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HELIOSEISMIC MEASUREMENTS OF THE SOLAR INTERNAL ROTATION

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Helioseismic measurements of the solar internal rotation.

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1 Introduction.

The rotation of the solar interior plays a major role in understanding the dynamics of the solar convection zone, and the generation of the solar magnetic field. Furthermore, it is possible that spin-down from an initial rapid rotation in the early phases of solar history might have led to hydrodynamical instabilities that could have caused mixing and hence affected the evolution of the Sun. Finally, it has been speculated that parts of the Sun may rotate so rapidly as to cause a centrifugal distortion, and hence a perturbation of the gravitational field outside the Sun, of sufficient magnitude to have a significant effect on the motion of Mercury; this could invalidate tests of the theory of general relativity based on the precession of the perihelium of Mercury's orbit. However, before the advent of helioseismology, essentially nothing was known about the rotation beneath the surface of the Sun.

As discussed by Stix (these proceedings) detailed observations have been made of the rotation of the solar surface, and its variation with latitude. Somewhat different values are obtained from Doppler observations of the surface plasma motion, and from observations of the advection of surface magnetic features. A typical result for the plasma rotation rate is

$$\frac{\Omega_p}{2\pi} = 452 - 49\mu^2 - 84\mu^4 \text{ nHz}, \quad (1.1)$$

where $\mu = \cos\theta$ (Snodgrass, 1984), whereas a typical rotation law for magnetic features is

$$\frac{\Omega_m}{2\pi} = 462 - 74\mu^2 - 53\mu^4 \text{ nHz} \quad (1.2)$$

(Snodgrass 1983). It should also be noted that Stenflo (1989) found a very different behaviour, with substantially smaller variation between equator and pole, from a correlation analysis of large-scale magnetic patterns. These differences may be a hint that the angular velocity varies with depth, although more quantitative estimates are difficult (see Schlusser 1987).

It is obviously impossible to carry out realistic numerical modelling of the complicated hydrodynamical processes that determine the rotation of the solar interior.

Simplified (although still quite complex) models of the solar convection zone suggest that rotation should be largely constant on cylinders, i.e. a function mainly of the distance to the rotation axis. Also, models of the solar spin-down indicate that the Sun could contain a rather large region in the radiative interior rotating at several times the surface rate; however, a major uncertainty in such models are the mechanisms that transport angular momentum from the radiative region to the convection zone. For example, even a weak magnetic field may be sufficient to cause an essentially complete redistribution of angular momentum, leading to uniform rotation, within the lifetime of the Sun.

The principal purpose of this review is to demonstrate that it has been possible through observation of solar oscillations to obtain a rather detailed picture of the solar internal rotation, and to present the main results. In section 2 I give a very brief description of the properties of solar oscillations, and discuss the effect of rotation on the frequencies. Section 3 gives an overview of the techniques used to analyse the observed frequencies, and section 4 discusses techniques to invert the observed data to determine the angular velocity as a function of position within the Sun. In section 5 I present the results of the observations, and the inferences about rotation that have been made from them. Finally section 6 discusses the results and, in the light of them, addresses the questions of the origin and effects of the rotation.

2 Rotation and solar oscillations.

2.1 Properties of solar oscillations.

To provide a background for the discussion of the effects of rotation, it is useful to review very briefly some important properties of solar oscillations. More extensive presentations can be found, for example, in the reviews by Deubner & Gough (1984), Christensen-Dalsgaard, Gough & Toomre (1985) and Christensen-Dalsgaard & Berthomieu (1990).

Small-amplitude oscillations of a spherical star can be separated into normal modes, each of which has a harmonic dependence on time, and depends on the spherical coordinates θ and φ (co-latitude and longitude) as a spherical harmonic. Thus the displacement for a single mode can be written

$$\delta \mathbf{r}(\mathbf{r}, \theta, \varphi, t) = \text{Re} \left\{ \left[\hat{\mathbf{r}}_l(\mathbf{r}) Y_l^m(\theta, \varphi) + \hat{\mathbf{r}}_n(\mathbf{r}) \left[\frac{\partial Y_l^m}{\partial \theta} \mathbf{a}_\theta + \frac{1}{\sin \theta} \frac{\partial Y_l^m}{\partial \varphi} \mathbf{a}_\varphi \right] \right] e^{-i\omega t} \right\}, \quad (2.1)$$

where $\mathbf{a}_r$, $\mathbf{a}_\theta$ and $\mathbf{a}_\varphi$ are unit vectors in the $\mathbf{r}$, θ and φ directions. Here $Y_l^m(\theta, \varphi) = c_{lm} P_l^m(\cos \theta) e^{im\varphi}$ is a spherical harmonic, P_l^m being a Legendre function and c_{lm} a normalization constant defined such that the integral of $|Y_l^m|^2$ over the

sphere is unity. The variation of the displacement with the distance r from the centre is determined by the *eigenfunctions* $\xi_r(r)$ and $\xi_\theta(r)$. The mode is characterized by three wavenumbers: n is the *radial order* which, roughly, gives the number of zeros in ξ_r ; ℓ is the *degree*, and m the *azimuthal order* of the mode. The degree of the mode is related to its horizontal wavenumber k_h and wavelength λ at radius r by

$$k_h = \frac{2\pi}{\lambda} = \frac{L}{r}, \quad (2.2)$$

where $L = \sqrt{\ell(\ell+1)}$. As is seen from equation (2.1), m , which is in the range $-\ell$ to ℓ , is twice the number of zeros around the equator.

We neglect processes that damp or excite the oscillations. Then ω is real, and it follows from equation (2.1) that the dependence of the displacement (or, more generally, any quantity associated with the mode) on longitude and time is given by

$$\cos(m\varphi - \omega t). \quad (2.3)$$

Thus for $m = 0$ the mode is a standing wave. Modes with $m > 0$, which are normally called *prograde*, propagate in longitude in the direction of increasing φ , whereas *retrograde* modes, with $m < 0$, propagate in the direction of decreasing φ . When ω is real, the eigenfunctions ξ_r and ξ_θ may also be chosen to be real.

In addition to the *angular frequency* ω , which is used in equation (2.1), the *cyclic frequency* $\nu = \omega/(2\pi) = 1/P$, is commonly used, particularly in discussions of observed frequencies; here P is the oscillation period.

In general the frequency $\omega = \omega_{\ell m}$ depends on all three wavenumbers. However if rotation or other departures from spherical symmetry are ignored, $\omega_{\ell m}$ does not depend on m . This follows from the fact that in this case there is no preferred axis in the star; since m depends on the choice of coordinate axis, the physics of the oscillations, and hence their frequencies, must be independent of m . For slow rotation a modal description as in equation (2.1) is still possible, provided that the rotation axis is chosen as coordinate axis.

The possible modes of oscillation can be separated into three classes: the p , the f and the g modes. The p modes are essentially standing acoustic waves; in solar models they have cyclic frequencies between about 250 and 5000 μHz . The f modes are surface gravity waves, whereas the g modes are standing internal gravity waves; g mode frequencies are below about 450 μHz in normal solar models. In the Sun the only modes that have been definitely observed are the so-called five minute oscillations, with frequencies between about 1500 and 5000 μHz ; these correspond to p modes or, at high degree, also f modes. There have also been reports of oscillations with periods of several hours; if caused by normal modes of the Sun, these must be g modes, but the identification or even the solar origin of these oscillations is still in some doubt.

Because of their high radial order or degree the p modes observed on the Sun can be described with reasonable accuracy by means of asymptotic theory. For the present purpose the main result of such an analysis is that p modes are trapped outside a *turning point*, the location $r = r_t$ of which is determined by

$$\frac{c(r_t)}{r_t} = \frac{\omega}{L}, \quad (2.4)$$

where c is the adiabatic sound speed. This corresponds to a point of total internal reflection; for $r < r_t$ the mode decays exponentially. At the surface the wave is reflected (provided the frequency does not exceed about 5500 μHz) by the steep density gradient. Thus the wave propagates in a series of "bounces" between the surface and the turning point. A mode of oscillation, which is a standing wave, is formed as an interference pattern between such bouncing waves. At low degree r_t as determined from equation (2.3) is small; the modes penetrate essentially to the solar centre, and hence their frequencies carry information about the entire Sun. However, the amplitudes are largest near the solar surface (see also Figure 1), and hence some care is required to extract that part of the information which derives from the deep solar interior.

The g modes are trapped beneath the solar convection zone, and have large amplitudes in the solar core. This makes them potentially very valuable as probes of conditions in the deep solar interior.

2.2 Rotational splitting.

We assume that the rotation is sufficiently slow to allow us to neglect the rotational distortion of the star, or more generally terms of order Ω^2 , where Ω is the angular velocity. Then the structure of the star is unchanged, and the effects of rotation can be treated as a small perturbation in the equations describing the oscillations.

That rotation causes a splitting of the frequencies can be seen from a purely geometrical argument. Assume the angular velocity Ω to be uniform, and consider an oscillation with a frequency ω_0 , independent of m , in the frame rotating with the star. I introduce a coordinate system in this frame, with coordinates (r', θ', φ') which are related to the coordinates (r, θ, φ) in an inertial frame through

$$(r', \theta', \varphi') = (r, \theta, \varphi - \Omega t). \quad (2.5)$$

Since in the rotating frame the perturbations depend on φ' and t as $\cos(m\varphi' - \omega_0 t)$ (cf. equation (2.3)), the dependence in the inertial frame is $\cos(m\varphi - \omega_m t)$, where

$$\omega_m = \omega_0 + m\Omega. \quad (2.6)$$

Thus an observer in the inertial frame finds that the frequency is split uniformly according to m .

This description is obviously incomplete. Even in the case of uniform rotation the effects of the Coriolis force must be taken into account in the rotating frame, causing a contribution to the frequency splitting (Cowling & Newing 1949; Ledoux 1949). Furthermore, in general the angular velocity is a function $\Omega(r, \theta)$ of position. Nevertheless, as shown below, the effect of the Coriolis force is often small, and equation (2.6) is approximately correct if Ω is replaced by a suitable average of the position-dependent angular velocity.

The general case was considered by Gough (1981), Hansen, Cox & van Horn (1977) and Cuyper (1980). The result is that the *rotational splitting*, i.e. the perturbation in the frequencies caused by rotation, can be written

$$\delta\omega_{n, \ell m} = m \frac{R_{n, \ell m}}{I_{n, \ell m}}, \quad (2.7)$$

where

$$R_{n, \ell m} = \int_0^R \sin\theta d\theta \int_0^R \left\{ \xi_r^2 P_\ell^m(\cos\theta)^2 + \xi_h^2 \left[\left(\frac{dP_\ell^m}{d\theta} \right)^2 + \frac{m^2}{\sin^2\theta} P_\ell^m(\cos\theta)^2 \right] - 2P_\ell^m(\cos\theta)^2 \xi_r \xi_h - 2P_\ell^m(\cos\theta) \frac{dP_\ell^m}{d\theta} \cos\theta \xi_h^2 \right\} \Omega(r, \theta) r^2 \rho dr - 2P_\ell^m(\cos\theta)^2 \xi_r \xi_h - 2P_\ell^m(\cos\theta) \frac{dP_\ell^m}{d\theta} \cos\theta \xi_h^2 \left\{ \Omega(r, \theta) r^2 \rho dr \right. \quad (2.8)$$

and

$$I_{n, \ell m} = \frac{2}{2\ell + 1} \left(\ell + \frac{|m|}{2} \right) \int_0^R \left[\xi_r^2 + \ell(\ell+1) \xi_h^2 \right] \rho(r) r^2 dr; \quad (2.9)$$

here ρ is the density.

2.3 Splitting for spherically symmetric rotation.

To proceed an explicit assumption about the variation of Ω with θ is required. For simplicity I shall first assume that Ω is independent of θ . In this case the integrals over θ in equation (2.8) only involve Legendre functions and may be evaluated analytically. The result is

$$\delta\omega_{n, \ell m} = m \frac{\int_0^R \Omega(r) \left[\xi_r^2 + L^2 \xi_h^2 - 2\xi_r \xi_h - \xi_h^2 \right] r^2 \rho dr}{\int_0^R \left[\xi_r^2 + L^2 \xi_h^2 \right] r^2 \rho dr}, \quad (2.10)$$

where I have introduced $L^2 \equiv \ell(\ell+1)$. It should be noticed that the integrands in equation (2.10) are given solely in terms of ξ_r , ξ_h and L , and therefore are independent of m . Hence in the case of spherically symmetric rotation, the rotational splitting is proportional to m .

It is convenient to write equation (2.10) as

$$\delta\omega_{n, \ell m} = m \beta_{n, \ell} \int_0^R K_{n, \ell}(r) \Omega(r) dr, \quad (2.11)$$

where

$$K_{n, \ell} = \frac{\left[\xi_r^2 + L^2 \xi_h^2 - 2\xi_r \xi_h - \xi_h^2 \right] r^2 \rho}{\int_0^R \left[\xi_r^2 + L^2 \xi_h^2 - 2\xi_r \xi_h - \xi_h^2 \right] r^2 \rho dr}, \quad (2.12)$$

and

$$\beta_{n, \ell} = \frac{\int_0^R \left[\xi_r^2 + L^2 \xi_h^2 - 2\xi_r \xi_h - \xi_h^2 \right] r^2 \rho dr}{\int_0^R \left[\xi_r^2 + L^2 \xi_h^2 \right] r^2 \rho dr}. \quad (2.13)$$

By using this definition it is ensured that the *rotational kernel* $K_{n, \ell}$ is unimodular, i.e.

$$\int_0^R K_{n, \ell}(r) dr = 1. \quad (2.14)$$

Hence for uniform rotation, where $\Omega = \Omega_0$ is constant,

$$\delta\omega_{n, \ell m} = m \beta_{n, \ell} \Omega_0. \quad (2.15)$$

In this case the effect of rotation is completely given by $\beta_{n, \ell}$. For high-order or high-degree p modes the terms in ξ_r^2 and $L^2 \xi_h^2$ dominate, and $\beta_{n, \ell} \approx 1$. Thus the rotational splitting between adjacent m -values is given simply by the angular velocity, as in equation (2.6). Physically, the neglected terms in equation (2.13) do indeed arise from the Coriolis force. For high-order g modes, on the other hand, the displacement is predominantly horizontal and the terms containing ξ_r can be neglected; hence

$$\beta_{n, \ell} \approx 1 - \frac{1}{L^2}. \quad (2.16)$$

In particular, the splitting of high-order g modes of degree 1 is only half the angular velocity.

Returning to the case where Ω depends on r , it should be noted that the integral in equation (2.11) provides a weighted average

$$\langle \Omega \rangle = \int_0^R K_{n, \ell}(r) \Omega(r) dr \quad (2.17)$$

of $\Omega(r)$. For high-order p modes one can neglect the terms in ξ_h in equation (2.12) and use that, according to the asymptotic theory for p modes $\xi_r \sim (\rho c)^{1/2} r^{-1}$ outside the turning point. The result is

$$\delta\omega_{n, \ell m} \approx m \langle \Omega \rangle \approx m \frac{\int_{r_1}^R \Omega(r) \frac{dr}{c}}{\int_{r_1}^R \frac{dr}{c}}. \quad (2.18)$$

Thus in this case we obtain the intuitively appealing result that the rotational splitting is an average of the angular velocity, weighted by the sound travel time. A

more accurate expression can be obtained by including the term in $L^2 \xi_b^2$ and using the complete asymptotic expressions for ξ_r and ξ_b , or from ray theory (Gough 1984).

Figure 1 shows a few examples of kernels for high-order p modes. They are clearly large near the solar surface, as is also implicit in equation (2.18). Beneath the turning point the kernels get very small, but they are locally enhanced just above it. This effect arises from the term in ξ_b in equation (2.12); physically it corresponds to the fact that the waves travel approximately horizontally in this region, and hence spend a relatively long time there.

A kernel for a high-order g mode is shown in Figure 2. It is evident that observations of such modes would provide sensitive measures of the rotation of the

Figure 1. Kernels $K_{n,l}$ for the frequency splitting in p modes caused by spherically symmetric rotation (cf. equation (2.12)). In a) is plotted $RK_{n,l}(r)$ for a mode with $l=1$, $n=22$ and $\nu=3233 \mu\text{Hz}$. The maximum value of $RK_{n,l}(r)$ is 57. In b) is shown the same mode, on an expanded vertical scale, together with the modes $l=20$, $n=17$, $\nu=3367 \mu\text{Hz}$ (-----), and $l=60$, $n=10$, $\nu=3231 \mu\text{Hz}$ (-----). Notice that the kernels almost vanish inside the turning point radius r_1 , and that there is an accumulation just outside the turning point.

solar core. Unfortunately definite detection, let alone identification, of g modes in the Sun has yet to be made.

To get an impression of the contribution to the splitting made by different parts of the Sun, it is instructive to consider the integrated kernel

$$\int_0^1 K_{n,l}(r') dr' \quad (2.19)$$

Examples are shown in Figure 3 for a p and a g mode. In the p-mode case only about 1 per cent of the contribution comes from the inner 10 per cent of the radius, whereas the corresponding number for the g mode is about 45 per cent. This further emphasizes the difficulty in measuring the solar core rotation with p modes, and the great potential importance of g-mode observations.

Figure 2. Kernel $K_{n,l}$ for the frequency splitting caused by spherically symmetric rotation (cf. equation (2.12)), in a g mode with $l=2$, $n=10$ and $\nu=101 \mu\text{Hz}$.

Figure 3. Integrals of kernels for spherically symmetric rotation. The solid curve is for a p mode with $\ell = 1$, $n = 21$ and $\nu = 3096 \mu\text{Hz}$; the dashed curve is for a g mode with $\ell = 2$, $n = 10$ and $\nu = 101 \mu\text{Hz}$.

2.4 General rotation laws.

In the general case, where Ω depends on both r and θ , the rotational splitting may be computed from equations (2.7) - (2.9), by evaluating the two-dimensional integral in equation (2.4a). This integral is in general m -dependent, and so the splitting is no longer a linear function of m . To illustrate the properties of the splitting, I consider again the case of high-order p modes, keeping only the term in ξ_1^2 and approximating it as in the derivation of equation (2.18). This yields

$$\delta\omega_{\ell m} = m \frac{\int_0^\pi \sin\theta \left[P_\ell^m(\cos\theta) \right]^2 \int_{r_1}^R \Omega(r, \theta) \frac{dr}{c} d\theta}{\int_0^\pi \sin\theta \left[P_\ell^m(\cos\theta) \right]^2 d\theta \int_{r_1}^R \frac{dr}{c}}. \quad (2.20)$$

Thus the rotational splitting is once again a weighted average of the angular velocity, but now the weight depends both on the sound travel time and on the distribution of oscillation energy with latitude, as given by $(P_\ell^m)^2$. With increasing ℓ the sectoral modes, with $\ell = |m|$, get increasingly confined towards the equator. Thus,

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the rotational splitting of the sectoral modes provides a measure of the solar equatorial angular velocity. For general m , P_ℓ^m is confined approximately between latitudes $\pm \sin^{-1}[(1-m^2/L^2)^{1/2}]$. The variation of the extent of the P_ℓ^m with m allows resolution of the latitudinal variation of the angular velocity, in much the same way as the variation of the depth of penetration with ω/L allows resolution of the variation with radius.

To study the splitting without making the asymptotic approximation, it is convenient to consider a parameterized representation of $\Omega(r, \theta)$. Here I use an expansion of the form

$$\Omega(r, \theta) = \sum_{s=0}^{\infty} \Omega_s(r) \mu^{2s}, \quad (2.21)$$

where $\mu = \cos\theta$. Then the integrals over θ can be evaluated analytically, and the rotational splitting becomes (see also Cuypers 1980)

$$\delta\omega_{\ell m} = m \sum_{s=0}^{\infty} \int_0^R K_{\ell m s}(r) \Omega_s(r) dr. \quad (2.22)$$

Here

$$K_{\ell m s}(r) = \rho^2 I_{n, \frac{1}{2}}^{-1} \left[(\xi_1 - \xi_h)^2 + (L^2 - 2s^2 - 3s - 2)\xi_1^2 \right] Q_{\ell m} \quad (2.23)$$

$$+ s(2s-1)\xi_h^2 Q_{\ell m s-1} \Big\},$$

where

$$I_{n, s} = \int_0^R \rho^2 \left[\xi_1^2 + L^2 \xi_h^2 \right] dr, \quad (2.24)$$

and

$$Q_{\ell m s} = \frac{2\ell+1}{2} \frac{(\ell-|m|)!}{(\ell+|m|)!} \int_{-1}^1 \mu^{2s} [P_\ell^m(\mu)]^2 d\mu. \quad (2.25)$$

For spherically symmetric rotation, $\Omega_s = 0$ for $s > 0$. Since $Q_{\ell m 0} = 1$, the kernel $K_{\ell m 0}(r) = \rho_{\ell \ell} K_{\ell \ell}(r)$ is independent of m , and the splitting is uniform in m . Thus the results of section 2.3 are recovered.

It may be shown that the factor $Q_{\ell m s}$ is a polynomial in m^2 of degree s . Hence to equation (2.22) corresponds an expansion of the rotational splitting in odd powers of m . As discussed in section 3 it is convenient, from the point of view of data analysis, to write this in terms of Legendre polynomials as

$$\delta\omega_{\ell m} = \ell \sum_{j=0}^{\infty} \Delta\omega_{\ell \ell} P_{2j+1} \left[\frac{m}{\ell} \right]. \quad (2.26)$$

Here the expansion coefficients $\Delta\omega_{\ell \ell}$ of the splitting are related to the expansion coefficients Ω_s of the angular velocity through

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$$\Delta^1 \omega_{\ell, \ell} = \sum_{s=1}^{\max} \int_0^R K_{\ell, \ell, s}^1(r) \Omega_{\ell}(r) dr. \quad (2.27)$$

The kernels $K_{\ell, \ell, s}^1$ are obtained from expressions similar to equation (2.23), with the $Q_{\ell, ms}$ replaced by expansion coefficients of $Q_{\ell, ms}$ in terms of Legendre polynomials (see Brown *et al.* 1989 for details).

In the limit of large ℓ , the terms ξ_r^2 and $L^2 \xi_h^2$ dominate in equation (2.23). As in the case of spherically symmetric rotation, the effects of Coriolis forces are small and the rotational splitting is simply an average of $\Omega(r, \theta)$. Then the kernels $K_{\ell, \ell, s}^1(r)$ in equation (2.27) are approximately proportional to

$$H_{\ell, \ell, s}(r) = I_{\ell, \ell}^{-1} [\xi_r^2 + L^2 \xi_h^2] \rho(r) r^2; \quad (2.28)$$

note that this is also the expression for the kernel $K_{\ell, \ell}$ for spherically symmetric rotation in this limit (cf. equation (2.12)). As shown by Brown *et al.* equations (2.27) may be approximated by

$$\begin{aligned} \Delta^0 \omega_{\ell, \ell} &\approx \int_0^R \left[\Omega_0(r) + \frac{1}{5} \Omega_1(r) + \frac{3}{35} \Omega_2(r) + \dots \right] H_{\ell, \ell}(r) dr, \\ \Delta^1 \omega_{\ell, \ell} &\approx -\frac{1}{5} \int_0^R \left[\Omega_1(r) + \frac{2}{3} \Omega_2(r) + \dots \right] H_{\ell, \ell}(r) dr, \\ \Delta^2 \omega_{\ell, \ell} &\approx \frac{1}{21} \int_0^R \left[\Omega_2(r) + \dots \right] H_{\ell, \ell}(r) dr. \end{aligned} \quad (2.29)$$

These may be recombined to yield

$$\begin{aligned} \langle \Omega_0 \rangle &\approx \Delta^0 \omega_{\ell, \ell} + \Delta^1 \omega_{\ell, \ell} + \Delta^2 \omega_{\ell, \ell}, \\ \langle \Omega_1 \rangle &\approx -5 \Delta^1 \omega_{\ell, \ell} + 14 \Delta^2 \omega_{\ell, \ell}, \\ \langle \Omega_2 \rangle &\approx 21 \Delta^2 \omega_{\ell, \ell}, \end{aligned} \quad (2.30)$$

where the angular brackets denotes an average over r weighted with $H_{\ell, \ell}$. From equation (2.21) one finally obtains the corresponding radial average of $\Omega(r, \theta)$ as

$$\langle \Omega(\theta) \rangle \approx \Delta^0 \omega_{\ell, \ell} + (1 - 5\mu^2) \Delta^1 \omega_{\ell, \ell} + (1 - 14\mu^2 + 21\mu^4) \Delta^2 \omega_{\ell, \ell}, \quad (2.31)$$

where $\mu = \cos \theta$ (Brown *et al.* 1989; see also Morrow 1988). Thus by considering different combinations of the observed expansion coefficients, one can obtain some idea about the latitudinal variation of Ω . For example, $\Delta^0 \omega_{\ell, \ell} + \Delta^1 \omega_{\ell, \ell} + \Delta^2 \omega_{\ell, \ell}$ gives a measure of the equatorial angular velocity, whereas $\Delta^0 \omega_{\ell, \ell} - 4 \Delta^1 \omega_{\ell, \ell} + 8 \Delta^2 \omega_{\ell, \ell}$ measures the polar value. Note also that since the depth of penetration of the modes depends on ℓ through equation (2.3), the variation of these combinations with ℓ provides an initial impression of the variation of Ω with depth.

The expansions of the angular velocity and the splitting considered here is only one among many possible pairs. Other expansions were discussed by Hill *et al.* (1986) and Durney, Hill & Goode (1988).

3 Analysis of rotational splitting observations.

Only a brief summary of the analysis techniques can be given here. A more detailed discussion of techniques for observing solar oscillations was given by Brown (1988), while Harvey (1988a) reviewed observations of rotational splitting.

Early observations of the solar five minute oscillations dealt with modes of high degree. Furthermore, the data were generally averaged in one spatial dimension. In particular, averaging in latitude largely eliminates modes with $m \neq \pm \ell$. Two-dimensional Fourier analysis of the resulting signal, in longitude and time, therefore yields spectra that are dominated by the sectoral modes, with $|m| = \ell$. From this one may determine an average rotational splitting

$$\Delta^{(sec)} \omega_{\ell, \ell} = \frac{\omega_{\ell, \ell, \ell} - \omega_{\ell, \ell, -\ell}}{2\ell}, \quad (3.1)$$

which, according to equation (2.20) and the behaviour of P_{ℓ}^m , is dominated by the equatorial value of Ω .

From observations of the dependence of the oscillations on both latitude and longitude, modes at all values of ℓ and m , up to the limit imposed by the resolution, can be isolated. As discussed in section 2, this permits a determination of the latitudinal variation of the angular velocity. While it would in principle be possible to analyze directly the individual splittings $\delta \omega_{\ell, \ell, m}$, in practice the data have so far been presented in terms of fits to the m -dependence of the splittings. A particularly convenient representation is to expand the splitting in terms of Legendre polynomials, as

$$\delta \omega_{\ell, \ell, m} = 2\pi \ell \sum_{k=0}^{\max} a_k^{\ell} P_k \left[\frac{m}{\ell} \right], \quad (3.2)$$

where $P_k(x)$ is a Legendre polynomial. Since the P_k are orthogonal, the coefficients a_k^{ℓ} are statistically independent, as far as random errors are concerned. From equation (2.26) follows that rotation only contributes to the odd terms in this expansion. Even terms, if statistically significant, must be caused by departures from spherical symmetry in the equilibrium model, such as caused by rotational flattening, or from effects of magnetic fields which do not depend on the sign of m . (e.g. Dziembowski & Goode 1984; Gough & Taylor 1984; Gough & Thompson 1990a). Here we concentrate on the odd terms which, according to equation (2.26), are given by

$$a_{\ell}^{2\ell+1} = \frac{1}{2\pi} \Delta^1 \omega_{\ell, \ell}, \quad (3.3)$$

and hence are related to the expansion of the angular velocity through equation (2.27).

Modes of low degree are particularly interesting, since they penetrate to the core of the Sun. Unfortunately observation of splittings for such modes is difficult.

Spatially resolved observations in general suffer from a lack of an independent stable wavelength reference. As a result, the line position averaged over the solar disk is effectively used as a reference, precluding detection of modes of the lowest degrees. Observations that do not suffer from this problem have been obtained with the resonant scattering cell technique, applied to light integrated over the solar disk. Such measurements have provided data of very high quality for modes of degree 0 - 3. An additional problem, however, is that the observations provide little discrimination between modes of different m . If, as is usually assumed, rotation is the dominant source of departure from spherical symmetry, the axis of symmetry of the oscillations coincides with the rotation axis, which is always close to the plane of the sky. In that case observations in integrated light are virtually insensitive to modes where $\ell - m$ is odd, because for these the Legendre function is antisymmetric around the equator.¹ (For a discussion of the sensitivity of such observations, see, for example, Dziembowski 1977; Hill 1978; Christensen-Dalsgaard & Gough 1982). Even so, for each value of n and ℓ the spectrum contains $\ell + 1$ peaks, with a separation which for p modes is roughly $2\langle\Omega\rangle$, where $\langle\Omega\rangle$ is the average of the angular velocity defined in equation (2.17). Thus, if the angular velocity is nowhere much larger than its surface value, the separation in cyclic frequency is only of the order of 1 μHz . To resolve this separation requires observations spanning at least 15 days (e.g. Luomos & Deeming 1978). Furthermore, except for modes of frequency less than about 2000 μHz , the natural line width of the peaks is greater than 1 μHz (Libbrecht 1988a). In this case random variations in mode amplitude or phase may cause fine structure in the peaks which may be mistaken for rotational splitting (e.g. Sørensen 1988).

4 Inversion of the rotational splitting.

Observation of the rotational splitting for a mode of oscillation provides an integral measure of the angular velocity over the region where the mode is trapped. Because of the variation of the turning point radius r_t with degree and frequency, different modes sample different parts of the Sun. Hence, the variation of the splitting with degree, say, provides an indication of the variation of Ω with r . This also indicates that it may be possible to obtain localized information about Ω ; roughly speaking, the difference between splittings for modes with different r_t should be a measure of the rotation in the region between the turning points of the modes.

¹By considering the effect of rotation as an amplitude modulation, van der Raay, Pallé & Roca Cortes (1986) argued that all three $\ell = 1$ modes should be visible; however they apparently overlooked the cancellation caused by the latitude variation of the $m = 0$ mode.

The expression (2.11) for the splitting caused by spherically symmetric rotation is a particularly simple example of a relation between observable properties of oscillation frequencies and properties of the solar interior. The determination of $\Omega(r)$ from the $\delta\omega_{n\ell m}$ constitutes an *inverse problem*. Such problems have a vast literature, covering their application in, for example, geophysics and radiation theory (e.g. Parker 1977; Craig & Brown 1986; Deepak 1977; Tarrantola 1987). The application to the solar inverse problem was discussed by, for example, Gough (1978, 1985a) and Gough & Thompson (1990b). Christensen-Dalsgaard, Schou & Thompson (1990) made a systematic comparison of different inversion techniques, as applied to the problem of spherically symmetric rotation.

As background to the presentation in section 5 of results of inversions, it is instructive to consider briefly the properties of the technique of optimally localized averages, developed by Backus & Gilbert (1970). The inverse problem for a spherically symmetric angular velocity $\Omega(r)$ may be formulated as

$$\Delta_i = \int_0^R K_i(r) \Omega(r) dr, \quad (4.1)$$

where, for notational simplicity, i represent the pair (n, ℓ) by the single index i . Δ_i is the scaled rotational splitting $\pi^2 \delta\omega_{n\ell m}^{-1} \delta\omega_{n\ell m}$, so that the kernels K_i are normalized as in equation (2.14). The principle of the method is to construct a linear combination

$$\tilde{\Omega}(r_0) = \sum_i c_i(r_0) \Delta_i = \int_0^R K(r; r_0) \Omega(r) dr \quad (4.2)$$

of the observed data, where

$$K(r; r_0) \equiv \sum_i c_i(r_0) K_i(r). \quad (4.3)$$

The goal of the method is to choose coefficients $c_i(r_0)$ such as to make $K(r; r_0)$ approximate as far as possible a delta function $\delta(r - r_0)$ centred on r_0 . Then $\tilde{\Omega}(r_0)$ provides an approximation to $\Omega(r_0)$. If this can be done for all r_0 , an estimate of $\Omega(r)$ is obtained.

The coefficients $c_i(r_0)$ are determined by minimizing

$$\cos \eta \int_0^R (r - r_0)^2 K(r; r_0)^2 dr + \sin \eta \sum_{ij} E_{ij} c_i c_j, \quad (4.4)$$

where E_{ij} is the covariance matrix of the data, subject to the constraint

$$\int_0^R K(r; r_0) dr = 1. \quad (4.5)$$

This minimization problem leads to a set of linear equations for the c_i . The effect of the minimization is most easily understood for $\eta = 0$. Minimizing equation (4.4) subject to equation (4.5), ensures that $K(r; r_0)$ is large close to r_0 , where the weight function $(r - r_0)^2$ is small, and small elsewhere. This is precisely the required "delta-ness" of the combined kernel. However, with no further constraints, the optimization of the combined kernel may result in numerically large coefficients of opposite sign. Hence the variance in $\tilde{\Omega}$, which can be estimated as

$$\sigma^2(\tilde{\Omega}) = \sum_{ij} E_{ij} c_i c_j, \quad (4.6)$$

would be large. The effect of the second term in equation (4.4), when $\eta > 0$, is to restrict $\sigma^2(\tilde{\Omega})$. The size of η determines the relative importance of the localization and the size of the variance in the result. Hence η must be determined to ensure a trade-off between the localization and the error, and η is generally known as the *trade-off parameter*.

To characterize the properties of the inversion, it is convenient to consider a measure of the width of the averaging kernels $K(r; r_0)$ and the *error magnification*

$$\Lambda(r_0) = \left[\sum_i c_i (r_0)^2 \right]^{1/2}; \quad (4.7)$$

Λ is defined such that if the standard error $\sigma(\Delta_i)$ is the same for all the modes the standard error in the result of the inversion is

$$\sigma[\tilde{\Omega}(r_0)] = \Lambda(r_0) \sigma(\Delta_i). \quad (4.8)$$

By considering the *trade-off curve*, where Λ is plotted against the width for varying η , one may choose an appropriate value of η . It is evident that this value will depend on the properties of data, particularly the level of errors, and on the desired properties of the solution.

Figure 4. Averaging kernels $K(r; r_0)$ at selected radii $(r_0/R = 0.1, 0.2, \dots, 1.0)$ for inversion by means of optimally localized averages. The kernel at $r_0/R = 0.5$ is shown as a bolder curve. From Christensen-Dalsgaard, Schou & Thompson (1990).

An illustration of the use of this method is provided by the results obtained by Christensen-Dalsgaard *et al.* (1990). They considered a set consisting of about 830 modes at selected degrees between 1 and 200, and frequencies between 2000 and 4000 μHz . Examples of averaging kernels $K(r; r_0)$ are shown in Figure 4. The trade-off parameter was chosen such that the error magnification at $r_0 = 0.5R$ was close to 1. It should be realized that the kernels entering into the combination are of the form shown in Figure 1. Thus a very large degree of cancellation has been achieved of the dominant contribution from near the surface. Nevertheless, it is obvious that the averaging kernels are only approximate realizations of delta functions; structure on a scale smaller than roughly $0.05 R$ is not resolved.

This limitation is inherent in any inversion method. Indeed, it is evident that from a finite set of data one can never completely resolve the function $\Omega(r)$. Thus the solution must be constrained. The constraint that is invoked in the present method, and in most other inversion methods, is that the solution be smooth. This is ensured by the representation of the solution by the averaging kernels, whose shape is determined by the minimization in equation (4.4).

A second technique that has been used for the determination of the solar internal angular velocity is the regularized least-squares method (e.g. Phillips 1962; Craig & Brown 1986). Here $\Omega(r)$ is parameterized, often as a piecewise constant function, and the parameters are determined through a least-squares fit to the data by minimizing the sum of the squared differences between the observed splittings and the splittings computed from the parameterized representation of Ω . To obtain a smooth solution, in the quantity to be minimized one adds a multiplicity of the average of the square of Ω , or of its first or second derivative; the weight given to this term serves as a trade-off parameter, determining the balance between resolution and error for this method. Another inversion technique is spectral expansion, where Ω is approximated as a linear combination of the kernels (see for example Backus & Gilbert 1967; Gough 1985). Finally, from the asymptotic properties of p modes follows that the inverse problem can be formulated approximately as an integral equation of the Abel type, the solution to which can be written down analytically; this leads to a very simple, although approximate, technique for inversion of data consisting only of such modes (e.g. Gough 1984). - In these methods, also, there are parameters which determine the trade-off between resolution and error.

These methods are all linear, in the sense that the result of the inversion depends linearly on the data. For any such method, there exist coefficients $c_i(r_0)$ such that the solution $\tilde{\Omega}(r_0)$ may be written in terms of the observed data Δ_i as in equation (4.2), and hence, by using equation (4.1), may be expressed from the original angular velocity $\Omega(r)$ through an averaging kernel $K(r; r_0)$. It should be noticed that, apart from the choice of the trade-off parameter, the averaging kernels are independent of the data. As discussed extensively by Christensen-Dalsgaard *et al.* (1990), a quantitative comparison of different inversion methods can be carried out

in terms of the averaging kernels and the coefficients $c_i(r_0)$.

So far I have considered inversion for a function that depends on r alone. It is evidently desirable, however, to carry out inversion for a general angular velocity $\Omega(r, \theta)$ which is a function both of r and θ . As discussed in section 2.4, it is often convenient to expand $\Omega(r, \theta)$ in terms of suitable functions of θ , with coefficients Ω_n that are functions of r (cf. equation (2.21)). If the observed splittings are expanded as in equation (2.26), there result the relations (2.27) between the expansion coefficients of the splittings and the expansion functions $\Omega_n(r)$ of $\Omega(r, \theta)$; from these the Ω_n can be determined from r -dependent inversion, as discussed above. The practical implementation of this procedure was discussed, for example, by Korzenik *et al.* (1988) and Brown *et al.* (1989). Finally, for p modes a two-dimensional asymptotic inversion technique may be developed by noting that the dependence on P_0^2 in equation (2.20) can be approximated in such a way as to lead to an integral equation of the Abel type in latitude (Kosovichev & Parchevskii 1988; Gough 1990; Gough & Thompson 1990a,b). As in the one-dimensional case the radial dependence also leads to a similar integral equation. Thus the solution to the inverse problem can once again be written down analytically.

5 Results of rotational splitting observations.

In this section I discuss observed rotational splittings and the results obtained by analysing them. No complete critical review of the observations can be attempted here. However, it might be noted that Harvey (1988a) compared a number of different observations, as did Rhodes *et al.* (1988).

5.1 The equatorial rotation rate.

As discussed in section 3, the splitting between the prograde and retrograde sectoral modes provides a measure of the angular velocity in the neighbourhood of the equatorial plane. The first successful observations of this nature were carried out by Deubner, Ulrich & Rhodes (1979). They considered modes with surface wavenumbers k_s exceeding 0.2 Mm^{-1} , corresponding to degrees greater than about 140. The splitting increased somewhat with decreasing k_s , and hence increasing depth of penetration, indicating that the angular velocity increased with depth in the upper part of the convection zone. There were indications that this trend was reversed for the most deeply penetrating modes in the data. To relate the variation in the splitting to the depth in the solar convection zone Deubner *et al.* assigned a so-called *effective depth* to each mode, essentially assuming a linear variation of Ω with depth. According to this definition, Ω increased to a depth of about 15 Mm, the maximum value of $\Omega/2\pi$ being about 50 mHz higher than the surface value. It was pointed out by Gough (1978) that such an analysis of the depth variation could be

misleading. Gough demonstrated the power of inverse analysis for such data, by analysing artificial data for an assumed $\Omega(r)$; however, the analysis was not applied to the observed data.

Somewhat more extensive data, spanning up to five days, were obtained by Hill, Gough & Toomre (1984a). Hill *et al.* (1988) analyzed the data by means of optimally localized averages, to obtain the angular velocity as a function of depth. The results are shown in Figure 5, which is an average of four individual determinations. The most striking result is the increase in $\Omega/2\pi$ by about 20 mHz in the outer 2 Mm, followed by a gradual decrease. This result is at least qualitatively consistent with the earlier results obtained by Deubner *et al.*, given the uncertain definition of the depth scale in the latter analysis. It is also tempting to relate this behaviour to the difference between the Doppler and magnetic feature rotation rates (cf. equations (1.1) and (1.2)), on the assumption that the magnetic features are anchored at some distance beneath the visible surface. It should be noticed that the inferred surface value is about 25 mHz smaller than the value obtained from surface Doppler measurements. Hill *et al.* noted, however, that a very small uncorrected drift of the telescope would be sufficient to account for this discrepancy.

Figure 5. Equatorial rotation frequency $\Omega/2\pi$ as a function of depth (upper scale) and logarithm of pressure (lower scale). The curve was obtained as the average of four individual determinations; the bars denote the standard deviation in this average. From Hill *et al.* (1988).

Figure 6. Splittings $\overline{\Delta\nu_\ell} = \overline{\Delta\nu_\ell^{(ee)}/2\pi}$ between sectoral modes (cf. equation (3.1)), averaged over radial order n , as a function of the degree ℓ . The error bars represent the standard deviation of the mean. The horizontal line marks the surface equatorial angular velocity, and the surface angular velocity at latitudes 30° and 50° are also indicated at the right. From Duvall & Harvey (1984).

Figure 7. Equatorial rotation frequency $\Omega/2\pi$, as a function of fractional radius r/R , obtained by inverting the data shown in Figure 6. The inversion was carried out by representing Ω as a piecewise constant function and determining the constants by fitting equation (2.11) to the observed splittings. The error bars indicate the observational uncertainties. From Duvall *et al.* (1984).

Duvall & Harvey (1984) used an optical technique to obtain latitudinally averaged data covering essentially the entire solar disk. From these they were able to measure the rotational splitting for sectoral modes at degrees between 1 and 100. The results, after averaging over mode order n , are shown in Figure 6. At most degrees the splitting is slightly, but apparently significantly, below the value corresponding to the surface equatorial angular velocity. The exceptions are at the highest degrees included, in a peculiar feature at $\ell = 11$, and at $\ell = 1$ where, however, the error bars are substantial.

These data were inverted by means of several techniques by Duvall *et al.* (1984). Figure 7 shows the result of a least squares fit of a piecewise constant function to the data. As could have been expected from the splittings, Ω is below its surface value in most of the solar interior. The rapid variation at $r/R = 0.2 - 0.4$ arises principally from the local maximum in the splitting at $\ell = 11$. Finally, it is interesting that the central part of the Sun is found to rotate at roughly twice the surface rate; this, however, is almost entirely a result of the larger splitting at $\ell = 1$, and hence is subject to considerable uncertainty. I return to the rotation of the core in section 5.3 below.

If accepted, the behaviour of the angular velocity in the solar core may have far-reaching implications for the dynamics of the solar interior (Gough 1985b). The observations of Brown (1985) gave some support to the behaviour of the splitting found by Duvall & Harvey (1984); on the other hand Libbrecht (1986) found no maximum at $\ell = 11$. Thus a definite conclusion on the reality of this feature must await further observations.

Finally I note that Hill (1987) presented a summary of determinations of the equatorial rotation rate in the convection zone.

5.2 The dependence of rotation on latitude and depth.

As discussed in section 2, information about the latitude variation of the angular velocity can be obtained from observation of the m -dependence of the rotational splitting. The first to present such data was Brown (1985) (see also Brown 1986 for a more detailed description of the observing and analysis technique). From two-dimensional velocity images of the Sun he could isolate individual spherical harmonics² through suitable spatial filtering, and hence obtain spectra with contributions from only a small number of (ℓ, m) pairs. Rather than analyzing completely the resulting large number of spectra to determine individual frequencies $\nu_{\ell m}$, Brown employed a correlation technique, which effectively averages the splitting over mode order n . These averages were finally fitted by cubics in m/ℓ , the cubic

²In practice it is impossible to isolate completely a single spherical harmonic from data of this nature, since at most half of the solar surface is observed. Thus the filtered spectra contain contributions from a range of typically ± 2 in ℓ and m .

term providing a measure of the variation of angular velocity with latitude. The results showed that for $15 \leq \ell \leq 40$ the cubic term was substantially smaller than the cubic term that would result if the surface differential rotation, as in equation (1.1) or (1.2), persisted throughout the Sun. This suggests that the variation of Ω with latitude is smaller in the solar interior than on the surface. This conclusion was confirmed by Christensen-Dalsgaard (1986), who compared the observations with rotational splittings computed for a variety of rotation laws.

A more extensive dataset of this nature was obtained by Brown & Morrow (1987). They again considered averages of the splittings over n , fitting the results to an expression of the form given in equation (3.2) to determine coefficients $\bar{a}_k^2$, $k = 1, \dots, 5$, the coefficient $\bar{a}_5^2$ being only marginally statistically significant (cf.

Figure 8. Expansion coefficients for frequency splitting, averaged over mode order n . The dashed lines indicate the values obtained by assuming that the surface differential rotation extended throughout the Sun. In the panel for $\bar{a}_5^2$ the dotted and long-dashed lines show the result of assuming radius-independent rotation, and rotation on cylinders, respectively, for $r > 0.73 R$, with continuous match to the observed surface angular velocity; inside this radius the angular velocity was assumed to be constant, and equal to the surface equatorial value. From Brown & Morrow (1987).

Brown *et al.* (1989). The resulting odd coefficients are shown in Figure 8. It might be noticed, in particular, that $\bar{a}_5^2$ is significantly below the value corresponding to the surface rotation for $\ell \lesssim 50$. This again indicates that the latitude variation of Ω decreases with increasing depth.

Brown *et al.* (1989) carried out a detailed analysis of these data, both by comparing the observations with the splitting resulting from different rotation laws, and by applying different inversion techniques. The analyses gave broadly speaking similar results: the equatorial angular velocity decreases slightly, and the polar angular velocity increases substantially, with depth in the convection zone, and beneath the convection zone rotation may be independent of latitude. It should be emphasized, however, that the resolution of the inversions was relatively poor. Thus the extent of the transition region from surface-like to latitude-independent rotation could only be restricted to be less than about $0.2 R$.

Analyses of the Duvall & Harvey (1984) and the Brown & Morrow (1987) data were also presented by Kosovichev (1988), who obtained results similar to those described above.

The correlation technique used by Brown (1985) and by Brown & Morrow (1987) is not ideal. Because of the dependence of the turning point radius on frequency (cf. equation (2.3)) the average over n mixes modes that extend to different depths, and hence sense the rotation differently. This may make a significant contribution to the scatter in the observed splitting coefficients (see also Harvey 1988a).

Extensive sets of velocity images were analyzed by Libbrecht (1986, 1989). In contrast to Brown & Morrow, Libbrecht determined rotational splitting coefficients in the fit in equation (3.2) for individual n and ℓ . Furthermore, his longest time series extended over 100 days. As a result the scatter in the measured splitting coefficients was considerably smaller than in the previously discussed data; however, Libbrecht's results were essentially consistent with those obtained by Brown & Morrow, showing a similar decrease in $\bar{a}_5^2$ at low degree. To illustrate the dependence of the expansion coefficients on degree and frequency, Libbrecht (1989) fitted them to an expression of the form

$$\bar{a}_{\ell,2}^2 = a_k^* + b_k^*(\nu_{\ell,2} - \nu_0), \quad (5.1)$$

with $\nu_0 = 2500 \mu\text{Hz}$. The results are shown in Figure 9. It is evident that the a_k^* have a very well-defined dependence of ℓ , and hence on the turning-point position.

Libbrecht's data have been subjected to several independent inverse analyses. Figure 10 shows a representative result, obtained by Christensen-Dalsgaard & Schou (1988) by means of optimally localized averages. Given the range of degrees (5 - 60) included in the data, and the fact that the errors in the lowest-degree data were considerable, it was only possible to perform an inversion in the range $0.35 R$ to $0.8 R$. Furthermore the polar angular velocity, which as indicated by equation

Figure 10. Rotation frequency $\Omega/2\pi$ at the equator, latitude 45° and the pole, obtained by inverting Libbrecht's (1989) observed splittings. The dashed lines correspond to one standard deviation limits. The surface values are indicated by the straight lines starting at $r/R = 1$. Note that inversion for the higher-order expansion coefficient a_2 (cf. equation (2.21)), which is particularly important for the polar angular velocity, was only possible for $r > 0.6 R$; inside this it was essentially assumed to be constant. Thus the apparent constancy of the polar rotation frequency inside $0.6 R$ has no significance. From Christensen-Dalsgaard & Schou (1988).

that these two independent analyses gave very similar results. Dziembowski, Goode & Libbrecht (1989) discussed the least-squares inversion in more detail, with particular emphasis on the fact that the region with a sharp radial gradient in the angular velocity is a likely site for dynamo action which may be responsible for the solar magnetic activity (cf. section 6). A third independent inversion of the Libbrecht data was carried out by Thompson (1990). Using a least-squares technique he obtained results very similar to those of Christensen-Dalsgaard & Schou and Dziembowski & Goode. Furthermore Thompson made an accurate estimate of the average gradient of the angular velocity in the convection zone, by means of inversion using optimally localized averages. This showed that rotation in the convection zone is almost certainly not constant on cylinders, as might otherwise have been expected from hydrodynamical simulations of convection zone dynamics (see Stix, these proceedings, for a review).

Figure 9. Splitting coefficients a_l^* and b_l^* , resulting from a linear fit of the expansion coefficients a_l as a function of frequency (cf. equation (5.1)); note that the values of b_l^* assume that ν is in mHz). From Libbrecht (1989).

(2.31) is particularly sensitive to the highest-order expansion coefficients of the splitting, was relatively poorly determined. Nevertheless, the gross features of the result appear to be significant: the angular velocity is similar to the surface value in much of the convection zone, with a transition near the base of the convection zone to what may be latitude-independent rotation. It should be noted that this transition takes place within the resolution width of the inversion; thus, although with the present data the extent of the transition cannot be further constrained, the results of the inversion are consistent with it being essentially discontinuous.

These results are compared in Figure 11 with a regularized least-squares fit to the data carried out by Dziembowski & Goode (cf. Libbrecht 1988b). It is evident

Figure 11. Contour diagrams of rotational frequency $n/2\pi$ obtained by inverting Libbrecht's (1989) data. The left panel was obtained by Dziembowski & Goode using a least squares technique with first-derivative smoothing (see also Dziembowski, Goode & Libbrecht 1989). The right panel gives the results obtained by Christensen-Dalsgaard & Schou (1988) which were presented in Figure 10. From Libbrecht (1988b).

The data discussed so far were all obtained from observations of Doppler velocity. Duvall, Harvey & Pomerantz (1986) presented spatially resolved observations made in Ca II line intensities; the observations were carried out in 1981 at the South Pole. The data were analyzed by means of the cross-correlation technique of Brown (1985) and presented as averages over bins of 10 in degrees. In contrast to the Doppler velocity data little variation with ℓ of the expansion coefficients was found, indicating that the surface differential rotation persists throughout most of the Sun. However, similar data obtained at the South Pole in 1987 (Jefferies *et al.* 1988), which extended to $\ell = 150$, were essentially in agreement with the Doppler data at the degrees where the datasets overlapped (see also Harvey 1988a). The reason for the discrepancies with the earlier South Pole data is not clear. Since the Jefferies *et al.* data include modes at higher degree, inversion of the data, which has yet to be carried out, should enable determination of the angular velocity closer to the surface than was possible in the inversions shown in Figures 10 and 11.

Extensive sets of Doppler image observations have been obtained at the Mt Wilson Observatory. Tomczyk (1988) and Tomczyk *et al.* (1988) presented results of an analysis of the rotational splitting for part of the data, at degrees up to 120.

Results for sectoral modes, extending to higher degree, were obtained by Rhodes, Cacciani & Korzenik (1988). Comparisons, carried out by Harvey (1988a) and Rhodes *et al.* (1988), with the data obtained by Brown & Morrow (1987), Jefferies *et al.* (1988) and Libbrecht (1989) showed that for the Mt Wilson data a_2^2 was systematically higher than for the other sets, whereas a_2^2 was systematically lower; however, the sum $a_1^2 + a_2^2 + a_3^2$, which as indicated in section 2.4 is a measure of the equatorial rotation rate, was very similar for the Mt Wilson and the other sets. The reason for these systematic differences is so far not understood.

Rhodes *et al.* (1988) combined the different independent datasets and presented the results against an average location of the kernels, obtained as the midpoint between the lower turning point and the surface. Although this at best gives a very rough indication of the variation of angular velocity with depth, the results were in qualitative agreement with the results of inversions, indicating again a decrease in the angular variation of n towards the base of the convection zone. Korzenik *et al.* (1988) carried out inversions of both the individual observed datasets and a combined set, by means of the spectral expansion and the optimally localized averages techniques. Only data averaged in n at each ℓ were considered. The results differed in detail between the different sets, and from the inversions carried out by Christensen-Dalsgaard & Schou (1988) and Dziembowski *et al.* (1989). However, the dominant features were again similar to those discussed previously.

5.3 Rotation of the solar core.

To measure the rotation of the solar core from p-mode data requires observation of modes of low degree. As discussed in section 3, the best observations of such modes have been made through Doppler velocity measurements in integrated sunlight. From such measurements Claverie *et al.* (1981) found evidence of triplet structure for the $\ell = 1$ modes and quintuplet structure for the $\ell = 2$ modes. This was interpreted as due to rotation; the average splitting was found to be $0.75 \mu\text{Hz}$, i.e. substantially higher than the surface value of the angular velocity. Pallé *et al.* (1988) analyzed more extensive sets of data. To minimize the problems caused by the finite linewidth of the modes (*cf.* section 3) they selected subsets of the timestrings where a given mode appeared to maintain its phase and amplitude. The resulting spectra again showed triplet structure for several $\ell = 1$ modes, with an average splitting of $0.72 \mu\text{Hz}$. From an autocorrelation analysis the splitting for modes of degree 2 was found to be $0.55 \mu\text{Hz}$, whereas the splitting for $\ell = 3$ and 4 was estimated to be $0.47 \mu\text{Hz}$, i.e. close to the value expected from the surface rotation. It is interesting that these results are consistent with the values obtained by Duvall & Harvey (1984) from spatially resolved observations (*cf.* section 5.1).

A severe difficulty with the results of Claverie *et al.* (1981) and Pallé *et al.* (1988) is the detection of the $\ell = 1$, $m = 0$ mode which, as discussed in section 3, should be essentially invisible in observations of this nature; the same is true for

the $\ell = 2$, $m = \pm 1$ modes that were apparently detected by Claverie *et al.* It was suggested by Isaak (1982) that the observations could be explained if the oscillations had an axis of symmetry differing from the surface rotation axis, possibly due to a very strong magnetic field in the deep solar interior. However, except under very special circumstances such a configuration would lead to a far more complicated spectrum of oscillations (Dicke 1982, Gough 1982a, Gough & Taylor 1984; Gough & Thompson 1990a). An alternative explanation is that the observed fine structure in the peaks is predominantly due to amplitude variation or phase shifts. It is not inconceivable that such effects could mimic rotational splitting, even in the data of Pallé *et al.* who attempted to minimize them. Also it might be noted that in the examples of spectra shown by Pallé *et al.* the evidence for triplet structure is rather marginal; if these cases are disregarded it is possible that the measured splittings correspond to the difference between $m = -1$ and $+1$, in which case the corresponding averaged angular velocity $\langle \Omega \rangle$ (cf. equation (2.17)) might even be smaller than the surface value. Observations of low-degree p modes that provide some separation of the different values of m are clearly highly desirable.

Observation of g modes would provide far more sensitive measures of the core rotation. Delache & Scherrer (1983) found evidence for rotationally split $\ell = 1$ modes in differential Doppler velocity measurements; the splitting corresponded to an average rotation frequency $\langle \Omega \rangle / 2\pi$ of 1.2 μHz . From a statistical analysis of irradiance measurements, using the asymptotic properties of the modes, Fröhlich & Delache (1984a,b) found that $\langle \Omega \rangle / 2\pi$ was 1 to 1.2 μHz . Subsequent analysis of similar data (Fröhlich 1988) yielded $\langle \Omega \rangle / 2\pi = 1.05$ μHz . Pallé & Roca Cortés (1988) made a statistical analysis of several years' Doppler velocity measurements in integrated light and found that the average of $\langle \Omega \rangle / 2\pi$ was between 1.0 and 1.3 μHz ; on the other hand García, Pallé & Roca Cortés (1988) found no statistically significant evidence for g modes in more extensive datasets of this nature. Finally Henning & Scherrer (1988) identified several pairs of peaks in spectra of differential Doppler velocity observations as rotationally split $\ell = 1$ modes; the average splitting corresponded to $\langle \Omega \rangle / 2\pi = 1.6$ μHz . Although these results are interesting, it is probably fair to say that the significance of the detection of g modes has yet to be definitely established.

It is evident that there is considerable uncertainty concerning the observational situation for the splitting of modes that are sensitive to the rotation of the core. Nevertheless, it is of some interest to consider the consequences of the observed values that have been quoted. For example, it is immediately obvious from Figure 3 that to obtain a splitting of 0.7 μHz for the $\ell = 1$ p modes due to rapid rotation of a small core, a very high angular velocity would be needed. To illustrate this in more detail, I restrict myself to rotation that depends on r alone, and consider piecewise constant rotation laws on the form

$$\Omega(r) = \begin{cases} \Omega_c & \text{for } r < r_c \\ \Omega_s & \text{for } r > r_c; \end{cases} \quad (5.2)$$

for Ω_s the equatorial plasma rate given in equation (1.1) is used. Furthermore, I assume that the observed average splittings of p modes can be adequately represented by the splitting corresponding to the average of the kernels for modes of the appropriate ℓ within the frequency range 2500 to 3500 μHz , whereas the average splitting of g modes can be similarly represented by modes in the frequency range 50 to 100 μHz . Finally I impose the constraint that the average splitting of the $\ell = 1$ p modes be 0.7 μHz ; the required Ω_c can then be evaluated for any assumed radius r_c , and the splittings for other modes can be computed.

In Figure 12a the dashed line illustrates the resulting $\Omega_c(r_c)$. As was already inferred from Figure 3, quite rapid core rotation is required if the core is not extended. If, for example, the core extends over the inner 20 per cent of the solar radius, it would have to rotate at about 9 times the surface rate. Computed p -mode splittings, as functions of r_c , are shown in Figure 12b. Also shown are observed splittings for $\ell = 2, 3$ and 4 from Duvall & Harvey (1984), indicated by the intervals corresponding to the standard errors. For higher values of ℓ , the observations are generally at or below the surface angular velocity (cf. Figure 6). It is evident that the computed splittings are in general considerably outside the error limits of the observations. For $\ell = 2$ only a narrow range of r_c around 0.14 R appears to be marginally acceptable, and here Ω_c would have to be around 15 Ω_s . The splittings for $\ell = 3$ and 4 then deviate substantially more than 1 standard deviation from the observed value, however. The situation is further complicated if also the suggested observed splittings for g modes, corresponding to 2 - 3 times the surface angular velocity, are taken into account; for these modes most of the contribution to the splitting comes from the core, as was seen in Figure 3, and hence the average angular velocities $\langle \Omega \rangle$, as sensed by the g modes, are only slightly below Ω_c . This is confirmed by Figure 12a, where the solid curves show computed $\langle \Omega \rangle$ in units of Ω_s , as functions of r_c , for g modes. It is evident that the observations are only consistent with the model if r_c is as large as 0.5 R ; but this is incompatible with the p -mode data.

Obviously no definite conclusions can be drawn on the basis of this analysis, given that it is restricted to a very special form of the rotation law. Indeed, there is little doubt that an angular velocity distribution $\Omega(r)$ can be found which agrees with all the observed splittings, including those for g modes; the results obtained here suggest that it would have to be rather peculiar, however. If only the p -mode data are taken into account, it appears likely that reasonably plausible rotation laws can be found which are consistent with the data within their error bars. However, the constraints of the observed splittings for $\ell \geq 4$, which appear to be fairly well-determined, would limit the region of rapid rotation to radii below about 0.2 R , and consequently this region would have to rotate at many times the surface rate. Better data than now available, in particular definite and accurate measurements of

Figure 12. In panel a) the dashed curve shows the angular velocity Ω_c of the core, in units of the surface angular velocity Ω_s , as a function of the core radius r_c , for a rotation law of the form (5.2); Ω_c was determined such that the average splitting for $\ell = 1$ p modes was 0.7 mHz . The solid curves show the corresponding average angular velocities $\langle \Omega \rangle$ (cf. equation (2.17)), also in units of Ω_s , as sensed by $\ell = 1$ and $\ell = 2$ g modes. Panel b) shows $\langle \Omega \rangle / \Omega_s$ for p modes for this rotation law, again as a function of r_c . The thin continuous curves are for $\ell = 5 - 10$. The heavy curves give splittings for $\ell = 2$ (---), $\ell = 3$ (-----) and $\ell = 4$ (-----). Also shown, with the same line styles, are the one standard deviation intervals for the splittings observed by Duvall & Harvey (1984).

Individual splittings for $\ell = 1$ and 2, are needed before a more detailed determination of the rotation of the core can be made.

5.4 Measurements at the solar limb.

Apart from some fairly minor systematic differences the observations, discussed so far present a reasonably consistent picture. In particular, with the possible exception of the core, there is no indication that parts of the Sun are rotating at substantially more than the surface rate. However, completely discordant results have been obtained from observations of the solar limb intensity made by the SCLERA

group. Analysis of these data indicates that most of the solar interior rotates at more than 4 times the surface rate (e.g. Hill, Bos & Goode 1982). Furthermore evidence has been found for a substantial increase in the polar angular velocity, to well over the equatorial value, in the lower part of the convection zone (Hill *et al.* 1986). These results are inconsistent with the rotational splittings discussed in the preceding sections.

The reasons for this discrepancy are not clear. It should be noted, however, that analysis of the limb measurements provides a far less efficient separation between different modes than does the analysis of the two-dimensional velocity or intensity maps. Therefore, the resulting power spectra are extremely complicated, and the tasks of identifying individual modes and measuring the rotational splitting are correspondingly difficult. Given this, and the agreement between the remaining measurements, it appears likely that problems with the mode identification are responsible for the discrepant SCLERA measurements.

A more detailed discussion of these results was presented by Harvey (1988a).

6 Discussion.

It is evident that helioseismology has already provided a considerable amount of information about rotation in the solar interior. Although the results are incomplete for much of the Sun, and the details are still uncertain, it seems likely that the main features are robust. There appears to be an increase with depth of the equatorial angular velocity Ω_{eq} , by about 5 per cent, in the upper few Mm of the solar convection zone; in the rest of the convection zone Ω_{eq} probably decreases slightly. The latitude dependence of Ω is similar to the surface behaviour in much of the convection zone, with a transition, which could be quite rapid, near its base to a substantially smaller variation with latitude. The present data provide relatively little information about the latitude dependence in the radiative interior; however, they appear to be consistent with latitude-independent rotation in this region. More information is available about the variation of Ω_{eq} beneath the convection zone: in much of the radiative interior Ω_{eq} is slightly, but apparently significantly, smaller than its surface value, although the behaviour may be more complicated near the outer edge of the energy generating core. The rotation of the core is still very uncertain; there are indications from several independent sets of observations that it rotates at several times the surface rate, but the evidence cannot as yet be regarded as conclusive.

These results appear to be consistent with observations of the rotation on the solar surface. Thus the slightly higher angular velocity for magnetic features, compared with the surface plasma rate (cf. equations (1.1) and (1.2)) could be linked to the increase with depth in the equatorial angular velocity near the surface. Further observations of similar nature, reviewed by Schüssler (1987), also suggest

faster rotation in the upper part of the convection zone, whereas there are indications that the rotation in the deeper part of the convection zone could be slower than at the surface. Finally Stenflo (1989) found that the rotation of large-scale magnetic patterns varied little with latitude; assuming that such patterns originate near the base of the convection zone, he inferred a distribution of rotation within the convection zone which is similar to that obtained in Figure 10 from inversion of the rotational splittings.

It is of obvious interest to investigate how the observed rotation profile might have come about or is maintained. It is generally believed that solar-type stars rotate rapidly when they arrive on the main sequence but lose angular momentum from the surface through a magnetic stellar wind during their evolution. The distribution of rotation in the interior of the star then depends on the transfer of angular momentum to the convection zone, and hence to the surface where the braking takes place; however it is not unreasonable to expect that a rapidly rotating core would remain. Spruit, Knobloch & Roxburgh (1983) estimated the resulting rotation profile, by assuming that it would be such as to maintain marginal stability towards instabilities that would cause angular momentum transport. Pinsonneault *et al.* (1989) computed detailed evolution sequences of rotating stellar models, taking into account angular momentum loss from the surface and transport of angular momentum through assumed rotationally induced instabilities. These processes are not fully understood, and the calculation therefore depends on a number of unknown parameters; as a result there is considerable latitude in the results. However, in all cases Ω was a decreasing function of radius, and the angular velocity of the core exceeded the surface value by a large factor. In fact, calculations by Demarque & Guenther (1988), assuming such angular velocity profiles, of rotational splittings for $\ell = 7 - 10$ showed that these far exceeded the values observed by Duvall & Harvey (1984). Thus the models appear to be inconsistent with the observations, and additional mechanisms for angular momentum transport are required. Indeed, it was pointed out by Rosner & Weiss (1985) that even a very modest magnetic field in the radiative interior would transport angular momentum so efficiently that rotation would become essentially rigid. A more detailed discussion of the possible mechanisms for angular momentum transport, with special emphasis on magnetic effects, was given by Spruit (1987).

Pinsonneault *et al.* (1989) pointed out that if the Sun were to be rotating approximately rigidly, as appears to be indicated by the helioseismic data, the rotation period will increase rapidly, to more than 200 days, when the Sun evolves into a subgiant. This seems to be in conflict with inferences about subgiant rotation made by Noyes *et al.* (1984) who found that the rotation period was roughly constant, and below 50 days, for subgiants of effective temperature down to about 4900 K. On the other hand the subgiant data were in good agreement with the computations of Pinsonneault *et al.* It is evident that there is no formal conflict between the solar helioseismic results and the subgiant data: there is no assurance that

either the Sun, or the subgiants, are representative, and hence there is no compelling reason to expect them to behave similarly. It should also be pointed out that the subgiant rotation periods were determined by means of an indirect method based on chromospheric activity and measured rotation of main-sequence stars; this may not hold equally for subgiants. Nevertheless, there are indications of a very interesting problem here: It is obviously unsatisfactory to be forced to claim that the Sun is special in terms of its rotational properties. Yet, given the observed rotational splittings, it seems highly unlikely that the Sun can hide a sufficient amount of angular momentum to enable it to maintain relatively rapid rotation as it evolves away from the main sequence. Indeed, for the simple model of rotation discussed in section 5.3 (*cf.* equation (5.2)), imposing again an average splitting of 0.7 μHz for the $\ell = 1$ modes and demanding that the model be consistent with the other splitting observations, the angular momentum is less than 1.5 times the value corresponding to uniform rotation; this is insufficient to maintain rapid rotation when the Sun evolves into a subgiant. - Better, and more extensive, data on the rotation of stars away from the main sequence are badly needed.

The inferred angular velocity distribution in the convection zone is in conflict with results of detailed hydrodynamical simulations of convection zone dynamics, which tend to predict rotation that is constant on cylinders, *i.e.* depending only on the distance from the rotation axis (*e.g.* Gilman & Miller 1986; Glatzmaier 1985). Gilman, Morrow & DeLuca (1989) argued that the observed distribution may be explained through angular momentum transport between the convection zone and the radiative interior, through some as yet unspecified mechanism; the details of this model remain to be worked out, however. It should also be noted that simplified models incorporating anisotropic viscosity or inhomogeneous heat transport (Pidatella *et al.* 1986; see also Stix 1987) have produced rotation profiles that are superficially similar to those obtained from the helioseismic measurements.

The centrifugal distortion of the Sun due to rotation perturbs the gravitational field outside the Sun. It has been speculated that this might make a significant contribution to the precession of the orbit of Mercury, possibly invalidating the apparent agreement between the observed value and the predictions of Einstein's theory of general relativity. It is in principle possible to measure the distortion of the visible solar surface, and from this to infer the distortion of the equipotential surfaces, as measured by the quadrupole moment J_2 . Such measurements have in the past indicated a substantial oblateness (Dicke & Goldenberg 1974), although more recent measurements have found smaller values (Hill & Stebbins 1975; Dicke, Kuhn & Libbrecht 1987). In any case the determination of the shape of the visible surface is complicated by the variation in solar activity, and hence possibly in surface brightness, between equator and poles. On the other hand, given a helioseismic determination of $\Omega(r, \theta)$ it would be straightforward to calculate J_2 (Gough 1982b). Indeed, the main areas of uncertainty in the current determination of the angular velocity are in the solar core and very near the surface, both of which regions have

little effect on J_2 . An additional problem is the lack of definite information about the latitude variation of rotation in the radiative interior; however if, as seems plausible, the latitude dependence is modest, J_2 is unlikely to be substantially different from the value $J_2 = 1.7 \times 10^7$ which results from the equatorial angular velocity determined by Duvall *et al.* (1984) assuming no latitude dependence. In that case there is essentially perfect agreement between the observed orbit of Mercury and the predictions of general relativity (e.g. Narlikar & Rana 1985; see their "Note added in proof").

The rotation in the solar convection zone bears directly on those models of the generation of solar magnetic activity that are based on dynamo action spread out through the convection zone. As reviewed, for example, by Gilman & Deluca (1986), such models require an increase of the angular velocity with depth in the convection zone, to reproduce the observed migration of active regions towards the equator during the solar cycle; this appears to be inconsistent with the helioseismic measurements. This difficulty can be avoided by considering dynamo action in the region of convective overshoot beneath the convection zone. Here, according to results shown in Figure 10, there is a slight decrease in Ω with depth at low latitudes, and a substantial increase with depth at high latitudes. As discussed by Gilman, Morrow & Deluca (1989), who made a careful analysis of the implications on dynamo models of the helioseismic measurements of the angular velocity, it may be possible under these circumstances to explain the observed surface migration of magnetic activity.

As a further contribution to understanding the generation of solar activity, and the dynamics of the convection zone, measurements of other large-scale velocity fields would be very valuable. In particular one might be able to detect the flow associated with giant convective cells, which are predicted by numerical models of the convection zone. Gough & Toomre (1983) considered the effects of such large-scale convective eddies on the oscillations in a simplified model. Some evidence for day-to-day variations in the horizontal velocities inferred by inverting frequency splittings of high-degree modes was found by Hill, Gough & Toomre (1984a); this was tentatively attributed to the result of advection of giant cells past the region of the solar surface where the oscillations were observed. Hill, Gough & Toomre (1984b) carried out a sensitivity analysis, which provided some support for the reality of the results. Further evidence for such flows was obtained by Hill (1988a, 1988a). However, it should be pointed out that the analysis could be quite sensitive to beating between unresolved modes, or possibly to effects of atmospheric seeing. Thus the detection of convective velocity fields must so far be regarded as preliminary.

Measurements of this nature are probably at the limit of the capabilities of present observations. Indeed, despite their impressive achievements, helioseismic investigations of solar internal rotation are still in their infancy. Coming developments in the observation of rotational splittings should permit a far more accurate

determination of $\Omega(r, \theta)$ throughout the Sun.

Observations of solar oscillations, and analyses of the data, are long-term projects which are continually being pursued by several independent groups. As a result there is a steady increase in the material available for inverse analyses. In the near future one may hope in particular to obtain improved data for high-degree modes; this should enable a better resolution of the equatorial angular velocity near the surface and, one may hope, information about the variation with latitude in this region. Additional data for modes of moderate degree are also becoming available, which may allow us to limit further the extent of the region near the base of the convection zone where the transition to almost latitude-independent rotation takes place.

In the slightly longer term, dramatic improvements in the observational situation can be expected as new instruments become operational during the coming decade. Two networks of observing stations using integrated light, Doppler velocity techniques have already begun observations (Aindow *et al.* 1988; Fosserat 1988), and in both cases further stations will be added. The GONG project (Hill 1989b; see also Harvey 1988b, Hill 1988b, Pintar 1988) will establish six stations for Doppler velocity observations with a spatial resolution which allows observation of modes of degree up to about 200. This network is expected to become operational in 1992. Finally, the satellite SOHO (Domingo 1988), to be launched in 1995, will carry instruments for unresolved (Damoé 1988) and highly resolved (Scherer, Hoeksema & Bogart 1988) Doppler velocity measurements, as well as instruments for measuring solar irradiance and intensity (Fröhlich *et al.* 1988). The data from these instruments will allow resolution of the depth and latitude variation of rotation in far greater detail than is now possible; undoubtedly this will cause the results presented in this review to appear extremely crude. Nevertheless it may not be unreasonable to hope that their main features will still be found to be valid.

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References.

- Aindow, A., Elsworth, Y. P., Isak, G. R., McLeod, C. P., New, R. & van der Raay, H. B., 1988. *Seismology of the Sun & Sun-like Stars*, p. 157, eds Domingo, V. & Rolfe, E. J., ESA SP-286.
- Backus, G. & Gilbert, F., 1967. *Geophys. J.*, **13**, 169.
- Backus, G. & Gilbert, F., 1970. *Phil. Trans. Roy. Soc. London*, **A266**, 123.
- Brown, T. M., 1985. *Nature*, **317**, 591.
- Brown, T. M., 1986. *Seismology of the Sun and the distant Stars*, p. 199, ed. Gough, D. O., D. Reidel Publ. Co., Dordrecht.

Brown, T. M., 1988. *Proc. IAU Symposium No 123*, p. 453, eds Christensen-Dalsgaard, J. & Frandsen, S., D. Reidel Publ. Co., Dordrecht.

Brown, T. M. & Morrow, C. A. 1987. *Astrophys. J.*, **314**, L21.

Brown, T. M., Christensen-Dalsgaard, J., Dziembowski, W. A., Goode, P., Gough, D. O., Morrow, C. A., 1989. *Astrophys. J.*, **343**, 526.

Christensen-Dalsgaard, J., 1986. *Cool Stars, Stellar Systems and the Sun. Lecture Notes in Physics*, **254**, p. 145, eds Zeilik, M. & Gibson, D. M., Springer.

Christensen-Dalsgaard, J. & Berthomieu, C., 1990. *In Solar interior and atmosphere*, eds Cox, A. N., Livingston, W. C. & Matthews, M., Space Science Series, University of Arizona Press, submitted.

Christensen-Dalsgaard, J. & Gough, D. O., 1982. *Mon. Not. R. astr. Soc.*, **198**, 141.

Christensen-Dalsgaard, J., Schou, J., 1988. *Seismology of the Sun & Sun-like Stars*, p. 149, eds Domingo, V. & Rolfe, E. J., ESA SP-286.

Christensen-Dalsgaard, J., Gough, D. O. & Toomre, J., 1985. *Science*, **229**, 923.

Christensen-Dalsgaard, J., Schou, J. & Thompson, M. J., 1990. *Mon. Not. R. astr. Soc.*, in the press.

Claverie, A., Isaak, G. R., McLeod, C. P. & van der Raay, H. B., 1981. *Nature*, **293**, 443.

Cowling, T. G. & Newling, R. A., 1949. *Astrophys. J.*, **109**, 149.

Craige, I. J. D. & Brown, J. C., 1986. *Inverse problems in astronomy: a guide to inversion strategies for remotely sensed data*, Adam Hilger, Bristol.

Cuypers, J. 1980. *Astron. Astrophys.*, **89**, 207.

Damé, L., 1988. *Seismology of the Sun & Sun-like Stars*, p. 367, eds Domingo, V. & Rolfe, E. J., ESA SP-286.

Deepak, A., 1977. *Inversion Methods in Atmospheric Remote Sounding*, Academic, New York.

Delache, P. & Scherrer, P. H., 1983. *Nature*, **306**, 651.

Demarque, P. & Guenther, D. B., 1988. *Seismology of the Sun & Sun-like Stars*, p. 99, eds Domingo, V. & Rolfe, E. J., ESA SP-286.

Deubner, F.-L. & Gough, D. O., 1984. *Ann. Rev. Astron. Astrophys.*, **22**, 593.

Deubner, F.-L., Ulrich, R. K. & Rhodes, E. J., 1979. *Astron. Astrophys.*, **72**, 177.

Dicke, R. H., 1982. *Nature*, **300**, 693.

Dicke, R. H. & Goldenberg, H. M., 1974. *Astrophys. J. Suppl.*, **27**, 131.

Dicke, R. H., Kuhn, J. R. & Libbrecht, K. G., 1987. *Astrophys. J.*, **318**, 451.

Domingo, V., 1988. *Seismology of the Sun & Sun-like Stars*, p. 363, eds Domingo, V. & Rolfe, E. J., ESA SP-286.

Durney, B. L., Hill, F. & Goode, P. R., 1988. *Astrophys. J.*, **326**, 486.

Duvall, T. L. & Harvey, J. W., 1984. *Nature*, **310**, 19.

Duvall, T. L., Dziembowski, W. A., Goode, P. R., Gough, D. O., Harvey, J. W. & Leibacher, J. W., 1984. *Nature*, **310**, 22.

Duvall, T. L., Harvey, J. W. & Pomerantz, M. A., 1986. *Nature*, **321**, 500.

Dziembowski, W. A., 1977. *Acta Astron.*, **27**, 203.

Dziembowski, W. A. & Goode, P. R., 1984. *Mem. Soc. astr. Ital.*, **55**, 185.

Fossat, E., 1988. *Seismology of the Sun & Sun-like Stars*, p. 161, eds Domingo, V. & Rolfe, E. J., ESA SP-286.

Fröhlich, C., 1988. *Proc. IAU Symposium No 123*, p. 83, eds Christensen-Dalsgaard, J. & Frandsen, S., Reidel, Dordrecht.

Fröhlich, C. & Delache, Ph., 1984a. *Mem. Soc. astr. Ital.*, **55**, 99.

Fröhlich, C. & Delache, Ph., 1984b. *Solar seismology from space*, p. 183, eds Ulrich, R. K., Harvey, J., Rhodes, E. J. & Toomre, J., NASA, JPL Publ. 84-84.

Fröhlich, C., Andersen, B. N., Berthomieu, C., Crommelynck, D., Delache, Ph., Domingo, V., Jimenez, A., Jones, A. R., Roca Cortés, T., Wehrli, Ch., 1988. *Seismology of the Sun & Sun-like Stars*, p. 371, eds Domingo, V. & Rolfe, E. J., ESA SP-286.

García, C., Pallé, P. L. & Roca Cortés, T., 1988. *Seismology of the Sun & Sun-like Stars*, p. 353, eds Domingo, V. & Rolfe, E. J., ESA SP-286.

Gilman, P. A. & Deluca, E. E., 1986. *Cool Stars, Stellar Systems and the Sun. Lecture Notes in Physics*, **254**, p. 163, eds Zeilik, M. & Gibson, D. M., Springer.

Gilman, P. A. & Miller, J., 1986. *Ap. J. Suppl.*, **61**, 585.

Gilman, P. A., Morrow, C. A. & Deluca, E. E., 1989. *Astrophys. J.*, **338**, 528.

Glatzmaier, G., 1985. *Ap. J.*, **291**, 300.

Gough, D. O., 1978. *Proc. Workshop on solar rotation*, p. 255, eds Belvedere, G. & Paterno, L., University of Catania Press.

Gough, D. O., 1981. *Mon. Not. R. astr. Soc.*, **196**, 731.

Gough, D. O., 1982a. *Nature*, **298**, 350.

Gough, D. O., 1982b. *Nature*, **298**, 334.

Gough, D. O., 1984. *Phil. Trans. R. Soc. London*, **A 313**, 27.

Gough, D. O., 1985a. *Solar Phys.*, **100**, 65.

Gough, D. O., 1985b. *Future missions in solar, heliospheric and space plasma physics*, p. 183, eds Rolfe, E. & Battrick, B., ESA SP-235, ESTEC, Noordwijk.

Gough, D. O., 1990. *In Astrophysical Fluid Dynamics*, eds Zahn, J.-P. & Zinn-Justin, J., Elsevier Science Publisher B.V., in the press.

Gough, D. O. & Taylor, P. P., 1984. *Mem. Soc. astr. Ital.*, **55**, 215.

Gough, D. O. & Thompson, M. J., 1990a. *Mon. Not. R. astr. Soc.*, in the press.

Gough, D. O. & Thompson, M. J., 1990b. *In Solar interior and atmosphere*, eds Cox, A. N., Livingston, W. C. & Matthews, M., Space Science Series, University of Arizona Press, submitted.

Gough, D. O. & Toomre, J., 1983. *Solar Phys.*, **82**, 401.

Hansen, C. J., Cox, J. P. & van Horn, H. M., 1977. *Astrophys. J.*, **217**, 151.

Harvey, J., 1988a. *Seismology of the Sun & Sun-like Stars*, p. 55, eds Domingo, V. & Rolfe, E. J., ESA SP-286.

Harvey, J., 1988b. *Seismology of the Sun & Sun-like Stars*, p. 203, eds Domingo, V. & Rolfe, E. J., ESA SP-286.

Henning, H. M. & Scherrer, P. H., 1988. *Seismology of the Sun & Sun-like Stars*, p. 419, eds Domingo, V. & Rolfe, E. J., ESA SP-286.

Hill, F., 1987. *The internal solar angular velocity*, p. 45, eds Durney, B. R. & Sofia, S., Reidel, Dordrecht.

Hill, F., 1988a. *Astrophys. J.*, **333**, 996.

Hill, F., 1988b. *Seismology of the Sun & Sun-like Stars*, p. 209, eds Domingo, V. & Rolfe, E. J., ESA SP-286.

Hill, F., 1989a. *Astrophys. J.*, **343**, L69.

Hill, F., 1989b. *Inside the Sun*, Proc. IAU Colloquium No. 121, eds Berthomieu G. & Cribier M., in the press.

Hill, F., Gough, D. O. & Toomre, J., 1984a. *Mem. Soc. astr. Ital.*, **55**, 153.

Hill, F., Gough, D. O. & Toomre, J., 1984b. *Solar seismology from space*, p. 95, eds Ulrich, R. K., Harvey, J., Rhodes, E. J. & Toomre, J., NASA, JPL Publ. 84-84.

Hill, F., Gough, D. O., Toomre, J. & Haber, D. A., 1988. *Proc. IAU Symposium No 123*, p. 45, eds Christensen-Dalsgaard, J. & Frandsen, S., D. Reidel Publ. Co., Dordrecht.

Hill, H. A., 1978. *The New Solar Physics*, chapter 5 ed. Eddy, J. A., Westview Press, Boulder CO.

Hill, H. A. & Stebbins, R. T., 1975. *Astrophys. J.*, **200**, 471.

Hill, H. A., Bos, R. J. & Goode, P. R., 1982. *Phys. Rev. Lett.*, **49**, 1794.

Hill, H. A., Rabaey, G. F., Yakowitz, D. S., & Rosenwald, R. D., 1986. *Astrophys. J.*, **310**, 444.

Isaak, G. R., 1982. *Nature*, **296**, 130.

Jefferies, S. M., Pomerantz, M. A., Duvall, T. L., Harvey, J. W. & Jaksha, D. B., 1988. *Seismology of the Sun & Sun-like Stars*, p. 279, eds Domingo, V. & Rolfe, E. J., ESA SP-286.

Korzenik, S. G., Cacciani, A., Rhodes, E. J., Tomczyk, S. & Ulrich, R. K., 1988. *Seismology of the Sun & Sun-like Stars*, p. 117, eds Domingo, V. & Rolfe, E. J., ESA SP-286.

Kosovichev, A. G., 1988. *Pis'ma Astron. Zh.*, **14**, 344 (English translation: *Sov. Astron. Lett.*, **14**, 145).

Kosovichev, A. G. & Parchevskii, K. V., 1988. *Pis'ma Astron. Zh.*, **14**, 473 (English translation: *Sov. Astron. Lett.*, **14**, 201).

Ledoux, P., 1949. *Mem. Soc. R. Sci. Liège*, 4th ser., **9**, 263.

Libbrecht, K. G., 1986. *Nature*, **319**, 753.

- Libbrecht, K. G., 1988a. *Astrophys. J.*, **334**, 510.
- Libbrecht, K. G., 1988b. *Seismology of the Sun & Sun-like Stars*, p. 131, eds Domingo, V. & Rolfe, E. J., ESA SP-286.
- Libbrecht, K. G., 1989. *Astrophys. J.*, **336**, 1092.
- Loumos, G. L. & Deeming, T. J., 1978. *Astrophys. Space Science*, **56**, 285.
- Morrow, C.A. 1988. *Seismology of the Sun & Sun-like Stars*, p. 137, eds Domingo, V. & Rolfe, E. J., ESA SP-286.
- Narlikar, J. V. & Rana, N. C., 1985. *Mon. Not. R. astr. Soc.*, **213**, 657.
- Noyes, R. W., Hartman, L. W., Ballunas, S. L., Duncan, D. K. & Vaughan, A. H., 1984. *Astrophys. J.*, **279**, 763.
- Pallé, P. L. & Roca Cortés, T., 1988. *Proc. IAU Symposium No 123*, p. 79, eds Christensen-Dalsgaard, J. & Frandsen, S., D. Reidel Publ. Co., Dordrecht.
- Pallé, P. L., Pérez Hernandez, F., Régulo, C. & Roca Cortés, T., 1988. *Seismology of the Sun & Sun-like Stars*, p. 125, eds Domingo, V. & Rolfe, E. J., ESA SP-286.
- Parker, R. L., 1977. *Ann. Rev. Earth Planet. Sci.*, **5**, 35.
- Pidatella, R. M., Stix, M., Belvedere, G. & Paterno, L., 1986. *Astron. Astrophys.*, **156**, 22.
- Phillips, D. L., 1962. *J. Assoc. Comput. Mach.*, **9**, 84.
- Pinsoneault, M. H., Kawaler, S. D., Sofia, S. & Demarque, P., 1989. *Astrophys. J.*, **338**, 424.
- Pintar, J. A., 1988. *Seismology of the Sun & Sun-like Stars*, p. 217, eds Domingo, V. & Rolfe, E. J., ESA SP-286.
- Rhodes, E. J., Cacciani, A. & Korzenik, S. G., 1988. *Seismology of the Sun & Sun-like Stars*, p. 81, eds Domingo, V. & Rolfe, E. J., ESA SP-286.
- Rhodes, E. J., Cacciani, A., Korzenik, S. G., Tomczyk, S., Ulrich, R. K. & Woodard, M. F., 1988. *Seismology of the Sun & Sun-like Stars*, p. 73, eds Domingo, V. & Rolfe, E. J., ESA SP-286.
- Rosner, R. & Weiss, N. O., 1985. *Nature*, **317**, 790.
- Schüssler, M., 1987. *The internal solar angular velocity*, p. 303, eds Durney, B. R. & Sofia, S., Reidel, Dordrecht.
- Scherrer, P. H., Hoeksma, J. T. & Bogart, R. S., 1988. *Seismology of the Sun & Sun-like Stars*, p. 375, eds Domingo, V. & Rolfe, E. J., ESA SP-286.
- Snodgrass, H. B., 1983. *Astrophys. J.*, **270**, 288.
- Snodgrass, H. B., 1984. *Solar Phys.*, **94**, 13.
- Spruit, H. C., 1987. *The internal solar angular velocity*, p. 185, eds Durney, B. R. & Sofia, S., Reidel, Dordrecht.
- Spruit, H. C., Knobloch, E. & Roxburgh, I. W., 1983. *Nature*, **304**, 520.
- Stenflo, J. O., 1989. *Astron. Astrophys.*, **210**, 403.
- Stix, M., 1987. *The internal solar angular velocity*, p. 329, eds Durney, B. R. & Sofia, S., Reidel, Dordrecht.
- Sørensen, J. M., 1988. *Seismology of the Sun & Sun-like Stars*, p. 41, eds Domingo, V. & Rolfe, E. J., ESA SP-286.
- Tarrantola, A., 1987. *Inverse Problem Theory*, Elsevier, Amsterdam.
- Thompson, M. J., 1990. *Solar Phys.*, in the press.
- Tomczyk, S., 1988. *A measurement of the rotational frequency splitting of the solar five-minute oscillations*, PhD dissertation, University of California, Los Angeles.
- Tomczyk, S., Cacciani, A., Korzenik, S. G., Rhodes, E. J. & Ulrich, R. K., 1988. *Seismology of the Sun & Sun-like Stars*, p. 141, eds Domingo, V. & Rolfe, E. J., ESA SP-286.
- van der Raay, H. B., Pallé, P. L. & Roca Cortés, T., 1986. *Seismology of the Sun and the distant stars*, p. 215, ed. Gough, D. O., Reidel, Dordrecht.