roots α_i . Consequently $\mathcal{N}$ is a singular matrix. One of its eigenvalues is zero while the remaining ones coincide with the eigenvalues of $\mathcal{M}$. In the next sections we systematically analyze the ATFT for all the Lie algebras. The mass spectrum is given in Table.2 and Table.3. Let us start with the ADE series.

4.1 $A_n^{(1)}$ series

For the $A_n^{(1)}$ series the matrix $\mathcal{N}$ reduces to the Cartan matrix of the affine Lie algebra. Then the characteristic equation associated to $\mathcal{N}$ is

$$Q_{n+1}(x) = \begin{vmatrix} 2-x & -1 & 0 & \dots & 0 & 0 & -1 \\ -1 & 2-x & -1 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2-x & -1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -1 & 2-x & -1 \\ -1 & 0 & 0 & \dots & 0 & -1 & 2-x \end{vmatrix} \quad (1.3)$$

Define $2y = 2 - x$. It can be proved (see for instance [32]) that

$$Q_{n+1} = 2(T_{n+1}(y) - 1) \quad (1.4)$$

where $T_{n+1}(y)$ is the Chebyshev polynomial of the first kind

$$T_{n+1}(\cos \theta) = \cos(n+1)\theta \quad (1.5)$$

In Appendix A we give some properties of these polynomials.

The mass spectrum of the $A_n^{(1)}$ series is given by the n non-zero roots of the equation $T_{n+1}(y) = 1$, i.e.

$$m_k^2 = 4 \sin^2 \frac{k\pi}{n+1} \quad k = 1, 2, \dots, n \quad (1.6)$$

In Appendix A we prove that the coefficients p_i entering the characteristic equation for the $A_n^{(1)}$ series have the following form

$$p_i = (-1)^{i+1} \frac{2(n+1)}{2(n+1)-i} \binom{2(n+1)-i}{i}, \quad i = 1, 2, \dots, n \quad (1.7)$$

and correspondingly we find the following expression for the coefficients r_i entering the recursive equations for S_k

$$r_i = -\frac{p_i}{p_n} \frac{p_{n-i}}{p_n} = (-1)^n 2^i (n+1)^{i-2} p_{n-i} \quad (1.8)$$

Then we prove our statement that the r_i 's for the $A_n^{(1)}$ series are integer number. Consequently S_k is an integer sequence. The first examples of these sequences are

$$\begin{aligned} n=1 & \dots & S_k &= 1 \\ n=2 & \dots & S_k &= 2^k \\ n=3 & \dots & S_k &= 2^{k-1}(1+2^k) \\ n=4 & \dots & S_k &= 2[(5+\sqrt{5})^{k-1} + (5-\sqrt{5})^{k-1}] \\ n=5 & \dots & S_k &= 3^{k-1}[1+2^{k-1}] + 2^{2k-1} \end{aligned} \quad (1.9)$$

4.2 $D_n^{(1)}$ series

For this series the expression of $\mathcal{N}$ is the following

$$\mathcal{N} = \begin{vmatrix} 4 & -2 & 0 & \dots & 0 & 0 & -2 & -2 \\ -2 & 4 & 2 & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & -1 & 0 & \dots & 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & \dots & 0 & 0 & 0 & 2 \end{vmatrix} \quad (1.10)$$

The characteristic equation has the form

$$\|\mathcal{N} - x \cdot 1\| = 2^{n+2}(y-1)(2y-1)^2 U_{n-2}(y) = 0 \quad (1.11)$$

where we put $x = t(1-y)$. U_n is the Chebyshev polynomial of the second kind (see Appendix A). The roots of (4.11) are given by

$$\begin{aligned} y_{n+1} &= 1 \rightarrow x_{n+1} = 0 \\ y_n &= \frac{1}{2} \rightarrow x_n = 2 \\ y_{n-1} &= \frac{1}{2} \rightarrow x_{n-1} = 2 \end{aligned} \quad (4.12)$$

and $U_{n-2}(y) = 0$, i.e.

$$y_k = \cos \frac{k\pi}{n-1} \rightarrow x_k = 8 \sin^2 \frac{k\pi}{2(n-1)} \quad k = 1, 2, \dots, n-2 \quad (4.13)$$

The first root in (4.12) is irrelevant for the computation of the mass spectrum. We obtain the mass spectrum given in Table 2.

Using the properties of the Chebyshev polynomials given in Appendix A, we find the following expression for the parameters r_i

$$r_i = 2(-1)^{i+1}[2(n-1)]^{i-1} \times \left[\left(\frac{n-1+i}{n-2-i} \right) + 2 \left(\frac{n-2+i}{n-1-i} \right) + \left(\frac{n-3+i}{n-i} \right) \right] \quad (4.14)$$

They are integer numbers. Therefore the *mass formulae* for the $D_n^{(1)}$ series is a sequence of integers.

4.3 F_n series

The analysis of this exceptional series is carried out considering each of the characteristic equations separately. They can be found in [8].

4.3.1 $F_6^{(1)}$ Toda Field Theory

This theory is related to the high temperature phase of the 3-state Potts model with annealed vacancies [17,18]. The characteristic equation for the mass spectrum is

$$\begin{aligned} \|\mathcal{M} - x \cdot 1\| &= x^6 - 24x^5 + 246x^4 - 936x^3 + 2052x^2 - 2460x + 864 = \\ &= [x^2 - 12x + 24][x^2 - 6x + 6]^2 \end{aligned} \quad (4.15)$$

The existence of two doublets of degenerate particles is evident. The mass spectrum can be found in Table 2. The physical content of this theory was discussed in [17,18]. In [8] the factorization

form of the polynomial equation (4.15) was used to write in a compact form the bootstrap fusions of the model. For the coefficients r_i we have

$$\begin{aligned} r_1 &= 60 \\ r_2 &= -1368 \\ r_3 &= 14976 \\ r_4 &= -82944 \\ r_5 &= 221184 \\ r_6 &= -r_5 \end{aligned} \quad (4.16)$$

Since they are integers, the *mass formulae* for F_6 are integer numbers as well. Their close expression is

$$S_k = 2^{k-1}(2^k + 1)[(3 + \sqrt{3})^{k-1} + (3 - \sqrt{3})^{k-1}] \quad k = 0, 1, \dots \quad (4.17)$$

4.3.2 $F_7^{(1)}$ Toda Field Theory

This theory describes the high temperature phase of the tricritical Ising model. The relativistic field theory and the scattering matrix associated to it are discussed in full detail in [5,8,17]. The characteristic equation for the mass excitations is

$$\begin{aligned} \|\mathcal{M} - x \cdot 1\| &= x^7 - 36x^6 + 504x^5 - 3552x^4 + \\ &+ 13536x^3 - 27648x^2 + 27648x - 10368 = \\ &= [x - 6][x^3 - 18x^2 + 72x - 72] \times \\ &= [x^3 - 12x^2 + 36x - 24] \end{aligned} \quad (4.18)$$

The mass spectrum is given in Table 2. The Z_2 symmetry of the affine F_7 diagram classify the particles into odd and even ones [5]. The factorization form of the polynomial equation (4.18) may be used to write down the bootstrap fusions of the scattering processes [8]. The integer coefficients r_i are easily computed

$$\begin{aligned} r_1 &= 96 \\ r_2 &= -3156 \\ r_3 &= 60912 \\ r_4 &= -575424 \\ r_5 &= 2939328 \\ r_6 &= -7558272 \\ r_7 &= -r_6 \end{aligned} \quad (4.19)$$

4.3.3 $E_n^{(1)}$ Toda Field Theory

This theory describes the Ising model at the critical value of the temperature but in presence of an external magnetic field, as shown initially in [1] and successively analyzed also in [8, 11, 14, 17]. For E_8 we have

$$\begin{aligned} \|\mathcal{M} - xI\| &= x^8 - 60x^7 + 1110x^6 - 18100x^5 + 1237140x^4 + \\ &\quad - 518100x^3 + 1166100x^2 - 1296000x + 518100 - \\ &\quad [x^4 - 30x^3 + 240x^2 - 720x + 720] \times \\ &\quad [x^4 - 30x^3 + 360x^2 - 1080x + 720] \end{aligned} \quad (1.20)$$

The masses can be found in Table 3. The corresponding integer coefficients r_i are

$$\begin{aligned} r_1 &= 150 \\ r_2 &= -8100 \\ r_3 &= 216 \cdot 10^3 \\ r_4 &= -3186 \cdot 10^3 \\ r_5 &= 27 \cdot 10^6 \\ r_6 &= -1296 \cdot 10^5 \\ r_7 &= 324 \cdot 10^6 \\ r_8 &= -r_7 \end{aligned} \quad (1.21)$$

We have then shown that the ADE series give for the S_k integer sequences. In the following we consider the non-simply laced ATFT.

4.4 The non-simply laced ATFT

4.4.1 B_n series

The matrix $\mathcal{N}$ for the $B_n^{(1)}$ series is given by

$$\mathcal{N} = \begin{vmatrix} 1 & -2 & 0 & \dots & \dots & 0 & 0 & -2 & -2 \\ -2 & 1 & 2 & \dots & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -1 & 0 & 0 & \dots & \dots & 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & \dots & \dots & 0 & 0 & 0 & 2 \end{vmatrix} \quad (1.22)$$

Then we have

$$xB(x) = x \det(\mathcal{M} - xI) = \det(\mathcal{N} - xI') \quad (1.23)$$

where I is the $n \times n$ unit matrix and I' is the $(n+1) \times (n+1)$ unit matrix. Define $x = 1(1-y)$. We can write (1.23) in the following form

$$B_n(x) = 2^n (1-2y) U_n(y) \quad (1.24)$$

Therefore the classical mass spectrum is given by

$$m_n^2 = 2, \quad m_k^2 = 8 \sin^2 \frac{k\pi}{2n}, \quad k = 1, 2, \dots, n-1 \quad (1.25)$$

Using the properties of the Chebyshev polynomial of the second kind, we obtained for the coefficients of the recursive equations the following expression

$$r_k = (-1)^{k-1} \frac{(2n-1)^k}{n} \left[\left(\frac{n+k}{2k+1} \right) + \left(\frac{n+k-1}{2k-1} \right) \right] \quad (1.26)$$

They are not integers in general and therefore the mass formulae for this ATFT produce a sequence of non integer values.

The same result holds in the case of the $\tilde{B}_n$ twisted series. The coefficients r_k in this case are given by

$$r_k = (-1)^{k+1} \frac{(2n)^k}{n+1} \left(\frac{n+1+k}{n-k} \right) \quad k = 1, 2, \dots, n \quad (1.27)$$

They are not integers.

4.4.2 the C_n series

For the $C_n^{(1)}$ series the matrix $\mathcal{N}$ has the following form

$$\mathcal{N} = \begin{vmatrix} 2 & -1 & 0 & \dots & \dots & 0 & 0 & 0 & -2 \\ -1 & 2 & -1 & \dots & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \dots & -1 & 2 & -2 & 0 \\ -1 & 0 & 0 & \dots & \dots & 0 & -1 & 2 & 0 \\ -1 & 0 & 0 & \dots & \dots & 0 & 0 & 0 & 2 \end{vmatrix} \quad (1.28)$$

Define the determinant

$$x C_n(x) = x \det(\mathcal{M} - x \mathbf{1}) = \det(\mathcal{M}' - x \mathbf{1}') \quad (4.29)$$

Using the variable $y = 1 - x/2$ we can write $C_n(x)$ in terms of the Chebyshev polynomial of the second kind

$$C_n(x) = (-1)^n 2(y+1) U_{n-1}(-y) \quad (4.30)$$

The classical mass ratios are given by

$$m_k^2 = 4 \sin^2 \frac{k\pi}{2n}, \quad k = 1, 2, \dots, n \quad (4.31)$$

The expression for the coefficients r_k is

$$r_k = (-1)^{k+1} \frac{2^k(n+1)^k}{4n} \left[4 \binom{n+k}{2k+1} + \binom{n+k-1}{2k-1} \right] \quad (4.32)$$

They are not integers in general.

The $\tilde{C}_n$ twisted series has the following matrix $\mathcal{N}$:

$$\mathcal{N} = \begin{vmatrix} 1 & -\frac{1}{2} & 0 & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & \dots & \dots & 0 & 0 & 0 & 0 & -1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & -1 & 2 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & \dots & 0 & -1 & 2 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & \dots & \dots & 0 & 0 & 0 & 0 & 1 \end{vmatrix} \quad (4.33)$$

Put $2 - x = 2y$. The characteristic equation is then

$$\begin{aligned} \det(\mathcal{N}' - x \mathbf{1}') &= -x \det(\mathcal{M} - x \mathbf{1}) = \\ &= 2(y-1)[U_n(y) + U_{n-3}(y)] \end{aligned} \quad (4.34)$$

The mass ratios are easily computed

$$m_1^2 = 1, \quad m_k^2 = 4 \sin^2 \frac{k\pi}{2n-1} \quad k = 1, 2, \dots, n-1 \quad (4.35)$$

The coefficient p_k are obtained from the expansion of (4.34):

$$p_k = (-1)^{k+1} \left[\binom{2n+1-k}{k} + \binom{2n-2-k}{k-3} \right] \quad (4.36)$$

We have defined

$$\begin{pmatrix} m \\ -s \end{pmatrix} = 0 \quad (4.37)$$

when m and s are positive integers. The r_k 's are

$$r_k = (-1)^{k+1} \frac{(2n)^k}{2n-1} \left[\binom{n+1+k}{2k+1} + \binom{n-2+k}{2k+1} \right] \quad (4.38)$$

They are not integers in general.

4.4.3 The G_2 ATFT

We start discussing the untwisted G_2 ATFT. The mass matrix in the diagonal basis is

$$\mathcal{M} = \begin{pmatrix} 3 & 0 \\ 0 & 9 \end{pmatrix} \quad (4.39)$$

This theory can be obtained by folding of the ATFT constructed on $D_4^{(1)}$ [7, 11, 24]. The corresponding mass formulae are

$$S_k = \left(\frac{4}{3}\right)^{k-1} [3^{k-1} + 1] \quad k = 0, 1, \dots \quad (4.40)$$

which, except for some values, turn out to be non-integer numbers.

The mass matrix of the twisted G_2 is given by

$$\mathcal{M} = \begin{pmatrix} 1 & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & 3 \end{pmatrix} \quad (4.41)$$

Then the mass ratio is

$$\frac{m_2}{m_1} = 2 \cos \frac{\pi}{12} = \frac{1 + \sqrt{3}}{\sqrt{2}} \quad (4.42)$$

This theory can be obtained as folding of the $E_6^{(1)}$ ATFT [7, 11, 24]. The corresponding sequence of S_k is

$$S_k = [(3 + \sqrt{3})^{k-1} + (3 - \sqrt{3})^{k-1}] \quad (4.43)$$

They are integer numbers.

4.4.4 The F_4 ATFT

The untwisted F_4 ATFT in the diagonal basis has the following mass matrix

$$M = \begin{pmatrix} 3 - \sqrt{3} & & & \\ & 2(3 - \sqrt{3}) & & \\ & & 3 + \sqrt{3} & \\ & & & 2(3 + \sqrt{3}) \end{pmatrix} \quad (1.11)$$

The corresponding mass formulae are

$$S_k = \left(\frac{3}{2}\right)^{k-1} (2^{k-1} + 1) [(3 + \sqrt{3})^{k-1} + (3 - \sqrt{3})^{k-1}] \quad (1.15)$$

They give, generally, not integer values.

The twisted F_4 ATFT is recovered as folding of the $E_7^{(1)}$ theory [5,7,11,21]. The folding is with respect to the Z_2 symmetry of Dynkin diagram of the affine E_7 theory. The folding operation projects out the particles odd with respect to the Z_2 symmetry leaving the four even parts of $E_7^{(1)}$ as the basic fields of the twisted F_4 ATFT. Therefore, the characteristic equation for the F_4 ATFT can be read from the characteristic equation of $E_7^{(1)}$, namely

$$\mathcal{P}(x) = x^4 - 24x^3 + 180x^2 - 504x + 432 = 0 \quad (1.16)$$

The corresponding r_i parameters of the recursive equation (3.10) are integer numbers. Their values are

$$\begin{aligned} r_1 &= 28 \\ r_2 &= -240 \\ r_3 &= 778 \\ r_4 &= -778 \end{aligned} \quad (1.17)$$

Therefore the mass formulae for the twisted F_4 theory give an integer sequence.

5 Conclusions

We have systematically analyzed the characteristic equations of the ATFT which give the mass spectrum of these theories. For the ATFT based on the classical algebras the masses are related to the zeros of the Chebyshev polynomials. For the ADE models the mass spectrum is exact, i.e. it coincides with the mass spectrum obtained from the solution of the bootstrap equations of the S-matrix approach.

We have defined the mass formulae in order to analyze the finite terms of the tadpole diagrams in ATFT. These quantities allow us to compute the decoupling point of the tadpole graphs, i.e. the value of the parameter μ entering the regularized expressions of the Feynman diagrams such that we can discard completely the self-contractions of the fields of ATFT. We have derived the recursion equations the mass formulae satisfy. The mass formulae for the ADE series and the twisted versions of the exceptional G_2 and F_4 ATFT are sequences of integer numbers. Besides their own mathematical interest, in the case of the A_n series there is moreover an interesting coincidence with another expression derived in a different context. In fact the mass formulae S_θ of A_n series coincide with the dimensions of the vector-bundle $\mathcal{V}_\theta$ of the minimal models of CFT on a Riemann surface of genus g . We recall that Verlinde [28] obtained a formula expressing the dimension of $\mathcal{V}_\theta$ of a CFT in terms of the elements of the S-matrix of the modular group

$$\dim \mathcal{V}_\theta = \sum_{n=0}^{N-1} |S_{n0}|^{-2(p-1)} \quad (5.1)$$

In this equation N is the number of fields entering the Fusion Algebra of the CFT and the label 0 is reserved for the identity operator. The definition of the quantity on the r.h.s. implies that the sum on these elements of the S-matrix should be an integer quantity. Using the following expression of S for the minimal models

$$S_{lk} = \left(\frac{2}{n+1}\right)^{\frac{1}{2}} \sin \frac{(l+1)(k+1)}{n+1} \quad (5.2)$$

we see that (5.1) coincides exactly with the mass formulae of the A_n series, derived in Sect. 1.1. We have proved these are indeed sequences of integer numbers. It would be interesting to see if the mass formulae of the other ATFT are relevant in physics or in number theory or, simply, if they clarify the structure of the Lie algebras.

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Appendix A

In this Appendix we collect some useful properties of the Chebyshev polynomials [32].

They satisfy the following differential equation

$$(1-x^2) \mathcal{F}'' - x \mathcal{F}' + n^2 \mathcal{F} = 0 \quad n = 0, 1, 2, \dots \quad (\text{A.1})$$

The general solution of this differential equation is

$$\mathcal{F} = \begin{cases} A T_n(x) + B \sqrt{1-x^2} U_{n-1}(x) & \text{if } n = 1, 2, 3, \dots \\ A + B \sin^{-1} x & \text{if } n = 0 \end{cases} \quad (\text{A.2})$$

T_n are the Chebyshev polynomials of the first kind and U_n are the Chebyshev polynomials of the second kind.

The Chebyshev polynomials of the first kind enter the analysis of the A_n ATFF. They are given by

$$T_n(x) = \cos(n \cos^{-1} x) \quad (\text{A.3})$$

They satisfy the following recursive equation

$$\begin{aligned} T_{n+1}(x) &= 2x T_n(x) - T_{n-1}(x) \\ T_0(x) &= 1 \\ T_1(x) &= x \end{aligned} \quad (\text{A.4})$$

and their expression in power of x is given by

$$2T_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \frac{n}{n-k} \binom{n-k}{k} (2x)^{n-2k} \quad (\text{A.5})$$

where

$$\left\lfloor \frac{n}{2} \right\rfloor = \begin{cases} \frac{n}{2} & \text{if } n \text{ even} \\ \frac{n-1}{2} & \text{if } n \text{ odd} \end{cases} \quad (\text{A.6})$$

An important relation is the following

$$T_n(2x-1) = T_n(\sqrt{x}) \quad (\text{A.7})$$

Using this formula we obtain for Q_{n+1} (defined in eq. (4.1))

$$\begin{aligned} Q_{n+1} &= 2(T_{n+1}(1-\frac{x}{2})-1) = \\ &= (-1)^{n+1} x \sum_{k=0}^n T_k(-1)^k \frac{2(n+1)}{2(n+1)-k} \binom{2(n+1)-k}{k} x^{n-k} \end{aligned} \quad (\text{A.8})$$

Factorizing the x in front, which is the zero root of the matrix $\mathcal{M}$, we get the characteristic equation of the mass spectrum of the $\mathcal{M}_n^{(1)}$ series

$$\|\mathcal{M} - x \cdot 1\| = x^n - p_1 x^{n-1} - \dots - p_n \quad (\text{A.9})$$

where

$$p_k = (-1)^{k+1} \frac{2(n+1)}{2(n+1)-k} \binom{2(n+1)-k}{k}, \quad k = 1, 2, \dots, n \quad (\text{A.10})$$

Using this expression, finally we get for the parameters r_i

$$r_k = (-1)^n 2^k (n+1)^{k-2} p_{n-k} \quad (\text{A.11})$$

The analysis of the $D_n^{(1)}$ series needs the Chebyshev polynomials of the second kind. Their expression is

$$U_n(x) = \frac{\sin[(n+1) \cos^{-1} x]}{\sin(\cos^{-1} x)} \quad (\text{A.12})$$

They have the following properties

$$\begin{aligned} (i) \quad U_m(x) &= 2x U_{m-1}(x) - U_{m-2}(x) \\ U_0(x) &= 1 \\ U_1(x) &= 2x \\ (ii) \quad U_{2m+1}(x) &= 2x U_m(2x^2-1) \end{aligned} \quad (\text{A.13})$$

Using these properties, we get for the coefficients p_i of the characteristic equation of the $D_n^{(1)}$ series the following expression

$$p_k = (-1)^{k+1} 2^k \left[\binom{2n-1-k}{k-2} + 2 \binom{2n-2-k}{k-1} + \binom{2n-3-k}{k} \right] \quad (\text{A.14})$$

and correspondingly the expression (4.14) for the coefficients r_i .

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Figure 1: one loop mass correction.

Figure 2: Tadpole diagrams for the self-energy of the particle ϕ_i .

$\Lambda_{2r}^{(1)}/\mathcal{Z}_2$	$\Rightarrow$	$\Lambda_{2r}^{(2)}$
$D_{r+1}^{(1)}/\sigma$	$\Rightarrow$	$H_r^{(1)}$
$D_{r+2}^{(1)}/\mathcal{Z}_2$	$\Rightarrow$	$D_{r+1}^{(2)} \equiv \hat{H}_r$
$\Lambda_{2r-1}^{(1)}/\mathcal{Z}_2$	$\Rightarrow$	$C_r^{(1)}$
$D_{2r}^{(1)}/\mathcal{Z}_2$	$\Rightarrow$	$\Lambda_{2r-1}^{(2)} \equiv \hat{C}_r$
Length square • = 1 o = 2 ⊙ = f		

Table 1a: Foldings of simply-laced Dynkin diagrams: the principal series.
Near to the roots are the numbers q_i .

$l_1^{(1)}/\sigma$ 	$G_2^{(1)}$
$l_6^{(1)}/Z_3$ 	$D_4^{(3)} \equiv \tilde{G}_2$
$l_6^{(1)}/Z_2$ 	$F_4^{(1)}$
$l_7^{(1)}/Z_2$ 	$F_4^{(2)} \equiv \tilde{F}_4$
length square	$\bullet = 1 \text{ or } 2/3 \quad \circ = 2$

Table 1b: Foldings of simply-laced Dynkin diagrams: the exceptional series. Near to the roots are the numbers q_i .

$2M \sin(\frac{\pi i}{2r+1}), \quad 1 \leq i \leq 2r$ $D_{r+1}^{(1)}$	$A_{2r}^{(1)}$ $4M \sin(\frac{\pi i}{2r+1}), \quad 1 \leq i \leq r$
$M, M, 2M \sin(\frac{\pi i}{2r}), \quad 1 \leq i \leq r-1$ $D_{r+2}^{(1)}$	$B_r^{(1)}$ $M, 2M \sin(\frac{\pi i}{2r}), \quad 1 \leq i \leq r-1$
$M, M, 2M \sin(\frac{\pi i}{2r+2}), \quad 1 \leq i \leq r$ $A_{2r-1}^{(1)}$	$D_{r+1}^{(2)} \equiv \tilde{B}_r$ $\sqrt{2}M \sin(\frac{\pi i}{2r+2}), \quad 1 \leq i \leq r$
$2M \sin(\frac{\pi i}{2r}), \quad 1 \leq i \leq 2r-1$ $D_{2r}^{(1)}$	$C_r^{(1)}$ $2M \sin(\frac{\pi i}{2r}), \quad 1 \leq i \leq r$
$M, M, 2M \sin(\frac{\pi i}{2(r-1)}), \quad 1 \leq i \leq 2r-2$ $D_1^{(1)}$	$A_{2r-1}^{(2)} \equiv \tilde{C}_r$ $\frac{M}{\sqrt{2}}, \sqrt{2}M \sin(\frac{\pi i}{2(r-1)}), \quad 1 \leq i \leq r-1$
$M, M, M, \sqrt{3}M$ $F_6^{(1)}$	$C_2^{(1)}$ $M, \sqrt{3}M$
$m_a = m_\pi = M$ $m_b = m_\pi = 2M \cos(\frac{\pi}{12})$ $m_c = 2M \cos(\frac{\pi}{4})$ $m_d = 4M \cos(\frac{\pi}{12}) \cos(\frac{\pi}{4})$	$D_1^{(2)} \equiv \tilde{G}_2$ m_π, m_d
$F_7^{(1)}$ $m_1 = M$ $m_2 = 2M \sin(\frac{2\pi}{9})$ $m_3 = 2M \cos(\frac{\pi}{9})$ $m_4 = 2M \cos(\frac{\pi}{18})$ $m_5 = 4M \cos(\frac{\pi}{18}) \sin(\frac{2\pi}{9})$ $m_6 = M/2 \sin(\frac{\pi}{18})$ $m_7 = 4M \cos(\frac{\pi}{18}) \cos(\frac{\pi}{9})$	$F_1^{(1)}$ m_a, m_b, m_c, m_d
	$F_8^{(2)} \equiv \tilde{F}_4$
	m_2, m_1, m_5, m_7

Table 2: masses of models related by folding.

$E_8^{(1)}$
$m_1 = M$
$m_2 = 2M \cos(\frac{\pi}{5})$
$m_3 = 2M \cos(\frac{\pi}{30})$
$m_4 = 4M \cos(\frac{\pi}{5}) \cos(\frac{2\pi}{30})$
$m_5 = 4M \cos(\frac{\pi}{5}) \cos(\frac{2\pi}{15})$
$m_6 = 4M \cos(\frac{\pi}{5}) \cos(\frac{\pi}{30})$
$m_7 = 2M \cos^2(\frac{\pi}{5}) \cos(\frac{2\pi}{30})$
$m_8 = 8M \cos^2(\frac{\pi}{5}) \cos(\frac{2\pi}{15})$

Table 3: $E_8^{(1)}$ mass spectrum.