Higher Order Gravitational Deflection and Soft Bremsstrahlung in Planckian Energy Superstring Collisions

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ABSTRACT

The leading behaviour of Planckian energy superstring scattering in D space-time dimensions was described in previous papers by an eikonal function which, at large distances, could be interpreted in terms of an effective Aichelburg-Sexl metric yielding the first order Einstein deflection. We compute here the subleading terms of such eikonal function up to two loops and, for $D = 4$, we find: a finite and positive classical correction to the Einstein deflection, implying a deviation from the external metric picture; an IR divergent absorptive part which admits a Bloch-Nordsieck interpretation in terms of soft graviton bremsstrahlung. The derivation of the above results is based on a novel treatment of the IR behaviour of quantum gravity around $D = 4$, proving the absence of inelastic Coulomb singularities and the lack of renormalization of the infinite Coulomb phase.

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1. INTRODUCTION

Classical and quantum gravity effects in string theory have been recently investigated in a series of papers [1-8] by appealing to gedanken Planckian energy collisions. The asymptotic regime that allowed the analysis was

$$E > M_p \gg M_s \quad (1.1)$$

where

- $E = \sqrt{s}$ is the c.m. energy of the collision of gravitons or low mass string states;
- M_p is the Planck mass related to the Newton constant G and the Planck length λ_P by

$$\lambda_P = \hbar/M_p = (G\hbar)^{\frac{1}{D-2}} \quad (1.2)$$

D being the number of space-time dimensions ($D = 10 - D_{compact}$) and

- M_s is the string mass (the mass of the first massive state) related to the fundamental string length parameter λ_s by:

$$\lambda_s = \hbar/M_s = (2\alpha' \hbar)^{1/2} \quad (1.3)$$

The ratio of the string and Planck masses is, at least perturbatively, an arbitrary parameter of the theory. Indeed, for closed or heterotic strings

$$\left(\frac{M_s}{M_p}\right)^{D-2} = \left(\frac{\lambda_P}{\lambda_s}\right)^{D-2} = \alpha \quad (1.4)$$

where α is the grand unified coupling, related to the expectation value of the dilaton and to the compactified volume. Its assumed smallness last inequality of (1.1) allows a meaningful string loop expansion in our asymptotic regime.

The physics of the collisions was found [1] to depend crucially on D and on the relative magnitude of three scales;

- $b = \frac{\sqrt{s}}{E}$, the impact parameter of the collision (ℓ is the total angular momentum);
- $R = (GE)^{\frac{1}{D-1}}$, the classical Schwarzschild radius associated with the energy E
- λ_s , the string length parameter.

These three scales, even within our asymptotic regime (1.1), can be in rather arbitrary relative ratios. A leading multiloop resummation was performed in [1] for large distances (impact parameter)

$$b \gg R, \lambda_s \quad (1.5)$$

giving rise to an operator eikonal S matrix that could be interpreted as a deflection in a background shock wave metric [6] (of an Aichelburg-Sexl type).

For smaller distances, the onset of non-leading contribution - and thus the physics - depends crucially [1,5] on the relative size of R and λ_s .

For $R < \lambda_s$ (which includes the limit of very small Newton constant at fixed, large energy) the analysis done [1] at $b \gg R$ can be apparently extended [5], through $b = \lambda_s$, down to $b = 0$. No important classical corrections are seen to emerge as if the finite string size would screen the strong, classical gravitational effects expected at $b = O(R)$. An interesting outcome of this analysis was the irrelevance of scales (impact parameters) smaller than λ_s , which naturally leads [5] to previously conjectured enlarged forms of the Uncertainty [9] and/or Equivalence [10] principles in String-Gravity. The results find support from other considerations as well [11,12].

For $R > \lambda_s$ (which includes the limit of very high energy at fixed, small α , or fixed Newton constant) the situation is quite reversed; short-distance string-size effects are now screened behind large effects occurring at $b = O(R) > \lambda_s$. These effects have a classical interpretation, e.g. in terms of an Einstein-like gravitational deflection. They represent corrections of order $(\frac{R}{\lambda_s})^{2n}$ to the leading result and stem from non-leading contributions of genus $g = 2n$ diagrams.

The aim of this paper is precisely to start a systematic study of these corrections in the regime $R > \lambda_s$ and to learn about their effect on the classical (Einstein) deflection and on particle production (bremsstrahlung). The calculations turn out to involve some subtle (and, to the best of our knowledge, new) infra-red problems which need careful treatment and interpretation.

The first correction to the eikonal ($O(R/b)^2$) comes at the two-loop level and was identified in I as originating from a so-called H diagram. The Regge-Gribov method for evaluating it, outlined in I, will be carried through in the present paper. The calculation of an apparently similar diagram by string loop methods has been carried out in Ref. [8]. We shall discuss in Section 6 the difference between the two calculations and results.

Our results for the H -diagram are then used to compute the non-leading eikonal function ϕ_2 , while emphasizing the need for a careful analysis of the exponentiation

mechanism at both one and two-loop level. In doing so, we shall see how the IR singularities of the H diagram cancel in $Re \delta_2$ leading to finite and calculable classical effects (e.g., on the deflection angle). We also explicitly show how the IR singularities of $Im \delta_2$ admit a Bloch-Nordsieck interpretation, i.e., suppress the elastic process to the advantage of soft bremsstrahlung.

This method can be extended in principle to multiloops hence to higher orders in $(R/b)^2$ by using the appropriate Regge-Gribov vertices of Ref. [13]. Only a resummation of these non-leading terms in the eikonal may allow to expose qualitatively new classical effects such as the formation of collapsed objects (Kerr black holes) and of horizons as well as of semiclassical phenomena, like Hawking's radiation. We shall also restrict ourselves to $D = 4$, the theoretically interesting case of $D = 3$ being under consideration.

The outline of the paper is as follows:

In Section 2 we shall recall the general eikonal structure which is expected to hold at small scattering angles and the expansion of the eikonal in a series of "classical" corrections. We shall identify the first correction as originating from a subleading contribution of the two-loop H diagram, noticing, however, that the exact determination of the correction also involves a non-leading analysis at one-loop.

In Section 3, we start the analysis of the H diagram and expose the appearance of infra-red (IR) singularities. In order to solve this IR problem, we digress, in Section 4, on the structure of soft-graviton emission amplitudes, using string theory results [13] which extend Weinberg's old method [14] to the whole kinematical region we have to consider. A Bloch-Nordsieck interpretation of the IR divergences is obtained.

In Section 5, armed with the above results, we go back to the full two-loop calculation and determine the non-leading correction to the eikonal with the help of crossing and analyticity arguments applied to the one and two-loop amplitudes. We thus compute both the IR singular absorption due to $Im \delta_2$ and the IR finite contribution to the Einstein deflection angle.

In Section 6 we summarize our results and discuss the comparison of our H diagram calculation with that of Ref. [8]. A few technical details are left to Appendices A and B.

2. THE EIKONAL EXPANSION

Let us recall the asymptotic S matrix obtained in [1] by computing and resumming in a closed superstring theory - the leading contributions of the diagrams depicted in Fig. 1. It had the operator eikonal form:

$$\hat{S}_{\text{leading}}(\underline{b}, \underline{E}) = e^{2i\hat{\delta}_0(\underline{b}, \underline{E})} \quad (2.1)$$

$$\hat{\delta}_0(\underline{b}, \underline{E}) = \iint_0 \frac{d\sigma_a d\sigma_d}{(2\pi)^2} : a_{tree}(\underline{b} + \hat{X}_a(\sigma_a) - \hat{X}_d(\sigma_d), \underline{E}) : \quad (2.2)$$

where $X_{a,d}$ represent the two colliding string operators (without zero modes) and a_{tree} is the tree graviton-graviton amplitude given by

$$a_{tree}(\underline{b}, \underline{E}) = \hat{\delta}_0 \Big|_{\hat{X}_{a,d}=0} = \frac{G_S}{2(\pi^2)^2} \int_0^{\frac{b/4\alpha'Y}{t}} dt t^{\epsilon-1} e^{-t} \xrightarrow{\epsilon \rightarrow 0} \frac{G_S}{2(\pi^2)^2} \frac{\Gamma(\epsilon)}{\epsilon} \quad (2.3)$$

$2\epsilon \equiv D-4, Y = \ell g s$

This expression has an infrared singularity at $D = 4$ corresponding to the infinite Coulomb phase [14]. We can thus write

$$\delta_0|_{D=4} = G_S \log\left(\frac{1}{b\lambda}\right) \quad (2.4)$$

where λ is an infrared (IR) cut-off. Since the λ dependence appears in S multiplicatively in the form of a numerical phase factor, it is unobservable. An important point in the following will be that this infinite phase is not affected by subleading (and possibly b-independent) higher loop corrections.

We also note that in the $(\alpha' \rightarrow 0)$ field theory limit δ_0 , for any D is proportional to the transverse propagator:

$$\delta_0 = \frac{G_S}{(2\pi)^2} \epsilon \cdot D_\epsilon(b) \quad (2.5)$$

where

$$D_\epsilon(\underline{b}) = \left(\frac{1}{2\pi}\right)^{\epsilon+1} \int d^2 \underline{k} \frac{e^{i\underline{b} \cdot \underline{k}}}{k^2} \quad (2.6)$$

so that

$$\nabla_\epsilon^2 D_\epsilon(\underline{b}) = -\delta(\underline{b}) \quad (2.7)$$

Let us now classify the possible subleading corrections. We recall from [1] that they

are expected to appear in the generalized eikonal functions in the form

$$\mathcal{J} = \mathcal{J}_0 \left(1 + \sum_{p,q} a_{p,q} \left(\frac{\lambda^2}{b^2} \right)^p \left(\frac{R}{b} \right)^{2q(D-3)} \right) (1 + O(\alpha)) \quad (2.8)$$

Terms with $p \geq 1$ ($q \geq 1$) represent quantum-string (classical) corrections. In this language all terms with $q = 0$ are resummed in closed form in Eqs. (2.1) and (2.2) and have been discussed in [1]. As already mentioned in the introduction we shall concentrate here on the classical corrections, i.e., on terms with $p = 0, q \neq 0$ which dominate in the region $R \gg \lambda_s$. In (2.8) we have anticipated the fact that only even powers of $R^{D-3} \sim GE$ enter the eikonal expansion. This is because GE is the only classical gravitational length in our essentially massless limit ($E \gg M_{ext}$) and because analyticity excludes non-integer powers of $s = E^2$. Thus the generic form

$$\mathcal{J}_0 a_{0,q} \left(\frac{R}{b} \right)^{2q(D-3)} \sim \frac{G^q}{b^q} \left(\frac{G^2}{b^{D-3}} \right)^q \quad (2.9)$$

can emerge from a tree level $2q$ Regge-graviton vertex that connects the external legs as shown in Fig. 2.

In particular the first classical correction δ_2 is related to the two-loop H diagram of Fig. 3. In this diagram two Regge graviton trajectories $\alpha(q_1^2)$ and $\alpha(q_4^2)$ are attached to the upper string and the other two $\alpha(q_2^2)$, $\alpha(q_3^2)$ to the lower one. By this, we mean the contributions coming from the leading trajectory Regge poles $J_i = \alpha(q_i^2)$ the angular momentum variables of the channels in q . Such poles yield the dominant terms when the subenergies $s_1 \sim s_4$ and $s_2 \sim s_3$ are large. The precise construction of the corresponding amplitude will be given in the next section.

It may seem surprising that the first non-trivial contribution is at two-loop level. This is because the one-loop correction to the eikonal $\delta_1(b, s)$ turns out (cf. Section 5) to be a pure quantum effect, an $O(\alpha)$ correction to $\alpha_{1,0}$ of (2.8), i.e.,

$$\delta_1 = \alpha \delta_0 \frac{1}{b^2} \sim \delta_0 \frac{G^2}{b^2} = \frac{G^2}{b^2} \quad (2.10)$$

This can be safely neglected in the eikonal for all $b \gg \lambda_s$ but its precise evaluation will turn out to be important in order to determine δ_2 the two-loop correction to the eikonal - from a computation of the corresponding amplitude. This is so because the scattering amplitude to $O(G^3)$, as coming from expanding the eikonal, will contain both $\delta_0 \delta_1$ and δ_2 , which from (2.4), (2.10) and (2.9) are of the same order.

We shall discuss in Section 5 a precise method based on analyticity and crossing symmetry allowing us to disentangle the individual non-leading terms of the eikonal from the non-leading expression of the scattering amplitude.

3. THE H -DIAGRAM: GENERAL CONSIDERATIONS

In order to attempt the computation of the H -diagram defined before and depicted in Fig. 3, it is useful to recall [1,4] that such diagrams represent contributions to asymptotic string theory scattering amplitude obtained with Regge-Gribov techniques [15]. They should not be confused with ordinary Feynmann diagrams in the sense that they may contain contributions of different regions of modular space of the string amplitude at the corresponding genus. At the two-loop level we are discussing, the Regge-Gribov H -diagram of Fig. 3 contains two particle - two Regge vertices in the upper and the lowest blobs while the middle blob represents a four-Reggeon tree amplitude. The first ones were already encountered in the study of the leading terms and have been obtained [1] from string theory by factorizing a six-point tree amplitude. Analogously, the four-Reggeon vertex was obtained in Ref. [13] by factorizing an eight-point string tree amplitude in the appropriate asymptotic limit.

Our strategy, whose details will be discussed at length in section 5, will be to compute first the three body discontinuity of the Reggeon diagram and then to use analyticity in order to reconstruct its real part. Finally, we shall obtain the real part of $\delta_2(b, s)$, i.e., in the two-loop contribution to the eikonal we are looking for.

Let us work in a convenient frame of reference in which P_1 and P_4 have large, $O(\sqrt{s})$ + components (P_1, P_3 large - components) where, as usual, $P_{\pm} = P_0 \pm P_{D-1}$. Splitting the q_i integrations into transverse (q_i) parts and longitudinal, we arrive at the expression

$$\begin{aligned} \int_{\mathcal{M}} A_{\mathcal{M}}(s, \theta^i) &= \frac{4}{\pi} (\beta \pi G)^2 \int \frac{d\tilde{q}_1 d\tilde{q}_2}{(2\pi)^{2D-4}} V_2^u(\tilde{q}_1, \tilde{q}_4) V_2^d(\tilde{q}_1, \tilde{q}_3) \int \frac{d\tilde{s}_1 d\tilde{s}_2}{s} \\ &\quad \left(\frac{\alpha' \tilde{s}_1}{2} \right)^4 \left(\frac{\alpha' \tilde{s}_2}{2} \right)^4 \left(\frac{\alpha' (t_1 + t_2)}{2} \right)^4 W\left(\tilde{q}_1, \frac{s}{s_1 s_2} \right) ; \quad t_i = -\tilde{q}_i^2 \end{aligned} \quad (3.1)$$

Here $\alpha'_c = \frac{1}{2} \alpha'$, set to one when not explicitly written, V_2^u and V_2^d are the aforementioned two-particle two-Reggeon vertices integrated over q_{1-}, q_{2+} , i.e., over the masses of the intermediate states in the upper and lowest blobs of Fig. 3, which are explicitly given

amplitude [13]. In the $\eta \gg \alpha'$ limit one finds

$$J_{\mu\nu}(q_1, q_2) \xrightarrow{q_2 \gg \Lambda} J_{\mu\nu}(q_1, q_2) = \frac{1}{2} \left(J_{\mu\nu}^{(12)}(1,2) - q_1^2 J_{\mu\nu}^{(12)}(1,2) \right) (3.7)$$

where

$$J_{\mu}(1,2) = q_1^2 \frac{p_{1\mu}}{p_1 \cdot q} - q_2^2 \frac{p_{2\mu}}{p_2 \cdot q} + (q_1 - q_2)_\mu \left[-q^2 \left(\frac{p_{1\mu}}{p_1 \cdot q} - \frac{p_{2\mu}}{p_2 \cdot q} \right) \right]$$

$$j_{\mu}(1,2) = \frac{p_{1\mu}}{p \cdot q} - \frac{p_{2\mu}}{p \cdot q} \quad (3.8)$$

By introducing (3.5) into (3.1) and integrating over s_1 with $s_2 = sq^2/s_1$ we arrive at the result

$$\frac{1}{5} \ln A_H = \frac{(8\pi G)^{3/2}}{4\pi(2\pi)^{2D+2}} \int_{y_0}^{Y-y_0} dy \int_{\frac{q_0}{2}}^{\frac{q_1}{2}} \pi dq \cdot e^{-d' y (q^2 + q_0^2)} e^{-d'' (Y-y) (q^2 + \tilde{q}_0^2)}$$
$$\cdot J_{\mu\nu}(q, q) J^{\mu\nu}(q_3, q) ; \quad V \equiv \log s, \quad y_0 \equiv \log \tilde{x}^2 \quad (3.9)$$

$$J_{\mu\nu} J^{\mu\nu} = (q^2)^{-2} f(\theta_{ij}) \prod_i q_i^2 \quad (3.10)$$

where the angular factor f is given by

$$\cos \theta_{ij}^{(i)} \equiv \tilde{q}_i \cdot \tilde{q}_j / |\tilde{q}_i| |\tilde{q}_j| \quad (3.11)$$

An important feature of the expression for JJ in Eq. (3.10) is that the $\Pi(\underline{q}_i)^2$ factor removes the poles appearing in Eq. (3.9). This phenomenon, already present in the five-point function, implies that the Coulomb-like singularity only appears in the elastic channel. An analogous cancellation of the pole was already noticed in [1] for diffractive string excitation.

A similar but less stringent cancellation occurs for open strings, corresponding to Yang-Mills theory, in which the squared current becomes [13]

$$J_{\mu}(1,2)J_{\mu}(3,4) \sim \frac{1}{q^2} \left[\underline{q}_1 \cdot \underline{q}_4 \underline{q}_2 \cdot \underline{q}_3 + \underline{q}_1 \cdot \underline{q}_2 \underline{q}_3 \cdot \underline{q}_4 - \underline{q}_1 \cdot \underline{q}_3 \underline{q}_2 \cdot \underline{q}_4 \right] \quad (3.12)$$

The linear zero in $\Pi \underline{q}_i$ occurring in this case may be assigned to the gauge invariance condition applied to the exchanged soft gluon in the $q_i \rightarrow 0$ limit. The additional zero occurring in the gravitational case can be attributed to the vanishing of the transferred "charge" which is itself of order q_i .

The angular factor f of (3.11) has a few interesting limits

- i) If $|\underline{q}| \ll |q_i|$, $f \sim \frac{q_i^2}{q_i^2}$ thus reducing the $\underline{q}^{-4}$ singularity in (3.10) to $\underline{q}^{-2}$. This has important implications for the structure of IR singularities.
- ii) f also vanishes if $\underline{q}$ is parallel to any of the $\underline{q}_i$, ($\theta_{12} = 0$, for instance). This property is related to the absence of collinear singularities in quantum gravity and has implications on the UV behaviour of the amplitude.
- iii) For $D = 4$, the $\underline{q}_i$'s are all coplanar and f takes the simpler form

$$f(\theta_{12}) = 2 \sin \theta_{12} \sin \theta_{34} \cos(\theta_{12} - \theta_{34}) \xrightarrow{q \rightarrow 0} \frac{2 \underline{q}^2}{|\underline{q}_1| |\underline{q}_4|} \sin \theta_{12} \sin \theta_{34} \quad (3.13)$$

where we have used the limit $\sin \theta_{12} \xrightarrow{q \rightarrow 0} \frac{2}{q_1} \sin \theta_{1q}$.

Thus, in $D = 4$, we obtain in impact parameter space the final expression:

$$\frac{1}{S} \int_{\text{Im}} \tilde{A}_H(s, b) = \frac{(8\pi G)^3 s \log s}{16\pi} \int \frac{d\underline{q} d\underline{q}_2}{(2\pi)^4} \left(\frac{d\underline{q}}{(2\pi)^2} \right)^2 e^{i\underline{q} \cdot \underline{b}} f(\theta_{ij}) \frac{(q^2)^2}{(2\pi)^2} \quad (3.14)$$

Let us discuss the convergence properties of the integrals in (3.14), i.e., the UV and IR singularities for $D = 4$. The q_1 and q_2 integrations (for fixed Q) show an UV

* The leading logs factors allow us to disregard the boundary term $y_0 \sim \log \underline{q}$ in the rapidity integration which would give rise to extra Q^2 -dependence of the integral. It can be shown that such term gives rise to $\frac{(\log s)^2}{s^2}$ contributions to $\text{Im} \delta_2$ (to be compared with the leading ones $\log s \frac{\log s^2}{s^2}$ -calculated here), but do not modify our final result for $\text{Re} \delta_2$.

single-logarithmic divergence regulated of course by the exponential factor in Eq. (3.9) which contains α' . It arises from the region $\frac{1}{\sqrt{\alpha'}} > |\underline{q}_2| > |\underline{q}_1| > |\underline{Q}|$ and yields

$$\int_{\text{Im}} A_H(\underline{Q}, s) \underset{UV}{\sim} \int_{|\underline{Q}|}^{\frac{1}{\sqrt{\alpha'}}} d\underline{q}_2 \int_{|\underline{Q}|}^{\frac{1}{\sqrt{\alpha'}}} d\underline{q}_1 \frac{1}{q^2} \sim \log \frac{1}{\alpha' Q^2} \quad (3.15)$$

This α' -dependence appears only at short distances in the Fourier transform, the large distance behaviour being $1/b^2$ and completely α' -independent.

On the other hand, despite the help of the angular factor, (always for $D = 4$), the $q_1 q_2$ integral in (3.14) - at fixed Q - has still a double log IR singularity coming from the region $\lambda < q, Q \ll q_1, q_2$, where λ is an IR cut-off. Indeed

$$\begin{aligned} \int_{\text{Im}} A_H(\underline{Q}, s) &\sim \int \frac{d\underline{q} d\underline{q}_2}{|\underline{q}_1| |\underline{q}_4|} \frac{\sin \theta_{12} \sin \theta_{34}}{q^2} \sim \\ &\sim \int \frac{d\underline{q}_1}{\underline{q}^2} \frac{d\underline{q}_2}{\underline{q}^2} \frac{1}{\lambda^2 \underline{q}^2} \sim (\log \frac{Q^2}{\lambda^2})^2 \end{aligned} \quad (3.16)$$

which implies a single log IR singularity in the Fourier transform:

$$\int_{\text{Im}} \tilde{A}_H(b, s) \sim \frac{1}{b^2} \log b \cdot \log s \quad (3.17)$$

In order to understand the physical meaning of this IR singularity, it is useful to interrupt here the discussion of the H -diagram and to open a digression on soft graviton bremsstrahlung. This will allow us to identify a Block-Nordsieck mechanism at work and thus to compute (see Section 5) well defined classical gravitational corrections.

4. GRAVITATIONAL BREMSSTRAHLUNG

We expect any high energy gravitation scattering to be accompanied by radiation of real gravitons which are mostly soft with respect to the overall momentum transfer.

In our case the role of momentum transfer is played by the transverse momenta q_i occurring at each rescattering. Therefore, we shall define the soft graviton region of the integration phase space by the restriction $q \equiv |\underline{q}| \ll |q_i| \equiv q_i$, besides the usual one

$$q_0 = \omega_q \ll \sqrt{s}.$$

In this limit, the one-soft-graviton emission amplitude associated with the exchange of momentum q_1 is given by the external line insertion formula (Fig. 4)

$$\frac{1}{s} A^{\mu\nu}(q, q) \xrightarrow{|q| \ll |q_1|} \frac{q^2}{q_1^2} J^{\mu\nu}(q) \quad (4.1)$$

where

$$\begin{aligned} J^{\mu\nu} &= g \sum_{i=1}^4 \eta_i \frac{p_i^\mu p_i^\nu}{p_i \cdot q} \approx \\ &\approx -g q_1 q \left(\frac{p_1^\mu p_1^\nu}{(p_1 \cdot q)^2} - \frac{p_2^\mu p_2^\nu}{(p_2 \cdot q)^2} \right) + g \left[q_1^4 \left(\frac{p_1^\mu}{p_1 \cdot q} - \frac{p_2^\mu}{p_2 \cdot q} \right) + \mu \leftrightarrow \nu \right] \end{aligned} \quad (4.2)$$

is the Weinberg current [14] for the scattering process, the weight η_i being $+1(-1)$ for incoming (outgoing) particles.

The approximate expression in (4.2) is obtained in the limit $q_1/E \ll \theta \equiv q/\omega_q$ in which the graviton emission angle is large compared to the scattering angle, and it coincides with the singular terms $O(1/q)$ of the current $J^{\mu\nu}$ given before (Eqs. (3.7),(3.8)) for $q \ll q_1$. Note that the latter limit is consistent with the above large angle condition for soft gravitons ($\omega_q \ll \sqrt{s}$).

In order to better understand this point, it is convenient to separate in $J^{\mu\nu}$ the most singular terms in the $|q| \rightarrow 0$ limit. This task is rather easy on the vector current J_μ of Eq. (3.8) that we write in the form [13]

$$J^\mu = J_0^\mu + \partial J^\mu \quad (4.3)$$

where

$$J_0^\mu = q_1^2 \frac{p_1^\mu}{p_1 \cdot q} - q_2^2 \frac{p_2^\mu}{p_2 \cdot q} \quad (4.4)$$

is the $O(q_1/q)$ part, to be identified with the external line insertions (Fig. 4), while the remaining part

$$\partial J^\mu = (q_1 - q_2)^\mu - q^2 \left(\frac{p_1^\mu}{p_1 \cdot q} - \frac{p_2^\mu}{p_2 \cdot q} \right) \quad (4.5)$$

contains $O(q_1)$ and $O(q)$ terms corresponding to the internal line insertion. The formula (4.5) can be confirmed by direct calculations in Yang-Mills theory.

A similar decomposition can be performed on the gluon emission current $J^{\mu\nu}$, by writing

$$J^{\mu\nu} = \frac{1}{2} (J^\mu J^\nu - q_1^2 q_2^2 j^\mu j^\nu) = J_{\text{ext}}^{\mu\nu} + \frac{1}{2} \partial J^\mu \partial J^\nu \quad (4.6)$$

thus defining the internal and external contributions apart from ambiguities coming from gauge terms. At this point, it is straightforward to show that

$$\frac{q}{q_1^2} J_{\text{ext}}^{\mu\nu} \xrightarrow{q/q_1 \rightarrow 0} J_{\text{W}}^{\mu\nu} + \text{gauge terms} \quad (4.7)$$

A gauge independent description of the currents $J^{\mu\nu}$ and $J^{\mu\nu}$ in $D=4$ is obtained by using a convenient basis of physical transverse traceless (TT) polarization tensors, given by

$$\epsilon_{\mu\nu}^{(a)} = \frac{1}{\sqrt{2}} (\epsilon_\mu^L \epsilon_\nu^T + \epsilon_\mu^T \epsilon_\nu^L), \quad \epsilon_{\mu\nu}^{(b)} = \frac{1}{\sqrt{2}} (\epsilon_\mu^L \epsilon_\nu^L - \epsilon_\mu^T \epsilon_\nu^T) \quad (4.8)$$

where

$$\begin{aligned} \epsilon_\mu^T &= (0, \underline{\epsilon}, 0) ; \quad \underline{\epsilon} \cdot \underline{q} = 0, \quad \underline{\epsilon}^2 = 1, \\ \epsilon_\mu^L &= |q|^{-1} (q_3, 0, q_0) ; \quad \epsilon^L \cdot \epsilon^T = \epsilon^L \cdot \underline{q} = 0 \end{aligned} \quad (4.9)$$

is in turn an orthonormal basis of transverse polarization vectors. We find

$$J_{\text{W}}^{(a)} \equiv J_{\text{W}}^{\mu\nu} \epsilon_{\mu\nu}^{(a)} = 2\sqrt{2} g \frac{\underline{q}_1 \cdot \underline{\epsilon}(q_1)}{|q_1|} = 2\sqrt{2} g \frac{q_1}{q} \sin \theta_q ; \quad J_{\text{W}}^{(b)} = 0 \quad (4.10)$$

and, more generally,

$$\begin{aligned} J^{(a)} \equiv J^{\mu\nu} \epsilon_{\mu\nu}^{(a)} &= -\sqrt{2} \frac{\underline{q}_1 \cdot \underline{q}_2}{q} \epsilon(q_1) \cdot \epsilon(q_2) = 2\sqrt{2} \left(\frac{q_1 q_2}{q} \right)^2 \\ &\cdot \sin \theta_{12} \cos \theta_2 ; \quad J^b \equiv \frac{1}{\sqrt{2}} [(\underline{q}_1 - \underline{q}_2) \cdot \underline{\epsilon}(q)]^2 = 2\sqrt{2} \left(\frac{q_1 q_2}{q} \right)^2 \sin^2 \theta_2 \end{aligned} \quad (4.11)$$

It is interesting to compare the behaviour for $q, q_i \rightarrow 0$ of the Weinberg current and of its generalization $J^{\mu\nu}$ given before (which happens to differ in normalization by some q_1^2 factor).

In the $q \ll q_1$ limit of soft emitted gravitons, we see from (4.6) that $(q_1^2)^{-1} J^a \rightarrow J_W^a$, as expected, with dominance of a single polarization ($J_W^b = 0$), similarly to what happens in the vector case. On the other hand, in the $q(q_2) \ll q$ limit, both J^a and J^b are $O(q_1^2/(0q_2^2))$. This gives rise to the cancellation of the Coulomb like singularities mentioned in Section 3 (Eq. (3.10)). In this case, the use of the Weinberg current is not justified and internal line insertions – which are embodied in $J^{\mu\nu}$ (Fig. 5) – become important [13]. The limits $q/q_1 \rightarrow 0$ and $q_1/q \rightarrow 0$ do not commute, but the current $J^{\mu\nu}$ is valid for any value of this ratio, while $J_W^{\mu\nu}$ is only justified for $q/q_1 \rightarrow 0$.

The one-graviton insertion formula ((4.1), (4.2)) can be generalized to many soft gravitons by factorization theorems [14]. Real emission and virtual corrections associated with a single exchange diagram (Fig. 4) can be summarized in a coherent state operator factor [16] in the form

$$A_{\text{soft}}(q, A_q) = \frac{g^2}{q^2} \mathcal{U}_{\text{soft}}^q(A_q) \quad (4.12)$$

where the coherent state operator

$$\begin{aligned} \mathcal{U}_{\text{soft}}^q &\equiv \exp\left(\frac{d^3q}{(2\pi)^3} 2\omega_q \theta(q, -q) J_W^{(a)}(q, q) (A_q - A_q^*)\right) \\ [A_q, A_q^*] &= (2\pi)^3 2\omega_q \delta(\vec{q} - \vec{q}') \delta(\omega_q - \omega_{q'}) \end{aligned} \quad (4.13)$$

contains creation (annihilation) operators $A^\dagger(A)$ of the dominant polarization J_W^a of Eq. (4.10).

Finally, by summing the leading s rescattering series for all graviton exchanges, each exchange containing already all emitted soft gravitons, we obtain the S -matrix operator in the eikonal form embodying soft bremsstrahlung

$$\begin{aligned} \hat{S}_{\text{eik}}(\underline{b}, A_q) &= e^{2i\hat{\mathcal{D}}_{\text{soft}}^q(\underline{b}, A_q)} \\ \hat{\mathcal{D}}_{\text{soft}}^q(\underline{b}, A_q) &= \frac{g^2}{4} \int \frac{d^3q}{(2\pi)^3} e^{i\underline{q} \cdot \underline{b}} \mathcal{U}_{\text{soft}}^q(A_q) \end{aligned} \quad (4.14)$$

Note that the eikonal operator $\hat{\mathcal{D}}_{\text{soft}}$ is hermitian because, by (4.13), $(\mathcal{U}_{\text{soft}}^q)^\dagger = \mathcal{U}_{\text{soft}}^{-q}$. Therefore, the S -matrix is explicitly unitary in the Fock space of soft gravitons. This means that the Block-Nordsieck theorem holds at fixed b , or, in other words, that

cross-sections including soft radiation are finite. Therefore, the IR singular emission of soft gravitons associated with gravitational scattering at fixed b explains the existence of b -dependent IR singularities in the elastic channel.

Let us apply the above arguments to the IR singular part of our two-loop problem. To this purpose, we must consider one or two emitted gravitons per exchange, as obtained by expanding U_{soft} up to second order, i.e.,

$$\begin{aligned} \mathcal{U}_{\text{soft}}^q &= 1 + 2\sqrt{2}gq_1 \int \frac{d^3q}{(2\pi)^3} \frac{\theta(q, -q)}{2\omega_q} \frac{e^{i\underline{q} \cdot \underline{b}}}{|\underline{q}|} (A_q - A_q^*) + \\ &+ (2\sqrt{2}gq_1)^2 \int \frac{d^3q}{(2\pi)^3} \frac{\theta(q, -q)}{2\omega_q} \frac{e^{i\underline{q} \cdot \underline{b}}}{|\underline{q}|} \frac{e^{i\underline{q}' \cdot \underline{b}}}{2\omega_{q'}} \frac{e^{i\underline{q} \cdot \underline{b}}}{|\underline{q}|} (A_q - A_q^*) + \dots \end{aligned} \quad (4.15)$$

The trivial ($U = 1$) term in this expansion gives rise in Eq. (4.14) to the leading eikonal $\delta_0 = -Gs \log b$ of Eq. (2.6), with its IR singular Coulomb phase. The $O(g)$ term will be shown below to contribute to the H -diagram. Instead, the $O(g^2)$ term gives rise to the emission (absorption) of graviton pairs for each exchanged graviton (Fig. 5), a process which is irrelevant at the order we are interested in, but also contains, through normal ordering, a contribution to elastic rescattering (Fig. 5) of the form

$$-\frac{g^2}{8\pi^2} \log^2 \log q_1^2 \quad (4.16)$$

where the $\log s [\log \frac{s}{\Lambda^2}]$ factor comes from rapidity [transverse momentum] integration and λ is an IR cut-off. The insertion of Eq. (4.16) into Eq. (4.14) eliminates the graviton pole (q_1^2 factor), but still gives rise to a large distance contribution to the eikonal, owing to the $\log \frac{s}{\Lambda^2}$ singularity, given by

$$\sigma_1^{\text{soft}}(b) = -\frac{G^2}{\pi} \log^2 s \int \frac{d^3q}{(2\pi)^3} e^{i\underline{q} \cdot \underline{b}} \log^2 \frac{1}{|\underline{q}|} = -\frac{2}{\pi} \frac{G^2}{b^2} \log^2 s \quad (4.17)$$

This is a one-loop correction of type (2.10) to the leading eikonal of Eq. (2.4).

The $O(g)$ term in the expansion (4.15) is the most interesting one, because it dominates real emission and also contributes virtual corrections at two-loop level. In

fact, by inserting (4.15) into (4.14), we obtain a contribution $g^3(A-A^+)$ to the operator eikonal which is given by

$$\int \frac{d^3 q_1}{(2\pi)^2} \frac{g^3}{\sqrt{2}q_1} e^{i k \cdot q_1} \int \frac{d^3 q}{2\omega_q} \frac{\Theta(q_1 - q)}{(2\pi)^2} \frac{\sin \theta_q}{q} (A_q - A_q^+) \quad (4.18)$$

and is then exponentiated in the S -operator. Since the elastic amplitude is obtained as in QED by normal ordering S_{eik} , the term (4.18) will contribute a c -number exponent of order g^6 which represents a two-loop correction to the eikonal function to be compared to the H -diagram:

$$\delta_2^{\text{soft}} = i g^2 \log s \left| \frac{d^3 q}{2(2\pi)^2 q^2} \right| \int \frac{d^3 q_1}{\sqrt{2}(2\pi)^2 q_1} \frac{\sin \theta_{q_1}}{q_1} e^{i b \cdot q_1} \quad (4.19)$$

This contribution is purely imaginary and therefore of absorptive type, i.e., it damps the elastic cross-section as usual IR form factors do. By using the integral

$$\int \frac{d^3 q_1}{q_1 2\pi |q_1|} \sin \theta_{q_1} e^{i b \cdot q_1} = i \sin \theta_{bq} \frac{J_0(bq)}{|b|} \quad (4.20)$$

the above expression can be explicitly evaluated to be

$$\delta_2^{\text{soft}} = 2 i \frac{G^2}{\pi b^2} \log s \cdot \log \frac{1}{b^2 \lambda^2} \quad (4.21)$$

It is important to note that (4.21) coincides with the IR singular contribution to the H -diagram amplitude (3.14). Indeed, owing to the absence of $q_1 = 0$ divergences, the singular part of (3.14) given in (3.16) comes from the integration region $q \ll q_1 \simeq q_2$ in which its integrand coincides with the one in (4.19) thanks to the validity of Eq. (3.13), for the angular factors in this regime. Thus, the logarithmic IR divergence present in Eq. (3.14) is due to virtual soft gravitons and is required by the singular bremsstrahlung predicted by Eq. (4.18).

To sum up, the asymptotic S -matrix, including soft graviton contribution up to two-loops is given by the formula

$$\hat{S}_{\text{eik}} = e^{2i(\delta_0 + \delta_1^{\text{soft}} + \delta_2^{\text{soft}})} \cdot \exp \left[2 i g^2 \left(\frac{d^3 q}{\sqrt{2}(2\pi)^2 |q|} \int \frac{d^3 q_1}{2\omega_{q_1}} \frac{\Theta(q_1 - q)}{|q_1|} \frac{\sin \theta_{q_1}}{(2\pi)^2} (A_q - A_q^+) \right) \right] \quad (4.22)$$

where δ_1^{soft} are given in Eqs. (4.17) and (4.21).

5. THE TWO-LOOP CONTRIBUTION TO THE EIKONAL: EXPLICIT COMPUTATION

With the understanding of the IR contribution achieved in Section 4, we are ready to compute the integral (3.14) and to discuss the evaluation of the full two-loop amplitude. Although the expression (3.14) contains the string cut-off α' , the final result in b -space will turn out to be UV finite in $D = 4$ and therefore α' -independent. We choose to work in $D = 4 + 2\epsilon$ ($\epsilon > 0$) dimensions to regulate the IR singularities, and we keep their dependence only to regulate some spurious UV singularities which occur in the intermediate steps of the calculation, setting $\alpha' = 0$ in the end ($GE \gg \lambda_0$).

It is useful to rewrite the general angular factor (3.9) in the form

$$(q^2)^{-2} f(\theta_{ij}) = \sum_{j=1}^4 F_j(q_i) \quad (5.1)$$

where the expressions

$$F_1 = \frac{q_1 \cdot q_2}{q_1^2 q_2^2} \frac{q_1 \cdot q_4}{q_1^2 q_2^2} \frac{q_1 \cdot q_3}{q_1^2 q_2^2} \frac{q_1 \cdot q_4}{q_1^2 q_2^2}, \quad F_2 = \frac{q_1 \cdot q_2}{q_1^2 q_2^2} \left(\frac{q_1 \cdot q_3}{q_1^2 q_2^2} - \frac{q_1 \cdot q_4}{q_1^2 q_2^2} \right) + \left(\frac{1 \leftrightarrow 2}{3 \leftrightarrow 4} \right)$$

$$F_3 = -\frac{q_1 \cdot q_2}{q_1^2 q_2^2} \frac{q_3 \cdot q_4}{q_3^2 q_4^2} + \left(\frac{1 \leftrightarrow 2}{3 \leftrightarrow 4} \right), \quad F_4 = -\frac{q_2 \cdot q_3}{q_2^2 q_3^2} \frac{1}{q_2^2} - \frac{q_2 \cdot q_4}{q_2^2 q_4^2} \frac{1}{q_2^2} \quad (5.2)$$

have at most four denominators (instead of the six appearing in the left-hand side of Eq. (5.1)), at the expense of introducing some spurious UV divergences for $q(q_2) \gg Q$.

The absorptive part of the H -diagram is then rewritten in the form

$$\text{Im } A_4(b) = \frac{\text{Im } \tilde{A}(yb)}{s} = \frac{8G^2}{(2\pi)^{1+2\epsilon}} \log s \sum_{j=1}^4 I_j \quad (5.3)$$

where the integral I_j are defined by

$$I_j \equiv \int \frac{d^3 q_1 d^3 q_2 d^3 q}{(2\pi)^{3(1+\epsilon)}} e^{i b \cdot q} F_j(q_i) \quad (5.4)$$

and the string α' -dependence is introduced by the replacement

$$1/q^2 \rightarrow \frac{e^{-\alpha' q^2}}{q^2} \quad (5.5)$$

with the appropriate y values, whenever required by the spurious UV behaviour mentioned before.

It is shown in Appendix A that all I_j 's are simply expressed, in terms of the transverse propagator of Eq. (2.6)

$$D(b) \xrightarrow{\epsilon \rightarrow 0} \int_0^1 d\alpha \frac{e^{i b \cdot x}}{(2\pi)^{1-\epsilon}} = \frac{(2\pi)^\epsilon}{G s} \sigma_0^\epsilon = \frac{\Gamma(\epsilon/2)}{2} \left(\frac{\epsilon}{b^2}\right)^\epsilon \quad (5.6)$$

with the α' -dependence understood whenever needed. We find

$$\begin{aligned} I_1 &= -2 \left[\sum D(b) \right]^2 [D(b) \frac{\Gamma(4+3\epsilon)}{\Gamma(4+2\epsilon)} \cos \pi \epsilon - D(0)] \\ I_2 &= 4 \left[\sum D(b) \right]^2 [D(b) - D(0)] \\ I_3 &= - \left[\sum D(b) \right]^2 D(b) (1-2\epsilon) \\ I_4 &= 2 \left[\sum D(b) \right]^2 D(0) \end{aligned} \quad (5.7)$$

We thus see that I_1 , I_2 and I_4 have an UV divergence (or α' -dependence) exhibited by $D(0)$ which however cancels out in the sum. This is expected on the basis of the analysis of the UV behaviour of Eq. (3.14), which has a single logarithmic UV divergence in Q -space, and is therefore finite in b -space. We also notice that the ϵ -dependent factor in I_1 behaves as

$$\frac{\Gamma(4+3\epsilon)}{\Gamma(4+2\epsilon)} \cos \pi \epsilon \xrightarrow{\epsilon \rightarrow 0} 1 + O(\epsilon^2) \quad (5.8)$$

so that I_1 and I_2 are IR finite (for $\epsilon \rightarrow 0$) and I_3 and I_4 are instead singular. Furthermore, I_1 is highly non-trivial for $\epsilon = 1, 2, \dots (D = 6, 8, \dots)$ as already noticed in [1], but gives no finite contribution in $D = 4$ because of the ϵ^2 behaviour in (5.8).

To sum up, by adding the I_j 's in (5.7), we obtain

$$\sum_{j=1}^4 I_j = \left(\sum D(b) \right)^2 D(b) (1+2\epsilon) + O(\epsilon^2) \quad (5.9)$$

and, by relating D to δ_0 through (5.6) and by using Eq.(5.3), the final result

$$\sum a_f(b) = \frac{2}{\pi} \log s \left(\frac{1}{3} \frac{\sum \sigma_0^2}{s} + 2 \frac{G^2}{b^2} \right) + O(\epsilon) + O(\alpha') \quad (5.10)$$

The expression (5.10) is UV finite in the $\alpha' \rightarrow 0$ limit, but is IR singular, the ϵ -pole being isolated in the derivative term $\sim \frac{1}{\epsilon} \delta_0^3$. This IR divergent part checks with the soft graviton contribution of Eq. (4.18) with the replacement

$$\frac{1}{3} \frac{\sum \sigma_0^2}{s} \rightarrow \frac{G^2}{b^2} \log \frac{1}{b^2 \lambda^2} \quad (5.11)$$

which is obtained from the expression $\delta_0 = -G s \log b \lambda$, translating the dimensional regularization into the λ cut-off.

Let us now come to the discussion of the full amplitude including the real part. So far, we have been careful in distinguishing the H -diagram amplitude $a_H(b, s)$ from the two-loop contribution to the elastic eikonal function $\delta_2(b, s)$ whose IR singular contribution has been given in Section 4. The former is defined by the diagram in Fig. 3 with the four Regge-graviton amplitude, while the latter is related to the S -matrix by exponentiation as follows

$$\begin{aligned} S_{el}(b, s) &\equiv \exp 2i \left(d_0(b, s) + d_1(b, s) + d_2(b, s) + \dots \right) = \\ &= 1 + 2i \left[d_0 + (i d_0^2 + d_1) + \left(-\frac{2}{3} d_0^3 + 2i d_0 d_1 + d_2 \right) + \dots \right] \end{aligned} \quad (5.12)$$

Here, in the square brackets, we have grouped together the tree, one-loop and two-loop contributions to the elastic amplitude. Their Fourier transform (FT):

$$\begin{aligned} A^{(0)} &= s(d_0)_{FT} \\ A^{(1)} &= s(i d_0^2 + d_1)_{FT} \\ A^{(2)} &= s \left(-\frac{2}{3} d_0^3 + 2i d_0 d_1 + d_2 \right)_{FT} \end{aligned} \quad (5.13)$$

are supposed to satisfy general properties of scattering amplitudes in particular $s \leftrightarrow u$ symmetry (crossing) and fixed t dispersion relations (analyticity).

We now proceed to use these tools for extracting $\delta_2(Re \delta_2, Im \delta_2)$ from the calculations we have carried out so far. As it turns out, we need an extra crucial information in order to carry out this goal, the absence of next-to-leading corrections to $Im A^{(1)}$, i.e.,

$$\begin{aligned} \text{Im } d_1 &= 0 \quad \text{to } O(s) \\ \text{Im } A^{(1)} &= s(d_0^2)_{FT} \end{aligned} \quad (5.14)$$

The proof of (5.14) is given in Appendix B and is based on an appropriate identification of the variable q up to next-to-leading order.

Now, knowing δ_0 and $\text{Im} A^{(1)}$ from (5.14), we can compute $\text{Re} A^{(1)}$ at first subleading level by analyticity and crossing symmetry. In fact, there are two $s-u$ symmetric analytic functions having s^3 (leading) and $s^2 \log s$ (next-to-leading) behaviour, namely

$$s^3 \log(-s) + u^3 \log(-u) \sim -s^3 \left(i\pi + \frac{3t}{s} \log s \right) \quad (5.15)$$

$$s^2 \log(-s) + u^2 \log(-u) \sim s^2 (-i\pi + 2 \log s) \quad (5.16)$$

and $A^{(1)}$ must be a combination of the two with coefficients which are real analytic functions of Q^2 . Since, by (5.14), $\text{Im} \delta_1 = 0$, the analytic structure (5.16) must be absent and $A^{(1)}$ must be proportional to (5.15). Thus we have, in impact parameter space,

$$A^{(1)}(b) \equiv \frac{1}{s} \tilde{A}^{(1)}(b, s) = i d_0^2 + d_1^2 = \left(i + \frac{3}{\pi} \nabla^2 \right) d_0^2(b, s) \quad (5.17)$$

where we have used the replacement $t \rightarrow -Q^2 = \nabla^2$ in the b transform. Therefore, we obtain

$$d_1(b, s) = \frac{3 \log s}{\pi s} \nabla^2 d_0^2 \underset{D=4}{\sim} \frac{6}{\pi} G s \frac{\lambda_P}{b^2} \log s \quad (5.18)$$

This IR finite, purely real, value of δ_1 gives rise to quantum corrections to the deflection angle (cf., Section 2), so that its direct contribution to the eikonal is not very important in the graviton dominated regime $b \geq R(E) \gg \lambda_s \gg \lambda_P$. However, its determination is needed at two-loop level because it generates in Eq. (5.13) the crossed term $2\delta_0 \delta_1$, which is infrared singular due to the δ_0 factor, and of the same order as δ_2 (cf. Section 2).

More precisely, we have

$$\begin{aligned} \text{Im} A^{(2)}(s, t) &= s \left(2 d_0^2 d_1 + \text{Im} d_2 \right)_{FT} \\ \text{Re} A^{(2)}(s, t) &= s \left(-\frac{2}{3} d_0^3 + \text{Re} d_2 \right)_{FT} \end{aligned} \quad (5.19)$$

where, since $2\delta_0 \delta_1$ is the whole two-body discontinuity, $\text{Im} \delta_2$ is to be identified with the three-body contribution, i.e., with $\text{Im} a_H$ computed before. Thus, by (5.6) we have

$$\text{Im} d_2 = \text{Im} a_H = \frac{\log s}{\pi s} \left(\frac{2}{3} \nabla^2 d_0^3 + \frac{4 G^2 s^2}{b^2} \right) + O(\epsilon) \quad (5.20)$$

where we have used the expression (5.10) for $\text{Im} a_H$. By inserting (5.18) and (5.20)

(5.19), we obtain the complete expression of $\text{Im} A^{(2)}$:

$$\text{Im} A^{(2)}(s, t) = s \left[\frac{4}{\pi} \frac{\log s}{s} \frac{2}{3} \nabla^2 d_0^3 + \frac{4}{\pi} \frac{G^2 s^2}{b^2} \log s \right]_{FT} \quad (5.21)$$

In order to determine $\text{Re} A^{(2)}$ we look, once more, for $s-u$ symmetric analytic functions having s^4 (leading) and $s^3 \log s$ (next-to-leading) behaviour and we find again two of them:

$$s^4 + u^4 \sim s^4 \left(1 - 2 \frac{Q^2}{s} \right) \quad (5.22)$$

$$s^3 (\log(-s))^2 + u^3 (\log(-u))^2 \sim s^3 \left(1 + \frac{2}{\pi} \frac{Q^2}{s} \log s \right) \quad (5.23)$$

We require $A^{(2)}$ to be a combination of the above analytic functions with the correct value of (i) the leading real part $(-\frac{2}{3} s (\delta_0^3)_{FT})$ and (ii) the subleading imaginary part (5.21). We find the unique answer

$$\begin{aligned} A^{(2)} &\equiv \frac{1}{s} \tilde{A}^{(2)}(s, b) = \left[\left(1 + 2 \frac{Q^2}{s} \right) - 2 \left(1 + \frac{2}{\pi} \frac{Q^2}{s} \log s \right) \frac{Q^2}{s} \right] \left(-\frac{2}{3} d_0^3 \right) + \\ &\quad + 2 \left(1 + \frac{2}{\pi} \log s \right) \frac{G^2 s^2}{b^2} \end{aligned} \quad (5.24)$$

From Eqs. (5.21) and (5.19) we read out the value of $\text{Re} \delta_2$, i.e., the next-to-leading contribution to the real part. The crucial property arising from (5.24) is that the IR singular contributions ($\sim \nabla^2 (\delta_0^3)$) cancel out between the analytic structures (5.22) and (5.23). The outcome is a finite result coming from the last term in Eq. (5.24), i.e.,

$$\text{Re} \delta_2 = 2 \frac{G^2 s^2}{b^2} \quad (5.25)$$

which yields the full two-loop contribution to the eikonal function

$$d_2(b, s) = \frac{2 G^2 s^2}{b^2} \left[1 + \frac{i}{\pi} \log s \left(\log \frac{1}{b^2 s} + 2 \right) \right] \quad (5.26)$$

The fact that $\text{Re} \delta_2$ is IR finite checks with the analysis of Section 4, which yields a purely imaginary IR singular contribution to the eikonal function.

Finally, in order to compute the Einstein deflection from (5.25) we use the definition of the two-dimensional momentum transfer in the brick-wall frame (Appendix B), i.e.

$$|Q| = \sqrt{-t} = \sqrt{s} \sin \frac{\theta}{2} \quad (5.27)$$

where θ is the c.m. scattering angle. The occurrence of $\sin \frac{\theta}{2}$ in Eq. (5.27) (instead of $\frac{1}{2} \sin \theta$ or $\frac{\theta}{2}$) is a non-trivial statement at the $O(\theta^3)$ accuracy which is needed here, and

stems from the requirement of absence of kinematical corrections at this order. From Eq.(5.25), by the usual saddle point argument [1], we get the classical deflection of massless scattering up to 3rd order

$$\sin \frac{\delta_L}{2} = -\frac{2}{\sqrt{s}} \frac{\partial}{\partial b} (d_0 + b d_1 + \dots) = \frac{R}{b} \left(1 + \left(\frac{R}{b} \right)^2 + \dots \right) \quad (5.28)$$

$$R = 2GE$$

We stress again the absence, in our massless scattering regime ($\sqrt{s} \gg m_{ext}$), of corrections of relative order R/b , due to analyticity in the s variable. We also note the positive sign of the two-loop contribution, consistent with the general idea that larger gravitational fields are being built up around $b = R$.

6. DISCUSSION

We have been able in the preceding sections to compute the two-loop contribution to the eikonal function $\delta_2(b, s)$ and to show that its IR singularities are under control: they appear only in the absorptive part ($Im\delta_2$) in which they reflect the soft graviton bremsstrahlung leaving a finite $Re\delta_2$ related to finite corrections to the Einstein deflection. An essential ingredient in taming the IR behaviour was the proof, given in Section 3, that the Coulomb phase is not modified by b -dependent divergences at two-loop level, i.e., that inelastic Coulomb-like singularities in the individual exchanged momenta q_i are absent.

We want to stress that a correct treatment of IR singularities associated with Coulomb or Newtonian scattering is a non-trivial and unavoidable issue of string theory and of any other theory containing gravity at large distances. The absence of inelastic Coulomb singularities found here is based on a delicate compensation between different terms in the square of the graviton emission current $J^{\mu\nu}$ entering the calculation. As noticed in Eq. (4.6), $J^{\mu\nu}$ contains both external and internal line insertions. While the external contributions dominate in the soft limit $q/q_i \rightarrow 0$ ($q_1 \sim q_2$ being the momentum of the exchanged graviton and q of the emitted one), the internal contribution dominates in the opposite (inelastic Coulomb) limit $q_1/q \rightarrow 0$ ($q_2/q \rightarrow 0$). In calculating the squared current (3.10) occurring in the amplitude (3.5) we have internal-internal, internal-external and external-external contributions (Fig. 6). The internal-internal terms do contain (a pair) of q_i^2 -poles, and therefore yields a spurious Coulomb singularity. The central point is that this is cancelled in Eq. (3.5) by the internal-external interference terms, leading to the result in Eq. (3.10) which is free of all q_i^2 -poles occurring in Eq. (3.5).

We see therefore how important it is that the H diagram be defined in the sense of Regge-Gribov, i.e., in terms of angular momentum poles, so that it contains all contributions to the current (internal and external). On the other hand, a definition of the H -diagram as a Feynmann diagram in terms of the momentum poles at $q_i^2 = 0$ or of the corresponding pinchings at the Riemann surface representing the string loop diagram, will isolate the internal-internal contribution which contains spurious Coulomb singularities and does not contain the normal bremsstrahlung terms.

It is important to keep in mind this distinction when comparing our results for the H -diagram with string loop calculations such as those of Ref. [8]. A direct comparison with Ref. [8] is difficult, because a complete calculation of the two-loop subleading terms is not obtained there. At the level of the H -diagram, our understanding is that the term isolated there is the one with the maximal singularities for $q_i^2 = 0$, to be identified with the internal-internal contribution in our language. This term alone will not have the correct IR behaviour and would lead to an uninterpretable b -dependent renormalization of the Coulomb phase. The remaining subleading terms (internal-external and external-external contributions) are to be looked for in other pinchings, including $1/s$ corrections to the leading three-graviton pinching.

Let us summarize the main results of this paper, which are essentially contained in Eq. (4.22) and in the expressions for the δ_i 's of Eqs. (2.4), (5.18) and (5.26). By combining them, we obtain the following expansion for the S -matrix in b -space as an operator in the Fock space of soft gravitons:

$$\hat{S} = e^{i \frac{G^2}{4} \log 4} e^{2i(\delta_0^F + \delta_1^F + \delta_2^F)} \cdot \exp \left(2ig^3 \int \frac{d^4 q}{(2\pi)^4} e^{i b \cdot q} \int \frac{d^4 q'}{(2\pi)^4} \frac{A(q, q')}{2u_q (2\pi)^3} \frac{1}{|q|} A_q^+ \right) \quad (6.1)$$

where

$$\begin{aligned} \delta_0^F &= -\frac{G^2 \log 6}{4} \\ \delta_1^F &= \frac{6}{\pi} \frac{G^2}{b^2} \log s \\ \delta_2^F &= \frac{2}{4} \frac{G^2}{b^2} \left(i \log s (\log \frac{1}{b})^2 + 2 \right) + 1 \end{aligned} \quad (6.2)$$

and $A_q[A_q^+]$ are absorption (emission) operators of soft gravitons in the leading polarization state $\epsilon^{(a)}$ of Eq. (4.3).

The IR singular Λ -dependence of $Im\delta_2$ can also be incorporated in the coherent state factor by undoing the normal ordering in the form of Section 4, i.e.,

$$\tilde{S} = e^{\frac{iG}{4} \delta_2 \log 2 : (\delta_0^\dagger + \delta_1^\dagger + R \delta_2)} \cdot \exp \left(2 i g s \log 2 \frac{e^{i b \cdot \tilde{q}_1}}{(2\pi)^3 \sqrt{2} |q_1|} \int \frac{d^3 q}{(2\pi)^3} \frac{\sin \theta_q}{2 \omega_q} \frac{A_2 - A_1}{|q|} \exp - \frac{8 G^2 s \log s}{4 \pi b^2} \right) \quad (6.3)$$

Apart from the last term, due to the finite part of $Im\delta_2$, this expression exhibits the unitarity of the S matrix in the space of soft gravitons, showing that a Bloch-Nordsieck mechanism is at work. The last absorptive factor in Eq. (6.3) is instead compensated by hard graviton emission, i.e., by the difference between the exact and the Weinberg current not explicitly written out here.

Finally, the finite expression for $Re\delta_2$ in (6.2) yields the two-loop correction to the deflection angle of Eq. (5.28), i.e.,

$$\sin \frac{\theta_d}{2} = \frac{R}{b} \left(1 + \left(\frac{R}{b} \right)^2 + \dots \right), \quad R \equiv 2 G E \quad (6.4)$$

Thus we see that the deflection angle increases with respect to both the first order Einstein value $\theta_E = \frac{2R}{b}$ and even more with respect to the Aichelburg-Sexl [17] value $(tg \frac{\theta_{AS}}{2} = \frac{R}{b})$. This increase is more sizeable for b approaching $R(E)$ thus pointing in the direction of a classical instability which may emerge from the resummation of higher order classical corrections.

APPENDIX A

H-Diagram Integrals

We illustrate here the manipulations needed to obtain the various terms of the H -diagram and in particular the cancellation of the UV divergences and the results in Eqs. (5.7)-(5.9).

From the basic Fourier transform

$$\left[(\tilde{Q}^2)^{-\alpha} \right]_{FT} \equiv \int \frac{d^{D-2} \tilde{Q}}{(2\pi)^{\frac{D-2}{2}}} \frac{e^{i \tilde{b} \cdot \tilde{Q}}}{(\tilde{Q}^2)^\alpha} = \frac{\Gamma(\frac{D-2}{2} - \alpha)}{2^\alpha \Gamma(\alpha)} \left(\frac{1}{b^2} \right)^{\alpha - \frac{D-2}{2}} \quad (A1)$$

and the definition

$$\begin{aligned} D_\epsilon^{(\alpha)}(\underline{b}) &= \left[(\tilde{Q}^2)^{-\alpha} \right]_{FT} = \frac{\Gamma(\frac{D-2}{2} - \alpha)}{2^\alpha \Gamma(\alpha)} \left(\frac{1}{b^2} \right)^{\alpha - \frac{D-2}{2}} \\ -\tilde{\nabla}^2 D_\epsilon^{(1)}(\underline{b}) &= (2\pi)^{1+\epsilon} d(\underline{b}) \quad ; \quad -\tilde{\nabla}^2 D_\epsilon^{\alpha+1} = D_\epsilon^\alpha \end{aligned} \quad (A2)$$

it is easy to get the six-denominator formula

$$\begin{aligned} & \left[\int \frac{d^3 q d^3 \tilde{q}}{(2\pi)^{D-2}} \left(\frac{1}{(q_1 + q_2)^2} \right)^2 \frac{1}{q_1^2 q_2^2 (Q - q_1)^2 (Q + q_2)^2} \right]_{FT} = \\ &= \int \frac{d^3 x d^3 y}{(2\pi)^{2(\epsilon+1)}} D_\epsilon^{(1)}(\underline{b}_1 - x) D_\epsilon^{(1)}(\underline{b}_2 - x) D_\epsilon^{(2)}(\underline{b}_1 - y) D_\epsilon^{(1)}(\underline{b}_2 - y) \Big|_{\substack{\underline{b}_1 = \underline{b}_2 = \underline{b} \\ \underline{b}_1 = \underline{b}_2 = \underline{b}}} \quad (A3) \end{aligned}$$

whose structure in configuration space is illustrated in Fig. 7. A similar formula holds for $\alpha' \neq 0$ with $D^\alpha(b)$ replaced by F.T. $(Q^2)^{-\alpha} e^{-\alpha' Q^2 y}$ with the appropriate y value.

From (A3) the various Fourier transforms occurring in (5.4) are obtained by partial derivation with respect to the variable $\underline{b}_i$ and by setting $\underline{b}_1 = \underline{b}_4 = 0$, $\underline{b}_2 = \underline{b}_3 = \underline{b}$ at the end. For instance, we get $(D^1(x) \equiv D(x) \text{ of Eq.(5.6)})$.

$$I_1 = \int \frac{d^3 x}{(2\pi)^{\epsilon+1}} \left[\tilde{\nabla} D_\epsilon(x) \right]^2 \left[\tilde{\nabla} D_\epsilon(\underline{b} - x) \right]^2 \quad (A4)$$

This integral is UV divergent around $x = 0$ and $x = \underline{b}$, the detailed behaviour being

$$I_1^{UV} = 2 \int \frac{d^3 x}{(2\pi)^{\epsilon+1}} \left(\tilde{\nabla} D(x) \right)^2 \left(\tilde{\nabla} D(\underline{b}) \right)^2 = 2 D(0) \left(\tilde{\nabla} D(\underline{b}) \right)^2 \quad (A5)$$

where we have used an integration by parts and the property (A2) of the transverse propagator.

In order to exhibit the UV regular part of I_1 we use the identity

$$\begin{aligned} I_1 &= 2 D(0) \left(\tilde{\nabla} D(\underline{b}) \right)^2 - \\ &\quad - \frac{\partial^2}{\partial \underline{b}_i \partial \underline{b}_j} \int \frac{d^3 x}{(2\pi)^{\epsilon+1}} \left(\tilde{\nabla} D(x) \right) \tilde{\nabla}_i D(x) \tilde{\nabla}_j D(\underline{b} - x) + d(\underline{b}) \text{term}, \end{aligned} \quad (A6)$$

which can be proved again by integration by parts. The second term in (A6) is UV finite and can be computed for $\epsilon > 0$ by standard integrations to yield

$$I_1 = \frac{\Gamma(\frac{D-2}{2})}{4} \left(\frac{1}{b^2} \right)^{1+\epsilon} \frac{\Gamma(\frac{D-2}{2} + \epsilon) \Gamma(\frac{D-2}{2} - \epsilon)}{\Gamma(\frac{D-2}{2} + \epsilon) \Gamma(\frac{D-2}{2} - \epsilon)} + 2 \left(\tilde{\nabla} D(\underline{b}) \right)^2 D(0) \quad (A7)$$

After use of some Γ -function identities, (A7) reduces to Eq. (5.7) of the text.

Similarly, by noting that the four terms in F_2 of Eq. (5.2) give pairs of identical contributions, we get

$$\bar{I}_2 = -2 \int \frac{d^4x}{(2\pi)^4} \left[\left(\nabla D_\xi(a) \right)^2 \left(\nabla D_\xi(b) \right) \left(\nabla D_\xi(b-x) \right) + x \leftrightarrow (b-x) \right] \quad (\text{A8})$$

After exhibiting the UV divergent part by manipulations similar to Eq. (A6), we obtain

$$\bar{I}_2 = \frac{\Gamma(1+\epsilon)}{4} \left(\frac{2}{b^2} \right)^{1+3\epsilon} \cdot \frac{4}{\epsilon} - 4 \left(\nabla D(b) \right)^2 D(o) \quad (\text{A9})$$

and finally

$$\begin{aligned} \bar{I}_3 &= 2 \left(\nabla D_\xi(b) \right) \left(\nabla D_\xi(b) \right) \left(\nabla D_\xi(b) \right) \left(\nabla D_\xi(b) \right) = \\ &= \frac{1}{4} \left(\frac{2}{b^2} \right)^{1+3\epsilon} \Gamma(1+\epsilon)^3 \frac{1-2\epsilon}{\epsilon} \\ \bar{I}_4 &= 2 \left(\nabla D_\xi(b) \right)^2 D_\xi(o) \end{aligned} \quad (\text{A10})$$

The above expressions prove Eq. (5.7) after use of the expression (5.6) for $D(b)$.

APPENDIX B

Subleading Terms in the Impact Parameter Transform

A full analysis of the two-loop exponentiation properties is complicated by the possible existence of subleading powers of s of kinematical origin. In fact, if we keep all terms of relative order q^2/s , even the leading graviton exchange is represented by a convolution

$$\int_o \otimes \int_o \otimes \dots \otimes \mathcal{D}_o = \frac{1}{s} A^{(n)}(s, Q^2) \quad (\text{B1})$$

whose phase space is not exactly diagonalized by a Fourier transform in the transverse momentum variables. More generally, the transverse variables themselves and the eikonal representation are ill-defined at non-vanishing scattering angle, if we look at corrections which are of relative order q^2/s .

Without attempting to solve this problem in general, we note that the Coulomb (or graviton) exchange does not give rise to kinematical corrections, provided we work in

the brick-wall frame or, in other words, we use a two-dimensional q -transform ($D=4$) with

$$\underline{Q}^2 = -t = s \sin^2 \frac{\theta}{2} \quad ; \quad (Q_o = Q_3 = 0) \quad (\text{B2})$$

θ being the c.m. scattering angle. This can be seen by writing in more detail the convolution (B1) in the form (Fig. 8)

$$\frac{1}{s} A^{(n)}(s, Q^2) = \int \prod_{i=1}^n \left[\frac{G s d\vec{q}_i}{(2\pi)^2 q_i} E \mathcal{D} \left(\vec{p}_i - \sum_{j=1}^i \vec{q}_j \right)^2 - \frac{\epsilon^2}{4} \right] \frac{1}{(\vec{Q} - \sum_{i=1}^n \vec{q}_i)^2} \quad (\text{B3})$$

where the δ -functions come from the two-body phase space for the intermediate particles, which also yields $q_o = 0$. By (B3), the following recursion relation holds (Fig. 8)

$$\begin{aligned} \frac{1}{s} A^{(n)}(s, Q^2) &= \frac{G s}{(2\pi)^2} \int \frac{d^3 Q'}{(\vec{Q} - \vec{Q}')^2} E \mathcal{D} \left((\vec{p}_1 - \vec{Q}')^2 - \frac{\epsilon^2}{4} \right) \frac{1}{s} A^{(n-1)}(s, Q'^2) = \\ &= \frac{G s}{(2\pi)^2} \int \frac{Q' dQ' d\phi}{Q^2 + Q'^2 - 2 Q Q' \cos \alpha' - \sin \alpha' \sin \alpha' \cos \phi} \frac{1}{s} A^{(n-1)}(s, Q'^2) \end{aligned} \quad (\text{B4})$$

where the on-shell δ -functions yield

$$\hat{\vec{p}}_1 \cdot \vec{Q} = \cos \alpha' = \frac{Q}{Q'} \quad \vec{p}_1 \cdot \vec{Q}' = \cos \alpha' = \frac{Q'}{\sqrt{s}} \quad (\text{B5})$$

It is now straightforward to check that the azimuthal average in (B4), due to (B5), is the same as for a two-dimensional momentum transfer, i.e.,

$$\begin{aligned} \langle [Q^2 + Q'^2 - 2 Q Q' \cos \alpha' - \sin \alpha' \sin \alpha' \cos \phi]^{-2} \rangle = \\ = \langle (Q^2 + Q'^2 - 2 Q Q' \cos \phi)^{-2} \rangle = \left\langle \frac{1}{(Q - Q')^2} \right\rangle \end{aligned} \quad (\text{B6})$$

where the identity

$$\frac{1}{2\pi} \int \frac{d\phi}{a + b \cos \phi} = (a^2 - b^2)^{-1/2} \quad (\text{B7})$$

has been used. Therefore (B4) is diagonalized by a two-dimensional impact parameter transform, provided we set $Q^2 = -t = Q'^2$ as in (B2).

This shows that the relativistic graviton exchange in $D=4$ does not give rise to subleading kinematical corrections in the brick-wall frame of Eq. (B2).

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FIGURE CAPTIONS

- Fig. 1. Regge-Gribov diagram resummed by the leading eikonal. Each blob represent a tree-diagram.
- Fig. 2. Regge-Gribov diagrams giving classical corrections. Again, each blob is a tree diagram.
- Fig. 3. The "H" diagram that provides the leading correction to the eikonal.
- Fig. 4. External (a) and internal (b) soft graviton emission diagrams.
- Fig. 5. Soft processes described by the expansion of U_{soft} up to second order: (a) one-graviton emission; (b) virtual graviton correction; (c) two-graviton emission.
- Fig. 6. The Feynman (or string loop) diagrams contained in the Regge-Gribov diagram of Fig. 3: (a) internal-internal; (b) internal-external; (c) external-external interferences.
- Fig. 7. Diagrammatic representation of: (a) the six-denominator quantity of Eq. (A3); (b), (c), (d), (e) the quantities I_i of Eq. (5.7).
- Fig. 8. (a) Representation of the phase-space integral encountered at leading order; (b) general recurrence relation in the brick-wall frame.

FIGURE 1

FIGURE 2

FIGURE 3

(a)

(b)

FIGURE 4

FIGURE 5

FIGURE 6

FIGURE 7

FIGURE 8