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w_∞ Gravity

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ABSTRACT

We construct a gauge theory of w_∞ gravity coupled to scalars. For scalar fields valued in some finite-dimensional Lie algebra, a complete realisation of gauged w_∞ can be given using a number of gauge fields equal to the rank of the Lie algebra. We show how the theory can be truncated to describe a gauge theory of W_N gravity, and we give the explicit result for W_3 . We show that the transformations of the scalar fields can be interpreted as a nonlinear realisation on the coset $w_{1+\infty}/w_\infty$.

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1. Introduction

The recent upsurge of interest in two-dimensional gravity was stimulated by the realisation that it could be related to certain exactly-solvable matrix models [1]. One reason for the interest in this subject is that it provides a way of studying string theories away from their critical dimensions. The Virasoro algebra is an underlying symmetry of two-dimensional gravity, whether or not the system is conformally invariant. It is known that there exist extensions of the Virasoro algebra, known as W_N algebras, that contain generators with spins $s = 2, 3, \dots, N$ [2]. It is therefore natural to look for extended two-dimensional systems that provide realisations of W_N symmetries. In fact, it has been suggested that some of the solutions of multi-matrix models may be associated to such higher-spin W -gravity theories [3]. Since these connections are not completely established, and indeed it is not yet clear precisely what W -gravity is, it is therefore of interest to investigate these issues further.

A recent advance in the construction of W -gravity theories was made by Hull [4], who began from a free action realising a chiral W_3 symmetry, and Noether coupled it to spin-2 and spin-3 background gauge fields. In this case it turns out that Noether coupling is by itself sufficient to complete the gauging. This work was extended by Schoutens, Sevinn and van Nieuwenhuizen [5], who constructed a non-chiral gauged theory of W_3 , in which both the left-handed and right-handed W_3 symmetries of the original action are made local. In this case the construction proves to be more subtle, and involves the definition of a nested set of covariant derivatives.

A so-far unrelated development has been the construction of $N \rightarrow \infty$ limits of W_N algebras. There are, of course, inequivalent ways of taking such a limit, which therefore lead to different algebras. The first of these that was discussed in the literature [6], which we denote by w_∞ , has a very simple algebraic structure. It is related [6] to the algebra of area-preserving diffeomorphisms on a cylinder [7]. Unlike the W_N algebras, w_∞ does not admit a central extension except in the Virasoro subsector. There is a more general $N = \infty$ algebra [7, 8], which we denote by W_∞ , that contains w_∞ as a contraction. W_∞ admits central terms for all conformal spins. Both w_∞ and W_∞ are linear algebras, unlike the finite- N W_N algebras.

Although they are much larger algebras, the linearity of w_∞ and W_∞ means that one might expect that their realisations as local symmetries of gauge theories could be simpler than in the case of the non-linear W_N algebra. In this paper we construct chiral and non-chiral w_∞ gauge-field theories that generalise the results of [4] and [5]. For scalar fields valued in some finite-dimensional Lie algebra $\mathcal{G}$, we shall find that a complete realisation of gauged w_∞ can be given using a number of gauge fields equal to the rank of $\mathcal{G}$. When $\mathcal{G} = SU(N)$ or $SL(N, R)$, we show that the theory can be truncated to a gauge theory of W_N . We give explicit results for the case of W_3 .

The w_∞ algebra can be enlarged to include generators of conformal spin 1. The resulting algebra, which we call $w_{1+\infty}$, can be utilised in a geometric description of the non-linear realisation of the w_∞ field theory. Specifically, the fundamental matter field of the theory is represented as the coordinates of the coset $w_{1+\infty}/w_\infty$. The algebra W_∞ can also be enlarged, to $W_{1+\infty}$, yielding a geometric interpretation of the non-linear realisation of the W_∞ algebra. In this case, the construction of an action that realises this symmetry remains an open problem. For the non-chiral w_∞ , there is an intriguing possible extension of this work in terms of symplectic diffeomorphisms of four-dimensional manifolds, which we mention in

the conclusions.

2. The chiral w_∞ theory

Our starting point is a free Lagrangian for a scalar field φ that takes its values in some Lie algebra $\mathcal{G}$:

$$\mathcal{L} = \frac{1}{2} \text{tr}(\partial_+ \varphi \partial_- \varphi), \quad (1)$$

where $x^\pm$ are the coordinates of the two-dimensional space. Here, and in the rest of the paper, we take the trace to be in the fundamental representation of $\mathcal{G}$. In practice, we shall primarily be concerned with the cases where the group $\mathcal{G}$ is $U(1)$ or $SU(N)$, or else the corresponding non-compact forms R or $SL(N, R)$. It is easy to see that this action is invariant under the following global transformations:

$$\delta\varphi = k_\ell(\partial_+ \varphi)^{\ell+1}, \quad (2)$$

where the k_ℓ 's are functions of x^+ only. We take the parameters k_ℓ to be singlets of the group $\mathcal{G}$. One can easily check that the transformations (2) close to form the algebra $[\delta(k_\ell), \delta(k_\ell)]\varphi = \delta(k)\varphi$, where $k = (\ell+1)k_\ell\partial_+ k_j - (j+1)k_j\partial_+ k_\ell$. By expanding the transformation parameters in terms of their Fourier modes e^{imx^+} , one recovers the w_∞ algebra:

$$[v_m^j, v_n^\ell] = [(\ell+1)m - (j+1)n]v_{m+n}^{j+\ell}. \quad (3)$$

Here v_m^ℓ is the m -th Fourier mode of the parameter k_ℓ , which corresponds to generators with conformal spin $\ell+2$.

To gauge this symmetry, we now allow the parameters k_ℓ to depend on x^- as well as x^+ . We must also introduce gauge fields A_ℓ , whose leading-order transformations are $\delta A_\ell = \partial_- k_\ell + \dots$, and add terms to the Lagrangian:

$$\mathcal{L} = \frac{1}{2} \text{tr}(\partial_+ \varphi \partial_- \varphi) - \sum_0^\infty \frac{1}{\ell+2} A_\ell \text{tr}(\partial_+ \varphi)^{\ell+2}. \quad (4)$$

Note that the currents $\text{tr}(\partial_+ \varphi)^{\ell+2}$ are all conserved by virtue of the free field equation $\partial_+ \partial_- \varphi = 0$. Varying the Lagrangian, we find it is invariant provided that the gauge fields transform as

$$\delta A_\ell = \partial_- k_\ell - \sum_j [(j+1)A_j \partial_+ k_{\ell-j} - (\ell-j+1)k_{\ell-j} \partial_+ A_j]. \quad (5)$$

A general feature of the above construction for scalar fields φ taking their values in some Lie algebra $\mathcal{G}$ is a sort of 'telescoping' of the number of truly independent gauge fields down to the rank of the group $\mathcal{G}$. We shall illustrate this phenomenon in the case of a single scalar φ , i.e. for $\mathcal{G} = U(1)$ or R .

The action $\int \mathcal{L}$ for the Lagrangian $\mathcal{L}$ of eq. (4) is invariant by construction under the symmetries of the w_∞ algebra (3), and the gauge transformations of the full set of gauge

fields A_ℓ (5) close by construction to form this algebra. But these are not the only local symmetries of the action when there is just one scalar. The action is also invariant under symmetries of the Stueckelberg type:

$$\begin{aligned} \delta A_0 &= - \sum_{\ell \geq 1} \frac{2}{\ell+2} \alpha_\ell (\partial_+ \varphi)^\ell \\ \delta A_\ell &= \alpha_\ell \quad \ell \geq 1. \end{aligned} \quad (6)$$

These express the fact that all the A_ℓ , $\ell \geq 1$ can be redefined away, i.e. the whole tower of gauge fields 'telescopes' down to just that for the Virasoro subalgebra. Moreover, the action is invariant under additional gauge symmetries involving A_0 alone:

$$\delta A_0 = \frac{\ell}{\ell+2} \partial_- \beta_\ell (\partial_+ \varphi)^\ell + \beta_\ell \partial_- (\partial_+ \varphi)^\ell + \frac{\ell}{\ell+2} \partial_+ \gamma_\ell (\partial_+ \varphi)^\ell + \gamma_\ell \partial_+ (\partial_+ \varphi)^\ell. \quad (7)$$

The Stueckelberg symmetries can be fixed by imposing the gauge conditions $A_\ell = 0$ for $\ell \geq 1$. Substituting these gauge conditions into the action we obtain

$$\mathcal{L} = \frac{1}{2} \partial_+ \varphi \partial_- \varphi - \frac{1}{2} A_0 (\partial_+ \varphi)^2. \quad (8)$$

After the gauge choice, the w_∞ symmetries are given by the original ones plus compensating Stueckelberg transformations, whose parameters are determined to be $\alpha_\ell = -\partial_- k_\ell + A_0 \partial_+ k_\ell - (\ell+1)k_\ell \partial_+ A_0$. If we add to them also special β and γ transformations with $\beta_\ell = k_\ell$ and $\gamma_\ell = -A_0 k_\ell$ we see that this realisation of the w_∞ algebra can be considered to be a set of Virasoro transformations with field-dependent parameters:

$$\begin{aligned} \delta\varphi &= k_\ell \partial_+ \varphi \\ \delta A_0 &= \partial_- k + k \partial_+ A_0 - A_0 \partial_+ k \end{aligned} \quad (9)$$

where $k = k_\ell(\partial_+ \varphi)^\ell$.

Thus, for a single scalar field φ , it is sufficient to gauge only the Virasoro subalgebra in order to have a gauged realisation of the full w_∞ algebra. Taken together with the β and γ transformations, we have the enlarged algebra

$$\begin{aligned} [\delta(k_1), \delta(k_2)] &= \delta(k_3 = k_2 \partial_+ k_1 - k_1 \partial_+ k_2) \\ [\delta(k), \delta(\beta_\ell)] &= \delta(\beta_\ell' = -k \partial_+ \beta_\ell + (\ell+1) \beta_\ell \partial_+ k) + \delta(\gamma_\ell = \beta_\ell \partial_- k) \\ [\delta(k), \delta(\gamma_\ell)] &= \delta(\gamma_\ell' = (\ell+2) \gamma_\ell \partial_+ k - k \partial_+ \gamma_\ell). \end{aligned} \quad (10)$$

In order to have a gauged realisation of w_∞ in which the higher gauge fields play a more essential rôle, it is necessary to consider a multi-component model, e.g. with $\mathcal{G} = SU(N)$, or $SL(N, R)$. In this case, there is also a variant of the above telescoping phenomenon. The non-trivial group indices now prevent one from eliminating the first $N-1$ gauge fields. Only for $\ell \geq N-1$ does one run out of independent Casimir invariants, in which case the trace of $(\partial_+ \varphi)^\ell$ factorises into $SU(N)$ invariants of lower order, permitting redefinitions to eliminate the higher gauge fields A_ℓ for $\ell \geq N-1$. The total number of independent currents is given in general by the rank of the group $\mathcal{G}$. To obtain a gauged realisation of w_∞ in which all

the higher gauge fields play an essential rôle, it would be necessary to consider a limit as $N \rightarrow \infty$ of the above models. This could be done, for example, by replacing the adjoint $SU(N)$ matrices by Poisson brackets defined with respect to x^+ and another continuous coordinate, thus realizing the area-preserving diffeomorphisms, which are themselves a form of infinite-spin algebra [9].

3. Non-chiral w_∞

The free Lagrangian (1) is in fact invariant under independent global transformations of the form (2), where k_ℓ are functions only of x^+ and will now be denoted $k_{-\ell}$, and also under $\delta\varphi = k_{+\ell}(\partial_-\varphi)^{\ell+1}$, where the $k_{+\ell}$ are functions only of x^- . Both of these sets of global symmetries can be gauged, by taking the parameters to depend upon both x^+ and x^- , and introducing gauge potentials $A_{\pm\ell}$. The gauged Lagrangian can conveniently be written in first-order form:

$$\mathcal{L} = -\frac{1}{2}\text{tr}(\partial_+\varphi\partial_-\varphi) - \text{tr}(J_+J_-) + \text{tr}\left[(\partial_-\varphi - \sum_{\ell=0}^{\ell+1} \frac{1}{\ell+2} A_{-\ell}(J_+)^{\ell+1})J_+\right] + \text{tr}\left[(\partial_+\varphi - \sum_{\ell=0}^{\ell+1} \frac{1}{\ell+2} A_{+\ell}(J_-)^{\ell+1})J_-\right], \quad (11)$$

where φ takes its values as before in the Lie algebra $\mathcal{G}$. The equations of motion for the auxiliary fields $J_\pm$ therefore give

$$J_\pm = \partial_\pm\varphi - \sum_{\ell=0}^{\ell+1} A_{\pm\ell}(J_\mp)^{\ell+1}. \quad (12)$$

These equations can be thought of as defining a set of nested covariant derivatives generalising those of [5].

It is straightforward to check that this Lagrangian is invariant under the $k_{-\ell}$ and $k_{+\ell}$ gauge transformations

$$\begin{aligned} \delta\varphi &= k_{+\ell}(J_-)^{\ell+1} + k_{-\ell}(J_+)^{\ell+1} \\ \delta A_{\pm\ell} &= \partial_\pm k_{\pm\ell} - \sum_{j=0}^{\ell} [(j+1)A_{\pm j}\partial_\mp k_{\pm\ell-j} - (\ell-j+1)k_{\pm\ell-j}\partial_\mp A_{\pm j}] \\ \delta J_\pm &= \partial_\pm(k_{\mp\ell}(J_\pm)^{\ell+1}). \end{aligned} \quad (13)$$

Note that $J_\pm$ is inert under the $k_\pm$ transformations respectively.

We now turn to a discussion of how these results can be consistently truncated to the W_3 case, corresponding to $G = SU(3)$ or $SL(3, R)$. (The generalisation to W_N will be obvious.) Just as we noted in section 2, the spin 2 and 3 gauge fields will play an essential rôle, while those with spins 4 and above can be set to zero. The most convenient way of doing this, while keeping track of the remaining invariances of the truncated theory, is to observe that there are generalisations of the Stueckelberg symmetries (6) in this case. We shall describe

these for the left-handed gauge fields; all the results hold independently *mutatis mutandis* for the right-handed fields. Specifically, for each $\ell \geq 2$ the gauge field A_ℓ in the Lagrangian (11) multiplies $(\text{tr} J^2 X + (\text{tr} J^3) Y)$, where X or Y are themselves products of $\text{tr} J^2$ and/or $\text{tr} J^3$ factors. (For $SU(3)$, $\text{tr} J^3$ must always be reducible like this.) Thus for $\ell \geq 2$ there is clearly a Stueckelberg symmetry $\delta A_\ell = \alpha_\ell$ together with $\delta A_0 = -\frac{2}{\ell+2}\alpha_\ell X$ and $\delta A_1 = -\frac{2}{\ell+2}\alpha_\ell Y$.

We can use these symmetries to set $A_{\pm\ell} = 0$ for $\ell \geq 2$. If we also set $k_{\pm\ell} = 0$ for $\ell \geq 2$ (since we wish to discuss only the W_3 symmetries; the model in fact still realises the full w_∞ local symmetry), then from (13) we see that $A_{\pm\ell}$, $\ell \geq 3$ remain zero for all $k_{\pm 0}$ and $k_{\pm 1}$ transformations, while $A_{\pm 2}$ remains zero provided we make a compensating $\alpha_{\pm 2}$ transformation given by $\alpha_{+2} = 2A_{+1}\partial_+ k_{+1} - 2k_{+1}\partial_+ A_{+1}$. Since $\text{tr} J^4 = \frac{1}{2}(\text{tr} J^2)^2$, this implies that $\delta A_{\pm 0}$ acquires an additional term of the form $\frac{1}{2}(k_{\pm 1}\partial_\mp A_{\pm 1} - A_{\pm 1}\partial_\mp k_{\pm 1})\text{tr} J_\mp^2$, while the transformation rule for $A_{\pm 1}$ receives no modifications. Putting all this together, we find the following result for the gauge theory of non-chiral W_3 :

$$\mathcal{L} = -\frac{1}{2}\text{tr}(\partial_+\varphi\partial_-\varphi) - \text{tr}(J_+J_-) + \text{tr}\left[(\partial_-\varphi - \frac{1}{2}A_{-0}J_+ - \frac{1}{3}A_{-1}J_+^2)J_+\right] + \text{tr}\left[(\partial_+\varphi - \frac{1}{2}A_{+0}J_- - \frac{1}{3}A_{+1}J_-^2)J_-\right] \quad (14)$$

with the symmetries

$$\begin{aligned} \delta\varphi &= k_{+0}J_- + k_{+1}J_-^2 + k_{-0}J_+ + k_{-1}J_+^2 \\ \delta A_{\pm 0} &= \partial_\pm k_{\pm 0} - A_{\pm 0}\partial_\mp k_{\pm 0} + k_{\pm 0}\partial_\mp A_{\pm 0} + \frac{1}{2}(k_{\pm 1}\partial_\mp A_{\pm 1} - A_{\pm 1}\partial_\mp k_{\pm 1})\text{tr} J_\mp^2 \\ \delta A_{\pm 1} &= \partial_\pm k_{\pm 1} - A_{\pm 0}\partial_\mp k_{\pm 1} + 2k_{\pm 1}\partial_\mp A_{\pm 0} - 2A_{\pm 1}\partial_\mp k_{\pm 0} + k_{\pm 0}\partial_\mp A_{\pm 1} \\ \delta J_\pm &= \partial_\pm(k_{\mp 0}J_\pm + k_{\mp 1}J_\pm^2). \end{aligned} \quad (15)$$

The action (14) also has the following additional β and γ symmetries that generalise (7):

$$\begin{aligned} \delta A_{\pm 0} &= \frac{\ell}{\ell+1}\partial_\pm \beta_\pm (\text{tr} J_\mp^2)^\ell + \beta_\pm \partial_\pm (\text{tr} J_\mp^2)^\ell + \frac{\ell}{\ell+1}\partial_\mp \gamma_\pm (\text{tr} J_\mp^2)^\ell + \gamma_\pm \partial_\mp (\text{tr} J_\mp^2)^\ell \\ \delta A_{\pm 1} &= \frac{\ell}{\ell+1}\partial_\pm \beta_\pm (\text{tr} J_\mp^2)^\ell + \beta_\pm \partial_\pm (\text{tr} J_\mp^2)^\ell + \frac{\ell}{\ell+1}\partial_\mp \gamma_\pm (\text{tr} J_\mp^2)^\ell + \gamma_\pm \partial_\mp (\text{tr} J_\mp^2)^\ell. \end{aligned} \quad (16)$$

The generalisation of our result to W_N is immediate.

Our non-chiral gauged W_3 model appears to be equivalent to that of ref. [5]. We have checked this to all orders in the spin 2 fields, and through cubic order in the spin 3 fields, by eliminating the auxiliary fields $J_\pm$ in our action, and the corresponding auxiliary fields $F_\pm$ in the action of ref. [5], and then comparing order by order in gauge fields. Our gauge fields are related to those of ref. [5] by $A_{-0} = h^{++} - e^{-1}h^{--}(B^{++})^2 + O(B^3)$ and $A_{-1} = e^{-2}B^{+++} + O(B^3)$, where $e \equiv \det(e_\mu^a)$ and the zweibein is defined by $e_\mu^a = \partial_+ x^a - h^{--}\partial_- x^a$ (and *idem* for $+$ and $-$ indices interchanged). Note that if we perform a chiral truncation of our W_3 theory, we get precisely the 8-scalar model of ref. [4].

In the case where there is just a single scalar φ , the discussion of α , β and γ symmetries in section 2 goes through independently for the left and right-handed sectors *mutatis mutandis*, the only change being that $\partial_\pm\varphi$ should be replaced by $J_\pm$ in the transformation rules. Note that $J_\pm$ is inert under these symmetries. The commutators of purely left-handed, and of purely right-handed, transformations are of the same form as in equation (10). Moreover, left-handed and right-handed transformations commute. As in the chiral case, therefore,

we see that the Stueckelberg parameters $\alpha_{\pm\ell}$ can be used to set all the gauge fields with spins ≥ 3 to zero, in the case where there is just one scalar field φ . In this case the Lagrangian, after eliminating auxiliary fields, can be cast into the more manifestly covariant form $\mathcal{L} = \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi$, where $g^{\mu\nu} = \frac{1}{2}(e_+^\mu e_-^\nu + e_-^\mu e_+^\nu)$ and $e = \det(e_\mu^\alpha) = (1 - A_{-04,0})^{-1}$.

4. Geometrical interpretation

The manner in which the w_∞ algebra is realised on the φ and A_ℓ fields can be described elegantly in terms of a coset construction which provides a geometrical picture of this non-linear realisation. Our starting point is to observe that the w_∞ algebra (3), with generators v_ℓ' for conformal spins $\ell+2 \geq 2$, can be enlarged to an algebra that we denote by $w_{1+\infty}$, by including also generators of conformal spin 1, i.e. v_ℓ' with $\ell = -1$.

The $w_{1+\infty}$ algebra can be linearly realised as the algebra of area-preserving diffeomorphisms of a cylinder. To see this, we observe that the generators v_ℓ' can be represented by functions on a two-dimensional space (x, y) ,

$$v_m' = -ie^{imx}y^{\ell+1}, \quad (17)$$

with the bracket in (3) being represented by the Poisson bracket

$$\{f, g\} = \partial_x f \partial_y g - \partial_y f \partial_x g. \quad (18)$$

The functions (17) are complete on the product of a circle and the real line, i.e. a cylinder. One can also represent the $w_{1+\infty}$ algebra in terms of the corresponding Hamiltonian vector fields $\hat{v}_m'$, defined by $\hat{v}_m' = \{v_m', \cdot\}$, i.e.

$$\hat{v}_m' = e^{imx}(my^{\ell+1}\partial_y + i(\ell+1)y^\ell\partial_x). \quad (19)$$

In this representation, the bracket in (3) is just the commutator of vector fields.

For simplicity, we shall describe the coset construction for the case of a single scalar field φ . We can view this field as the coordinates of the coset space $G/H = w_{1+\infty}/w_\infty$. Thus we may construct a coset representative by writing down a non-infinitesimal area-preserving transformation, and then replacing the parameter of the transformation by the field φ . For a general non-infinitesimal area-preserving transformation, the action on a scalar field $f(x, y)$ is given by

$$f \rightarrow \tilde{f} = e^{A\Delta_\Lambda} f = f + \{\Lambda, f\} + \frac{1}{2!}\{\Lambda, \{\Lambda, f\}\} + \dots \quad (20)$$

From (17), we see that the coset generators correspond to y -independent functions, and so the coset representative is obtained by setting $\Lambda(x, y) = \varphi(x)$ in (20). In terms of Hamiltonian vector fields, the coset representative is $\exp(\varphi'\partial_y)$, where $\varphi' = \partial_x\varphi$.

If we denote the coset generators by K , and the subalgebra generators by H , then one can see from (3) that the structure of the algebra is $[H, H] \subset H$, $[H, K] \subset H + K$, $[K, K] = 0$. Thus the coset $w_{1+\infty}/w_\infty$ is not a symmetric space, and the K generators do not even form a linear representation of H (except when the subalgebra element lies in the Virasoro

subalgebra). Although we can apply the general theory of non-linear realisations to this situation, we should not be surprised to find that some of the features are non-standard. In particular, the action of H on the coset will in general be non-linear.

Acting on the coset representative with a general G transformation g , one has

$$ge^{\varphi'\partial_y} = e^{\tilde{\varphi}'\partial_y}\tilde{h}. \quad (21)$$

For an infinitesimal transformation $g = 1 + \hat{\lambda}$, one therefore finds

$$e^{-\varphi'\partial_y}\hat{\lambda}e^{\varphi'\partial_y} = \delta\varphi' + \hat{\lambda}, \quad (22)$$

where $\hat{\lambda} \subset H$ and $\delta\varphi' = \tilde{\varphi}' - \varphi'$. In order to find $\delta\varphi'$ in (22), it is necessary to remove systematically the terms on the left-hand side that involve the subalgebra generators; the latter go to form $\hat{\lambda}$. Thus, taking $\lambda = k_\ell(x)y^{\ell+1}$, and so $\hat{\lambda} = k_\ell'y^{\ell+1}\partial_y - (\ell+1)k_\ell y^\ell\partial_x$, we find $\delta\varphi' = (k_\ell(\varphi'))^{\ell+1}$. Hence we have

$$\delta\varphi = k_\ell(\varphi')^{\ell+1}. \quad (23)$$

For $\ell \geq 0$ this precisely reproduces the realisation of w_∞ given in equation (2) (with $G = U(1)$ or R), where x is taken to be x^+ . When $\ell = -1$, the coset representative transforms inhomogeneously, $\delta\varphi = k_{-1}$; this is also a symmetry of the action (1). The result (23) can also be derived in the Poisson bracket language; in this case, the projection onto the coset generators is effected by setting $y \rightarrow 0$, so corresponding to (22) we have

$$\delta\varphi = (e^{-\text{Ad}_\varphi}\lambda)|_{y=0}, \quad (24)$$

where $\lambda = k_\ell y^{\ell+1}$.

In the case where φ takes its values in some non-abelian Lie algebra $\mathcal{G}$, we shall take over equation (24) unchanged, except that now φ is taken to be a matrix. This result is the same as one obtains from the above coset construction, provided that one works with the Poisson-bracket realisation of the algebra rather than the Hamiltonian-vector field realisation. (This is because the ordering of the coset generators is preserved in the Poisson brackets, unlike in the commutators.)

The W_∞ algebra [7], which contains w_∞ as a contraction, may also be realised non-linearly on a single scalar field by the use of a coset construction. Here too, the algebra may be enlarged to $W_{1+\infty}$ by the addition of generators with conformal spin 1 [8,10]. The $W_{1+\infty}$ algebra may be realised as the algebra of all smooth differential operators (of arbitrary degree) on the circle [11], and this provides us with a convenient starting point for a coset construction. The generators of conformal spin 1 correspond to the functions, whilst those of conformal spin 2 and above correspond to differential operators of degree ≥ 1 (with no non-derivative terms). The details of the realisation are described in ref. [11]. Thus we may take the coset representative for $G/H = W_{1+\infty}/W_\infty$ to be $e^{\varphi(x)}$, where x is the coordinate on the circle. From [11], the differential operators describing infinitesimal transformations

corresponding to the first few conformal spins are:

$$\begin{aligned}\lambda_{-1} &= k_{-1} \\ \lambda_0 &= k_0 \partial_x \\ \lambda_1 &= k_1 \partial_x^2 + \frac{1}{2} k_1' \partial_x \\ \lambda_2 &= k_2 \partial_x^3 + k_2' \partial_x^2 + \frac{1}{5} (k_2'' + k_2) \partial_x.\end{aligned}\quad (25)$$

Acting on the coset representative $e^{\varphi(x)}$, we obtain the result

$$e^{-\varphi} \lambda e^{\varphi} = \delta \varphi + \bar{\lambda}, \quad (26)$$

where $\bar{\lambda} \in H$. Thus the transformations of φ corresponding to (25) are:

$$\begin{aligned}\delta_{-1} \varphi &= k_{-1} \\ \delta_0 \varphi &= k_0 \varphi' \\ \delta_1 \varphi &= k_1 \varphi'' + \frac{1}{2} k_1' \varphi' + k_1 \varphi'^2 \\ \delta_2 \varphi &= k_2 \varphi''' + k_2' \varphi'' + \frac{1}{5} (k_2'' + k_2) \varphi' + 3k_2 \varphi' \varphi'' + k_2' \varphi'^2 + k_2 \varphi'^3.\end{aligned}\quad (27)$$

Note that the term of highest degree in φ' in each of the above equations coincides with the corresponding transformation rule in the w_∞ algebra given in equation (2). This is a reflection of the fact that w_∞ can be obtained from W_∞ by a process of contraction. Indeed, we could have introduced a parameter q in the realisation, corresponding to the one occurring in refs [7,8,10,11], which would give W_∞ when $q \neq 0$ and w_∞ when $q = 0$. The lower-order terms in (27) would all be multiplied by powers of q , and thus will disappear when q is set to zero.

There are several points that remain to be clarified concerning the realisation of the W_∞ algebra. One is that it is not immediately clear how to extend the above discussion to the case with more than one scalar field φ . Unlike the realisation of w_∞ , where we had the Poisson brackets that could be used to realise the $w_{1+\infty}/w_\infty$ structure without changing the order of the φ fields, the discussion of W_∞ involves commutators, which requires that one take into account the non-commutativity of the φ . A second problem that remains to be clarified is whether there is an action that is invariant under the above realisation of W_∞ . The usual Cartan construction of invariant differential forms is not by itself sufficient to make covariant derivatives because the $\partial_\pm$ derivatives are active operators in the W_∞ algebra. Nonetheless, the simple $\int \partial_+ \varphi \partial_- \varphi$ construction is invariant for the global w_∞ realisation, and although this is not itself invariant under the W_∞ realisation, one might still hope that an action may be found in this case.

5. Conclusions

In this paper, we have studied some issues in w_∞ gravity and have seen the emergence of a natural geometrical framework for its description in the coset construction of section 4.

Clearly, there are a number of directions in which this subject can be further developed. We should like to conclude briefly with an indication of some of them.

It is not too difficult to include fermions into the w_∞ -invariant action that we have given. For a single scalar field φ , one may include a single fermion ψ , with the Lagrangian now given by

$$\mathcal{L} = \frac{1}{2} \partial_+ \varphi \partial_- \varphi + \frac{1}{2} \psi \partial_- \psi - \frac{1}{2} A_0 \{ (\partial_+ \varphi)^2 + \psi \partial_+ \psi \} \quad (28)$$

and the Virasoro algebra is realised by

$$\begin{aligned}\delta \varphi &= k \partial_+ \varphi \\ \delta \psi &= k \partial_+ \psi + \frac{1}{2} (\partial_+ k) \psi \\ \delta A_0 &= \partial_- k + k \partial_+ A_0 - A_0 \partial_+ k.\end{aligned}\quad (29)$$

The β symmetry of eq. (7) now generalises to $\delta A_0 = \frac{L}{L+2} \partial_- \beta_L T_L + \beta_L \partial_- T_L$, with $T_L = (\partial_+ \varphi)^L + \frac{L}{2} \psi \partial_+ \psi (\partial_+ \varphi)^{L-2}$. The full set of w_∞ transformations is now given, analogously to (9), for $k = k_L T_L$.

Supersymmetry is clearly an element that could be added to this picture. Indeed, one might think that the existence of a supersymmetric extension to the w_∞ algebra [12,13] might of itself imply that one can straightforwardly supersymmetrise the w_∞ realisation given here. Unfortunately, this proves not to be the case; it does not seem possible to supersymmetrise directly even the global realisation (2) without involving an infinite number of negative conformal spins. The basic problem seems to be that although w_∞ can be supersymmetrised by the introduction of fermionic generators with spins $s \geq \frac{3}{2}$, the supersymmetrisation of $w_{1+\infty}$ necessarily involves an infinite number of negative conformal spins.

Another way in which the work of this paper can be extended is to construct kinetic terms for the gauge fields. Hopefully, the geometric construction of the w_∞ transformations can be used to motivate some higher-spin analogue of a two-dimensional gravity action. Furthermore, all the results so far are in the light-cone gauge, and it would be desirable to covariantise them. It is conceivable that some of the scalar fields, for example those that lie in the Cartan subalgebra of $\mathcal{G}$, may play an important rôle. Indeed, W_N can be realised on just the $N-1$ scalar fields corresponding to the Cartan subalgebra of $SU(N)$ or $SL(N, R)$, generalising the 2-scalar W_3 model in [4]. It may be that this realisation is related to the Toda-type W_N gravity [3].

In the case of the non-chiral bosonic w_∞ model that we discussed earlier, the fact that $w_{1+\infty}$ can be realised as the area-preserving algebra of a cylinder suggests the intriguing possibility of giving a four-dimensional interpretation of the model [14]. Since in our construction x^+ and x^- are both taken to be coordinates on circles, it follows that the two-dimensional "spacetime" is a 2-torus. In order to realise the left-handed and right-handed copies of w_∞ as described in section 4, we must introduce two additional coordinates, say y and $\bar{y}$, on the axes of the two copies of the cylinder. Thus we have a total of four coordinates, $(x^+, x^-, y, \bar{y})$. The area-preserving transformations on the two cylinders may be described by introducing the symplectic form

$$\Omega = dx^+ \wedge dy - dx^- \wedge d\bar{y}, \quad (30)$$

and using this to define a Poisson bracket $\{f, g\} = \Omega^{ij} \partial_i f \partial_j g$ in the four-dimensional space. The coordinates y and $\bar{y}$ play the rôle of "momenta" conjugate to x^+ and x^- , and so one can

think of the four-dimensional space as being the cotangent bundle $T^*(\Sigma)$ of the 2-torus Σ . $T^*(\Sigma)$ is a complex manifold, for which we may choose coordinates $z^1 = x^+ + ix^-$, $z^2 = y + iy$. In terms of these, Ω is the $(0,2) + (2,0)$ form

$$\Omega = \frac{1}{2}(dz^1 \wedge dz^2 + d\bar{z}^1 \wedge d\bar{z}^2). \quad (31)$$

Symplectic diffeomorphisms on $T^*(\Sigma)$, $\delta x^i = \{x^i, \kappa\}$, preserve the form of Ω , but deform the complex structure. Global non-chiral w_∞ transformations correspond to restricting κ to have the form $k_+ \epsilon(x^+) \bar{y}^{\ell+1} + k_- \epsilon(x^-) \bar{y}^{\ell+1}$. Their action on φ is given by (24), where $\text{Ad}_\varphi \kappa$ is now taken to be $\Omega^i \partial_i \varphi \partial_j \kappa$. If our construction of the non-chiral action were covariant, rather than being in a "light-cone" gauge, we should expect the action to have a larger group of invariances. A natural conjecture would be that it should include the algebra of general four-dimensional symplectic diffeomorphisms [14].

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