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Synchro-Beam Interaction

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Abstract

A symplectic mapping is presented, describing the effect of the beam-beam interaction in presence of synchrotron motion. In addition to the usual transverse kick, this mapping includes the longitudinal displacement of the collision point and the energy variation caused by the electric field due to the opposite bunch. These two effects are shown to play a concurrent role in preserving symplecticity. The case of weak-strong interaction in e^+e^- colliding ring is investigated by simulation.

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1 Introduction

Many years ago, Augustin remarked that a particle can lose or gain energy as a consequence of the beam-beam interaction [1]. Indeed, in the extreme relativistic limit, half of the force felt by the particle is due to the electric field of the opposite bunch and thus can have a component in the direction of the particle velocity. Later on, Piwinski developed a theory for synchro-betatron resonances driven by this effect in the case of collisions with a crossing angle [2]. Then one of the authors (M.B.) remarked that a variation of energy is possible even when the (average) crossing angle is zero [3]. Recently, he proposed a heuristic derivation of a new luminosity limit for e^+e^- storage rings, based on the comparison between the typical energy variation due to beam-beam collisions and the quantum-induced energy fluctuations over one machine revolution [4].

In this paper, we discuss the beam-beam interaction in weak-strong approximation. The evolution of a test particle in the weak bunch depends on the electromagnetic field created by all the particles in the strong bunch and the latter can be considered as an external *prescribed* field, since we neglect the reaction of the weak bunch on the particles of the strong one. Therefore the beam-beam interaction (in weak-strong approximation) and the interaction with bending and focusing magnets or RF-cavities of the storage ring must be treated on an equal footing. As a consequence, apart from radiation effects, the evolution of a test particle in the *six-dimensional* phase space is derivable from a Hamiltonian and the corresponding single-turn mapping must be *symplectic*. This condition is important both in particle tracking and for theoretical considerations; it implies the conservation of the six-dimensional phase-space volume element and any deviation from symplecticity can lead to unphysical effects, such as damping or antidamping of the particle oscillations as well as undue modifications of the synchro-betatron coupling.

We consider the case of a single interaction point (IP) and treat the strong bunch in *thin lens* approximation. Betatron and synchrotron transfer matrices allow us to describe the symplectic single-turn evolution, from IP to IP, in the absence of beam-beam interaction. Then, if the beam-beam collisions took always place at the IP, the associated map should also be symplectic. However, as a consequence of synchrotron oscillations, the point where a test particle collides with the strong bunch does not coincide with the IP and the corresponding longitudinal displacement changes from turn to turn. Therefore, we should require the symplecticity of the transformation consisting of a drift from the IP to the collision point, followed by the beam-beam map (i.e. transverse kick plus energy variation) and by a backward drift to the IP. This transformation, to be applied at the IP, will be called the ‘synchro-beam mapping’ and is discussed in Section 2 for a round beam with Gaussian radial distribution. There, we also consider the possible variation of transverse beam dimensions over the interaction region.

In Section 3, we describe the complete set of equations (including radiation effects) used in the simulation of the weak-strong beam-beam interaction for e^+e^- storage rings and Section 3.2 contains a discussion of some preliminary simulation results. They seem to confirm the importance of our symplecticity condition, at least when the r.m.s. value of the transverse betatron slopes is not negligible with respect to the relative energy spread: this is more likely the case for round beams and becomes generally true when approaching the beam-beam limit in the strong-strong interaction. Finally, in Section 4, we draw some conclusion about possible scaling laws and implications for future high-luminosity storage rings. A general discussion of symplecticity in the case of an arbitrary distribution for the strong bunch can be found in Appendix A, while Appendix B gives the formulae for a flat beam.

2 A Symplectic Map for Synchro-Beam Interaction

To make the discussion as simple as possible, we assume that the interaction region is dispersion-free and consider the case of a round beam with Gaussian radial distribution. Later on, we also assume that the motion of a test particle in the weak beam remains in a plane, so that we can reduce the number of transverse degrees of freedom of our problem from two to one. For the sake of definiteness, we assume that the strong beam consists of positrons (with positive charge e) while the test particle is an electron (with negative charge $-e$).

Canonical Variables

In order to discuss symplecticity, we use as independent variable the curvilinear abscissa s along the reference orbit of the storage ring and fix the set of canonical variables by choosing the normal-mode variables for the ideal linear machine, without beam-beam interaction. In the dispersion-free interaction region, these variables coincide with the following physical variables:

- x and y : [coordinates] horizontal and vertical positions.

- $x' \equiv dx/ds$ and $y' \equiv dy/ds$: [momenta conjugate to x and y] horizontal and vertical slopes.
- $z \equiv s - ct$: [coordinate] longitudinal displacement with respect to the center of the weak bunch (synchronous particle). If $z(s) > 0$, a particle of the weak bunch arrives at s ahead of the synchronous particle.
- $\epsilon \equiv (E - E_0)/E_0$: [momentum conjugate to z] relative energy deviation from the nominal energy $E_0 = \gamma mc^2$.

We denote all six variables by the vector

$$\mathbf{x} = (x, x', y, y', z, \epsilon)^t.$$

These variables are defined for a test particle in the weak beam.

Synchro-Beam mapping

As anticipated in the introduction, the symplectic mapping corresponding to ‘synchro-beam’ interaction is derived by treating the strong bunch in thin lens approximation and considering the case of a single interaction point in the ring at $s = 0$.

Imagine a particle arrives at the IP with positive z . This implies that the center of the other bunch is not yet at the IP: it is still at $s = z$. Thus, the real collision between this particle and the strong bunch takes place at $s = z/2$, to be called the collision point (CP). This effect can be described by a set of three subsequent transformations, namely

$$\mathbf{x}_{IP} \xrightarrow{D(z/2)} \mathbf{x}_{CP} \xrightarrow{B(z/2)} \mathbf{x}_{CP}^{ncw} \xrightarrow{D(-z/2)} \mathbf{x}_{IP}^{ncw}.$$

First the particle goes to $s = z/2$ by drift, D , then it undergoes the beam-beam interaction B at $s = z/2$ and finally it is brought back to the IP, again by drift. In the following, we consider the mapping corresponding to the concatenation of these three transformations and show that it is symplectic. Since the single-turn evolution along the arc of the storage ring, from IP to IP, can be assumed to be known and corresponds to betatron and synchrotron oscillations plus radiation effects, it is useful to include the back-drift in the ‘synchro-beam’ interaction.

2.1 Drift

The mapping corresponding to a drift of length $z/2$ can be written

$$\begin{pmatrix} x & y \\ x' & y' \end{pmatrix}_{CP} = \begin{pmatrix} 1 & z/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & y \\ x' & y' \end{pmatrix}_{IP}. \quad (2.1)$$

The longitudinal variables z and ϵ remain unchanged. However, since the length of the drift depends on the dynamical variable z , it is easy to see that the corresponding mapping in the six-dimensional phase space $\mathbf{x}$ is *not* symplectic. Indeed the Jacobian matrix

$$M(z/2; IP) \equiv \frac{\partial(\mathbf{x}_{CP})}{\partial(\mathbf{x}_{IP})}$$

is not a symplectic matrix, i.e.

$$M(z/2; IP) J M(z/2; IP)^t \neq J, \quad (2.2)$$

where M^t is the transpose of M and J is

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

2.2 Beam-Beam Map

The particle feels the transverse beam-beam kick at the CP ($s = z/2$):

$$x'^{new} = x' + \frac{(-e)}{E_0} \frac{1}{2} (\tilde{E}_x - c\tilde{B}_y) \quad (2.3)$$

$$= x' - \frac{e}{E_0} \tilde{E}_x, \quad (2.4)$$

where the subscript ‘CP’ is implied and a similar expression holds for y'^{new} . Here $\tilde{E}$ and $\tilde{B}$ denote the integrated electric and magnetic fields accompanying the strong bunch in thin lens approximation: for example, the horizontal components of the fields generated by the strong bunch when it is at s_* read

$$E_x(x, y, s) = \tilde{E}_x(x, y) \delta(s - s_*),$$

$$B_x(x, y, s) = \tilde{B}_x(x, y) \delta(s - s_*),$$

and so on. To first order in the betatron slopes x' and y' , the electric and magnetic fields give the same contribution to the transverse kick $\delta x'$. The factor $1/2$ in Eq. (2.3) comes from the fact that the test particle crosses the strong bunch in half the time it would take if the strong bunch did not move. The energy change is then given by

$$\epsilon^{new} = \epsilon - \frac{e}{E_0} \frac{1}{2} (x' \tilde{E}_x + y' \tilde{E}_y + \tilde{E}_z). \quad (2.5)$$

Longitudinal Electric Field

When the transverse dimension σ_* of the strong bunch can be considered constant in the interaction region, the integrated longitudinal component $\tilde{E}_z$ vanishes. However, if we take into account the fact that σ_* depends on s , the distribution of the strong bunch at the CP is tilted in the betatron phase space and there is a finite integrated component $\tilde{E}_z$, that we are now going to evaluate.

Neglecting quadratic terms in the betatron slopes x'_* and y'_* , the transverse components of the integrated electric field at (x, y) produced by a *single* particle of the strong bunch at (x_*, y_*) are

$$(\mathcal{E}_x(x - x_*, y - y_*), \mathcal{E}_y(x - x_*, y - y_*)) \equiv \frac{e}{2\pi\epsilon_0} \frac{(x - x_*, y - y_*)}{R^2},$$

where

$$R^2 = (x - x_*)^2 + (y - y_*)^2$$

and ϵ_0 is the dielectric constant of the vacuum. Since in the extreme relativistic limit the electric field is perpendicular to the velocity of the source particle, $\mathbf{v}_* = c(x'_*, y'_*, -1)$, we have $\mathcal{E} \cdot \mathbf{v}_* = 0$ and hence

$$\mathcal{E}_x x'_* + \mathcal{E}_y y'_* - \mathcal{E}_z = 0.$$

Thus the total integrated electric field produced by the strong bunch is

$$\tilde{E}_x = N_* \langle \mathcal{E}_x \rangle_*, \quad \tilde{E}_y = N_* \langle \mathcal{E}_y \rangle_*,$$

and

$$\tilde{E}_z = N_* \langle \mathcal{E}_z \rangle_* = N_* \langle x'_* \mathcal{E}_x + y'_* \mathcal{E}_y \rangle_*, \quad (2.6)$$

where $\langle \rangle_*$ denotes the average over the distribution ψ_* of the strong bunch

$$\langle A \rangle_* \equiv \int dx_* dx'_* dy_* dy'_* A \psi_*(x_*, x'_*, y_*, y'_*; s)$$

and N_* is the number of particles in the strong bunch.

In the case of a round beam with Gaussian distribution, where

$$\psi_*(x, x', y, y'; s) = \tilde{\psi}(x, x'; s) \tilde{\psi}(y, y'; s),$$

$$\tilde{\psi}(x, x', s) = \frac{1}{2\pi\epsilon_t} \exp \left[-\frac{x^2 + (-\alpha x + \beta x')^2}{2\epsilon_t \beta} \right], \quad (2.7)$$

we have

$$(\tilde{E}_x, \tilde{E}_y) = \frac{e N_*}{2\pi\epsilon_0} \frac{1 - \exp(-r^2/2\sigma_*)}{r} \frac{(x, y)}{r},$$

with $r = \sqrt{x^2 + y^2}$. Here β is the betatron function and $\alpha = -\beta'/2$ the usual Twiss parameter for the weak beam, while ϵ_t is the nominal transverse emittance, related to the radial beam size σ_* by

$$\sigma_* = \sqrt{\epsilon_t \beta}.$$

We assume that the optical functions and the nominal emittance are the *same* for the weak and for the strong beam (apart for the sign of α , as can be seen from Eq. (2.7)). Then, performing the integral appearing in Eq. (2.6), we obtain

$$\tilde{E}_z = -\frac{\alpha}{\beta} \frac{e N_*}{2\pi\epsilon_0} \exp \left(-\frac{r^2}{2\sigma_*} \right).$$

In Appendix A, we give a general formula to compute $\tilde{E}_z$ for any distribution ψ_* of the strong bunch.

Note that $\tilde{E}$ depends not only on x and y but also on z through σ^* at the CP ($s = z/2$):

$$\tilde{E} = \tilde{E}(x, y; \sigma_{CP}^*).$$

Indeed, since the strong bunch is in drift, we have

$$(\sigma_{CP}^*)^2 = (\sigma_{IP}^*)^2 \left(1 + \frac{z^2}{4\beta_{IP}^2} \right).$$

Beam-Beam Map at the CP

As a consequence of the transverse kick, x' and y' have a discontinuity at the CP. Therefore there is an ambiguity in Eq. (2.5), as for the treatment of x' and y' . This is a consequence of the thin lens approximation, equivalent to assuming that the longitudinal distribution of the strong bunch is a delta-function. To resolve the ambiguity, we use the following regularization procedure:

$$\delta(s) = \lim_{\Delta \rightarrow 0} \begin{cases} 0, & s > \Delta/2 \\ 1/\Delta, & \Delta/2 > s > -\Delta/2 \\ 0, & -\Delta/2 > s \end{cases}.$$

This regularization implies that we replace x' in Eq. (2.5) by

$$x' \longrightarrow \frac{x' + x'^{new}}{2}$$

and similarly for y' .

Here and hereafter we consider a particle with $(y, y') = 0$. The beam-beam map at the CP is then given by

$$\left. \begin{aligned} x_{CP}^{new} &= x_{CP}' - f(x_{CP}; \sigma_{CP}^*), \\ \epsilon_{CP}^{new} &= \epsilon_{CP} - \frac{1}{2} f(x_{CP}; \sigma_{CP}^*) \frac{x_{CP}' + x_{CP}^{new}}{2} - g(x_{CP}; \sigma_{CP}^*). \end{aligned} \right\} \quad (2.8)$$

$$f(x; \sigma_*) = \frac{e}{E_0} \tilde{E}_x = \frac{2N_* r_e}{\gamma} \frac{1}{x} [1 - \exp(-\frac{x^2}{2\sigma_*^2})],$$

$$g(x; \sigma_*) = \frac{1}{2} \frac{e}{E_0} \tilde{E}_z = -\frac{\alpha}{\beta} \frac{N_* r_e}{\gamma} \exp(-\frac{x^2}{2\sigma_*^2}),$$

where $r_e = e^2/4\pi\epsilon_0 mc^2$ is the classical electron radius.

Non Symplecticity of the Beam-Beam Map

Similarly to the drift, discussed in Sec. 2.1, the beam-beam map is *not* symplectic. This can be easily seen by testing the symplecticity condition Eq. (2.2) for the Jacobian matrix

$$M(CP^{new}; CP) \equiv \frac{\partial(\mathbf{x})_{CP}^{new}}{\partial(\mathbf{x})_{CP}}.$$

2.3 Recovery of the Symplecticity

Thus, neither the drift, Eq. (2.1), nor the beam-beam map, Eq. (2.8), are symplectic. However, if we concatenate these transformations as discussed before, we get a symplectic mapping. By applying successively Eqs. (2.1), (2.8) and (2.1) with $z/2$ replaced by $-z/2$, we arrive at the ‘synchro-beam’ mapping from $\mathbf{x}_{IP}$ to $\mathbf{x}_{IP}^{new}$, which can be written as follows:

$$\left. \begin{aligned} x^{new} &= x + \frac{1}{2}z f(X; \sigma_*(z/2)), \\ x'^{new} &= x' - f(X; \sigma_*(z/2)), \\ z^{new} &= z, \\ \epsilon^{new} &= \epsilon - \frac{1}{2}f(X; \sigma_*(z/2))[x' - \frac{1}{2}f(X; \sigma_*(z/2))] - g(X; \sigma_*(z/2)), \end{aligned} \right\} \quad (2.9)$$

where

$$\begin{aligned} X &= x + \frac{1}{2}zx', \\ f(x; \sigma_*) &= 8\pi\epsilon_t\eta\frac{1}{x}\left[1 - \exp\left(-\frac{x^2}{2\sigma_*^2}\right)\right], \\ g(x; \sigma_*) &= 4\pi\epsilon_t\eta\frac{1}{\sigma_*}\frac{d\sigma_*}{ds}\exp\left(-\frac{x^2}{2\sigma_*^2}\right). \end{aligned}$$

Here η is the nominal beam-beam parameter, which is proportional to the current of the strong beam

$$\eta = \frac{N_*r_e}{4\pi\gamma\epsilon_t},$$

and we have used

$$-\frac{\alpha}{\beta} = \frac{1}{\sigma_*}\frac{d\sigma_*}{ds}.$$

The symplecticity of this mapping can be easily checked by examining the Jacobian matrix

$$\frac{\partial(\mathbf{x})_{IP}^{new}}{\partial(\mathbf{x})_{IP}}.$$

This is also clear from the fact that Eqs. (2.9) are derivable from the following effective Hamiltonian H :

$$H = F(X; \sigma_*(z/2))\delta(s),$$

where F is the integral of f

$$F(X; \sigma_*) = \int_{-\infty}^X f(X; \sigma_*)dX.$$

As shown in Appendix A, this result is general and not restricted to the case of a round beam with Gaussian distribution.

We can say that, from the point of view of symplecticity, the energy variation due to the beam-beam interaction is complementary to the drift $D(z/2)$ and the back-drift $D(-z/2)$. It is now clear that if one considers the fact that the collision point is changing at every turn, one should also consider the energy variation and vice versa. Otherwise symplecticity is lost.

3 Weak-Strong Simulation

Let us describe the implementation of the synchro-beam mapping discussed above for particle tracking in weak-strong approximation. Here we consider the case of a round Gaussian beam, while the case of a flat Gaussian beam is discussed in Appendix B. The code includes betatron and synchrotron oscillations with damping and noise, associated with synchrotron radiation in the arcs of an e^+e^- storage ring. Thanks to the back-drift, $D(-z/2)$, we can apply the revolution matrix representing betatron and synchrotron oscillations just from IP to IP. This may be an important point, since the phase advance is defined for a fixed path length (s) and not for a fixed time interval (t). Had we used t as independent variable, the description would have been completely different (although equivalent).

3.1 The Program

In order to work with essential parameters only, we rescale the canonical variables as follows:

$$Q = \frac{x}{\sigma_x^0}, \quad P = \beta_{IP} \frac{x'}{\sigma_x^0},$$

$$Z = \frac{z}{\sigma_z^0}, \quad \Sigma = \frac{\epsilon}{\sigma_\epsilon^0},$$

where σ_x^0 , σ_z^0 and σ_ϵ^0 are the nominal values at IP of the transverse beam size, bunch length and relative energy spread, respectively.

The program tracks each particle as follows:

Oscillation

The transformations from IP to IP are just rotations with damping and quantum diffusions.

Thus we first apply the mapping

$$\begin{pmatrix} Q \\ P \end{pmatrix} \longrightarrow U(2\pi\nu_x) \begin{pmatrix} Q \\ P \end{pmatrix},$$

$$\begin{pmatrix} Z \\ \Sigma \end{pmatrix} \longrightarrow U(-2\pi\nu_s) \begin{pmatrix} Z \\ \Sigma \end{pmatrix},$$

where

$$U(\alpha) = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}.$$

Radiation

For transverse diffusion, the symmetric prescription discussed in [5] is used. The longitudinal diffusion is applied only to Σ . Thus,

$$Q \longrightarrow \lambda_x Q + \sqrt{1 - \lambda_x^2} \hat{r}_1,$$

$$\begin{aligned}
P &\longrightarrow \lambda_x P + \sqrt{1 - \lambda_x^2} \hat{r}_2, \\
\Sigma &\longrightarrow \lambda_s^2 \Sigma + \sqrt{1 - \lambda_s^4} \hat{r}_3,
\end{aligned}$$

where the $\hat{r}'$ s are independent, random Gaussian variables with unit standard deviation. Here the λ 's are damping factors defined by

$$\lambda_x = e^{-\delta_x}, \quad \text{and} \quad \lambda_s = e^{-\delta_s},$$

where the δ 's denote the corresponding damping decrements [6]. Their inverse values $\tau_x = 1/\delta_x$ and $\tau_s = 1/\delta_s$ represent the number of beam-beam collisions per betatron or synchrotron damping time, respectively.

Synchro-Beam Interaction

The mapping (2.9) for the synchro-beam interaction becomes

$$\begin{aligned}
Q &\longrightarrow Q - R_1 Z \delta P, \\
P &\longrightarrow P + \delta P, \\
Z &\longrightarrow Z, \\
\Sigma &\longrightarrow \Sigma + R_2 \delta P (P + \delta P/2) - G,
\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
\delta P &\equiv \kappa_r \frac{1}{Q + R_1 Z P} [\exp(-A_{col}^2) - 1], \\
G &\equiv \kappa_r R_1 R_2 \frac{Z}{1 + R_1^2 Z^2} \exp(-A_{col}^2), \\
A_{col} &= \frac{(Q + R_1 Z P)}{\sqrt{2(1 + R_1^2 Z^2)}}, \\
R_1 &= \frac{\sigma_z^0}{2\beta_{IP}}, \quad R_2 = \frac{\epsilon_t}{2\sigma_\epsilon^0 \beta_{IP}}
\end{aligned}$$

and

$$\kappa_r = 8\pi\eta.$$

The denominator of A_{col} represents the variation of the beam size (of the strong beam) due to the motion of the CP.

Let us note that the only dimensionless free parameters are the following:

- nominal beam-beam parameter η ,
- transverse and synchrotron tunes ν_x and ν_s ,
- transverse and synchrotron damping decrements δ_x and δ_s ,
- ratios R_1 and R_2 .

Figure 1: Tune dependence of the vertical beam size σ_y , normalized to its nominal value σ_{y0} . The parameters are the standard ones except for η , which is 0.01.

3.2 Results of Simulation

Due to CPU time restrictions, no extensive survey of the parameter space has been done yet and we only present some preliminary results. We have mainly studied the case of flat beam (see Appendix B), using the following standard set of parameters: $\eta = 0.05$, $\nu_x = 0.15$, $\nu_s = 0.085$, $\delta_y = 1/800$, $\delta_s = 2\delta_y$, $R_1 = 0.5$, $R_2 = 0.005$.

Dependence on ν

We vary the vertical tune ν_y for a fixed value of the beam-beam parameter η (equal to 0.01). The equilibrium vertical beam size is plotted as a function of ν_y in Fig. 1, where synchro-betatron side-bands appear clearly. This effect is still present for vanishing values of the parameter R_2 , expressing the relative magnitude of the energy variation due to the beam-beam interaction. Therefore it is mainly a consequence of the longitudinal modulation experienced by the collision point and of our thin lens approximation for the strong bunch. However, for finite values of R_2 , we have some indication that the synchro-betatron side-bands *do not disappear* even when the strong bunch is longitudinally distributed and consists of a sequence of many thin slices: this preliminary result seems to contradict the

Figure 2: The η dependence of σ_y for $\delta_y = 1/400$ and different values of R_1 .

conclusions of Ref. [7].

Dependence on η

If we consider only transverse motion, ignoring the energy variation and the modulation of the CP, it is already known that the transverse beam size grows up smoothly when increasing η . This means that there is no remarkable threshold value for η .

With the synchro-beam interaction, when we increase η , the beam size grows up slowly at the beginning but suddenly blows up by orders of magnitude at a certain point. For practical purposes, we define η_∞ as the value of η corresponding to an increase of the transverse beam size by a factor 100 over its nominal, unperturbed value. In Figs. 2 and 3 we plot the η -dependence of the beam size, corresponding to different values of the parameter R_1 , for a vertical damping decrement $\delta_y = 400$ and $\delta_y = 800$, respectively. For a given value of η , the equilibrium beam size is smaller in the case of rapid damping and, correspondingly, the threshold value η_∞ is larger.

Figure 3: The η dependence of σ_y for $\delta_y = 1/800$ and different values of R_1 .

Figure 4: The R_1 dependence of η_∞ for different values of $\delta_y = 1/\tau_y$.

Beam-Beam limit

We now consider the dependence of the threshold value η_∞ on R_1 and R_2 . Its dependence on ν_y and δ_y can be qualitatively deduced from the previous discussion.

Dependence on R_1

In Fig. 4, we show the dependence of η_∞ on R_1 for different values of δ_y . It appears that the product $\eta_\infty R_1$ is approximately constant (for fixed values of the δ 's, the ν 's and R_2). *For realistic values of the parameters, the dependence of this product on the δ 's, the ν 's and R_2 is rather weak.* Note that the disruption parameter D is given by

$$D = 8\pi\eta R_1.$$

This implies that the limiting value D_∞ is almost universal.

Dependence on R_2

In Fig. 5, we plot the threshold value η_∞ versus R_2 for different values of δ_y . It is remarkable that, at the beginning, the beam is more stable when increasing R_2 . However, beyond a

Figure 5: The R_2 dependence of η_∞ for different values of $\delta_y = 1/\tau_y$.

Figure 6: The R_2 dependence of η_∞ , σ_y and σ_z (just before blow up) for $\delta_y = 1/400$ and $R_1 = 0.5$.

certain value of R_2 corresponding to a maximum for η_∞ , the stability of the beam decreases and the beam-beam limit is reduced by almost a factor two for R_2 close to unity.

These results are a consequence of the symplecticity condition (satisfied by our synchro-beam mapping) on the synchro-betatron coupling. Indeed, as shown in Fig. 6, the bunch length at the beam-beam limit is an increasing function of R_2 . This means that the betatron oscillations can ‘release’ part of their energy to the synchrotron oscillations and, for small values of R_2 , this mechanism helps stabilizing the beam. Beyond a certain value of R_2 , the enhanced longitudinal modulation of the CP, associated with larger synchrotron oscillations, becomes the dominant effect and reduces the threshold value η_∞ . Let us notice that, for values of R_2 comparable to unity, the bunch lengthening tends to saturate. The corresponding decrease of the transverse beam size before blow up is then a consequence of the reduction of η_∞ . In Fig. 7 we show again the dependence of η_∞ , σ_y and σ_z on R_2 , but for a much lower value of R_1 . It appears that the peak of the threshold η_∞ is proportionally shifted to lower values of R_2 (it was at $R_2 = 0.05$ in Fig. 6, for $R_1 = 0.5$, and is shifted to $R_2 = 0.01$ in Fig. 7, for $R_1 = 0.1$).

Figure 7: The R_2 dependence of η_∞ , σ_y and σ_z (just before blow up) for $\delta_y = 1/400$ and a shorter nominal bunch length, corresponding to $R_1 = 0.1$.

Summary of Simulation Results

Our preliminary simulation results can be summarized as follows:

- For a fixed beam-beam parameter η , we observe clear synchro-betatron side-bands.
- For η small enough, the equilibrium beam size remains almost constant. It grows up suddenly at $\eta = \eta_\infty$.
- The threshold value η_∞ is smaller for slower damping.
- The threshold is lower for larger values of R_1 . The product $\eta_\infty R_1$, proportional to the disruption parameter D , is nearly constant and of the order of 0.1, for $R_2 = 0.005$.
- The threshold η_∞ has a peak corresponding to a value of R_2 roughly given by $R_1/10$. For larger values of R_2 it is considerably reduced. Meanwhile we observe a substantial increase of bunch length.
- For round beams we usually get a larger beam size than for flat beams.

Note that we use relatively fast damping. For more realistic values of the δ 's, the threshold seems to be smaller.

4 Discussion

We have proposed a symplectic ‘synchro-beam’ mapping, which describes the weak-strong beam-beam interaction in the 6-dimensional phase space. This mapping is valid in the extreme relativistic approximation and to first order in the betatron slopes of the particles in the weak and in the strong bunches. In order to include higher order effects, one should compute the electromagnetic field more carefully using Lienard-Wiechert potentials.

Although, when we started working on this problem, the main emphasis was on the derivation of a symplectic mapping, we have found as a ‘by-product’ the existence of a longitudinal electric field that contributes to the energy variation during beam-beam collisions (this probably means that Maxwell equations are partially built-in in the symplecticity condition). When the bunch length is comparable to the betatron function at interaction ($R_1 \sim 1$) and the beam-beam parameter is close to its threshold value ($\eta \sim 0.05$), the contribution of this longitudinal field to the energy variation is of the same order of the effect of the transverse electric field: from the mapping (3.1), we see that they are both proportional to the parameter R_2 .

In our weak-strong simulation, that includes synchrotron oscillations plus only one degree of freedom for betatron oscillations, we used fast damping and rather large values for R_2 . In existing low energy e^+e^- storage rings, damping decrements in the range 10^{-5} – 10^{-4} are rather frequent. This is not incompatible with the threshold values η_∞ we have found, since they tend to decrease for slower damping (see Fig. 5). As for the values of R_2 in existing or future machines, they are typically in the range 10^{-4} – 10^{-3} , although values as large as 10^{-2} may be expected for round beams. This is a consequence of the fact that, in the flat beam operation, the vertical emittance is usually made as small as possible. Indeed no storage ring has ever been operated at optimal coupling. It may be related to the fact that the optimal coupling would correspond to larger values of R_2 .

Six-dimensional Code

It is possible to perform 6-dimensional weak-strong simulation by using the Bassetti-Erskine formula [8]. Since our mapping can be applied at the nominal IP, it may be interesting to include it in some 6-dimensional tracking code, such as MAD or SAD to study pp or $\bar{p}p$ colliders, where symplecticity is more important than in the case of e^+e^- rings.

Strong-Strong Interaction

Our mapping can be used also in the strong-strong case. We should slice both the bunches longitudinally (in z-direction) and evaluate the effect of the collision between two slices [7]. The number of slices should be such that the changes of x' and y' should remain small enough, so that we can still neglect terms of order of $(x')^2$ and $(y')^2$.

Since our mapping can be derived from a potential $\mathcal{U}$ (see Appendix A) that is a function of $\mathbf{r} - \mathbf{r}'$, the conservation of energy and momentum after the collision between two slices is automatically insured.

Nonsymplectic Codes

As pointed out before, one should consider both of energy variation due to the beam-beam interaction and the modulation of the CP. Some existing code includes only the latter. When R_2 is large, as may be the case for a round beam, these codes are not reliable.

If we put $R_2 = 0$, it appears that we totally ignore the energy variation. However, $R_2 = 0$ is a singular case and we cannot go back to the original mapping for the physical variables. Conversely, we can say that we can ignore the energy change when σ_c^0 is infinitely large and this is not possible in e^+e^- rings.

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A More General Treatment

In this Appendix, we consider the general case of an arbitrary distribution

$$\psi_*(x_*, x'_*, y_*, y'_*; s)$$

for the strong bunch (that we still treat in thin lens approximation). We first prove the existence of a potential for the integrated electric field and then the symplecticity of the associated (Lie) mapping for the synchro-beam interaction.

Electric Field for an Arbitrary Distribution

We assume that the strong beam is not affected by the beam-beam force but do not assume a Gaussian distribution. The distribution function of the strong bunch ψ_* is described by the Vlasov equation corresponding to the drift space:

$$\frac{\partial \psi_*}{\partial s_*} + x'_* \frac{\partial \psi_*}{\partial x_*} + y'_* \frac{\partial \psi_*}{\partial y_*} = 0,$$

where $s_* = -s$. Now, from Eq. (2.6), we have

$$\hat{E}_z = N_* \int d\mathbf{r}_* d\mathbf{r}'_* [x'_* \mathcal{E}_x(\mathbf{r} - \mathbf{r}_*) + y'_* \mathcal{E}_y(\mathbf{r} - \mathbf{r}_*)] \psi_*(\mathbf{r}_*, \mathbf{r}'_*; s_*),$$

where

$$\mathbf{r} = (x, y)^t \quad \text{and} \quad \mathbf{r}' = (x', y')^t.$$

Since the transverse components of the integrated electric field created by a particle are derivable from a single-particle potential $\mathcal{U}$, i.e.

$$\mathcal{E}_x = -\frac{\partial \mathcal{U}}{\partial x} \quad \text{and} \quad \mathcal{E}_y = -\frac{\partial \mathcal{U}}{\partial y},$$

using the Vlasov equation and integrating by parts, we can express E_z as follows

$$\hat{E}_z = -N_* \int d\mathbf{r}_* d\mathbf{r}'_* \mathcal{U}(\mathbf{r} - \mathbf{r}_*) \frac{\partial \psi_*}{\partial s} \quad (\text{A.1})$$

$$= -N_* \frac{\partial}{\partial s} \int d\mathbf{r}_* d\mathbf{r}'_* \mathcal{U} \psi_*. \quad (\text{A.2})$$

We define the potential U as

$$U(\mathbf{r}; s) = \langle \mathcal{U}(\mathbf{r} - \mathbf{r}_*) \rangle_*,$$

so that we have

$$(\hat{E}_x, \hat{E}_y, \hat{E}_z) = -N_* \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial s} \right) U. \quad (\text{A.3})$$

In particular, for $s = z/2$, this implies

$$\frac{1}{2} \frac{\partial \hat{E}_z}{\partial x} = \frac{\partial \hat{E}_x}{\partial z}, \quad \text{and} \quad \frac{1}{2} \frac{\partial \hat{E}_z}{\partial y} = \frac{\partial \hat{E}_y}{\partial z}.$$

When ψ_* is Gaussian, all the s (or z) dependence can be parametrized in terms of the σ_* 's and we can write $U = U(x, y, \sigma_*(s))$.

Explicit Symplecticity

Now we show that the mapping for the synchro-beam interaction (from $\mathbf{x}_{IP}$ to $\mathbf{x}_{IP}^{new}$) can be written in terms of the Hamiltonian

$$H = F(x + zx'/2, y + zy'/2; z) \delta(s), \quad (\text{A.4})$$

where

$$F(X, Y, z) = \frac{-eN_*}{E_0} U(X, Y; s = z/2).$$

Indeed, this Hamiltonian induces the Lie mapping [9]:

$$\mathbf{x}_{IP}^{new} = \exp(: -F :) \mathbf{x}_{IP},$$

where the Lie operator $: A :$ associated with a function A is defined by

$$: A : \mathbf{x} \equiv [A, \mathbf{x}]$$

and the symbol $[,]$ denotes Poisson brackets.

In order to evaluate this Lie mapping, it is convenient to introduce a new set of canonical variables $\mathbf{x}_{new} = (X, P_X, Y, P_Y, Z, P_Z)^t$, given by

$$\begin{aligned} X &= x + zx'/2, & P_X &= x', \\ Y &= y + zy'/2, & P_Y &= y', \\ Z &= z, & P_Z &= c - \frac{(x')^2 + (y')^2}{4}. \end{aligned}$$

They are related to the original variables $\mathbf{x}$ by

$$\mathbf{x}_{new} = \exp\left(: -\frac{(x')^2 + (y')^2}{4} z :\right) \mathbf{x}.$$

In these new variables, it is easy to evaluate the Lie mapping. We find that (X, Y, Z) remain the same and

$$\exp(: -F(X, Y, Z) :) \begin{pmatrix} P_X \\ P_Y \\ P_Z \end{pmatrix} = \begin{pmatrix} P_X - f_X(X, Y, Z) \\ P_Y - f_Y(X, Y, Z) \\ P_Z - g(X, Y, Z) \end{pmatrix},$$

where

$$f_X(X, Y, Z) = \frac{\partial}{\partial X} F(X, Y, Z),$$

$$f_Y(X, Y, Z) = \frac{\partial}{\partial Y} F(X, Y, Z),$$

and

$$g(X, Y, Z) = \frac{\partial}{\partial Z} F(X, Y, Z).$$

We now use Eq. (A.3) to obtain

$$f_X = \frac{(-e)}{E_0} \hat{E}_x, \quad f_Y = \frac{(-e)}{E_0} \hat{E}_y,$$

and

$$g = \frac{(-e)}{2E_0} \hat{E}_z.$$

In particular, when ψ_* is Gaussian, we have

$$F(X, Y; Z) = \frac{-eN_*}{E_0} U(X, Y; \sigma_*(Z/2)) = F(X, Y; \sigma_*)$$

and thus

$$g(X, Y; Z) = \left(\frac{d\sigma_*^x}{dZ} \frac{\partial}{\partial \sigma_*^x} + \frac{d\sigma_*^y}{dZ} \frac{\partial}{\partial \sigma_*^y} \right) F(X, Y; \sigma_*).$$

Going back to the original variables $\mathbf{x}$, we get

$$\left. \begin{aligned} x^{new} &= x + \frac{1}{2} z f_X(X, Y; Z), \\ x'^{new} &= x' - f_X(X, Y; Z), \\ y^{new} &= y + \frac{1}{2} z f_Y(X, Y; Z), \\ y'^{new} &= y' - f_Y(X, Y; Z), \\ z^{new} &= z, \\ \epsilon^{new} &= \epsilon - \frac{1}{2} f_X(X, Y; Z) [x' - \frac{1}{2} f_X(X, Y; Z)] \\ &\quad - \frac{1}{2} f_Y(X, Y; Z) [y' - \frac{1}{2} f_Y(X, Y; Z)] \\ &\quad - g(X, Y; Z). \end{aligned} \right\} \quad (\text{A.5})$$

Under the assumption of the round Gaussian distribution and $(y, y') = 0$, this is reduced to Eq. (2.9).

It is clear that the synchro-beam mapping, from $\mathbf{x}_{IP}$ to $\mathbf{x}_{IP}^{new}$, is symplectic for any distribution ψ_* of the strong bunch, since it can be derived from the general Hamiltonian Eq. (A.4).

B Flat Beam Limit

In the flat beam limit, we assume that the beam is infinitely flat so that the horizontal motion can be ignored. We consider a sheet extending in the horizontal direction. Since the beam is assumed to be Gaussian in the horizontal direction, we average the x dependence over the bunch. In the mapping (2.9), x and x' should be replaced by the vertical variables y and y' and the function f is then given by

$$f(y; \sigma_*) = 2\pi^{3/2} \sqrt{\frac{\epsilon_y}{\beta_y}} \eta \operatorname{erf} \left(\frac{y}{\sqrt{2}\sigma_y} \right),$$

where ϵ_y and β_y are the nominal vertical emittance and nominal vertical beta-function at the IP. Here

$$\eta = \frac{N_* r_e}{\gamma} \frac{\beta_y}{2\pi \sigma_x^0 \sigma_y^0},$$

where β and σ 's are nominal values at the IP. Then we have

$$g(y; \sigma_*) = 2\sqrt{2}\pi \sqrt{\frac{\epsilon_y}{\beta_y}} \eta \exp\left(-\frac{y^2}{2\sigma_*^2}\right) \frac{d\sigma_*}{dz}.$$

If we introduce dimensionless variables in the same manner as for the round beam case, the expressions in the mapping (3.1) should then be replaced by

$$\delta P \equiv -\kappa_f \operatorname{erf}(A_{col}),$$

$$G \equiv \kappa_f R_1 R_2 \sqrt{\frac{2}{\pi}} \frac{Z}{\sqrt{1 + R_1^2 Z^2}} \exp(-X_{int}^2),$$

where R_1 and R_2 have similar expressions, obtained by replacing $\epsilon_t \rightarrow \epsilon_y$ and $\beta_{IP} \rightarrow \beta_y$, and

$$\kappa_f = 2\pi^{3/2}\eta.$$

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