CERN-TH.5712/90
CPH-A964.0490

**DOUBLING OF EQUATIONS AND UNIVERSALITY
IN MATRIX MODELS OF RANDOM SURFACES**

C. BACHAS and P. M. S. PETROPOULOS

*Theory Division, CERN
1211 Geneva 23, Switzerland*

and

*CPTh, ECOLE POLYTECHNIQUE
91128 Palaiseau Cedex, France*

Abstract

We prove that in the recently proposed scaling limit [1] in matrix models of random surfaces, the (singular piece of the) free-energy is obtained from the *sum* of solutions of *two* non-linear differential equations: $\epsilon_{\pm} z = \mathcal{D}^{(m)}(f \pm g)$. These are identical and universal modulo the two arbitrary parameters $\epsilon_{\pm}$, and the (common) normalization of the string coupling z . The doubling of equations implies a doubling of non-perturbative parameters. For even matrix-potentials one of the non-universal constants is fixed: $\epsilon_+ = \epsilon_-$, and the scaling function g vanishes to all orders in the loop expansion.

CERN-TH.5712/90
CPH-A964.0490
April 1990

The recently discovered double scaling limits [1] in the matrix-models of random discrete surfaces [2] have raised the hope for a non-perturbative understanding of string theory. After appropriate integration of angular variables, these matrix models describe the statistical mechanics of a gas of particles with logarithmic repulsive interactions moving in a one-dimensional potential well [3]. In the original treatments [1] this potential was for simplicity taken to be even, a restriction that hides part of the structure of the critical behaviour, and somewhat obscures the issue of universality. In this letter we will consider arbitrary potentials. We will prove that for a generic critical point corresponding to the case of pure gravity ($d = 0$) one obtains a pair of independent Painlevé equations:

$$\begin{aligned}\epsilon_+ z &= (f + g)^2 + \lambda(f + g)'' \\ \epsilon_- z &= (f - g)^2 + \lambda(f - g)''\end{aligned}\quad (1)$$

for two scaling functions $f \pm g$. The singular part of the free-energy is related to the scaling function f , and thus depends on *three* non-universal constants: ϵ_+ , λ (two of which can be absorbed in rescalings of f and the string coupling z), and *two* non-perturbative parameters. The scaling function g is related to the breaking of the parity-symmetry; it vanishes to all orders of perturbation theory ($z \rightarrow \infty$) when the potential is even in which case: $\epsilon_+ = \epsilon_-$.

The doubling of equations and of non-perturbative parameters also holds for higher order multicritical points. The number of non-universal constants remains equal to *three*, including the normalizations of z and f . The fact that one piece of (the singular) free-energy obeys a universal equation has been established independently by O. Alvarez and P. Windey [5]. Our result however is stronger: it shows that the *entire (singular) free-energy is universal*, modulo one extra parameter in addition to two normalizations.

We begin by reviewing rapidly the, by now well-known, orthogonal-polynomial method [4] for solving the one-hermitean-matrix model. The partition function reads:

$$\begin{aligned}Z &= \int dM e^{-\frac{N}{2} \text{Tr}(V(M))} \\ &= \int dx_1 \dots dx_N \prod_{i < j} (x_i - x_j)^2 e^{-\frac{N}{2} (V(x_1) + \dots + V(x_N))}\end{aligned}$$

where:

$$V(x) = \sum_{k=2}^K \frac{g_k}{k} x^k$$

is the potential, T the temperature, and $N \times N$ the size of the matrix M . One can define a set of polynomials of order n :

$$P_n(x) = x^n + \dots$$

which are orthogonal with respect to the measure

$$d\mu(x) = dx e^{-\frac{N}{2} V(x)},$$

namely:

$$\int d\mu P_n(x) P_l(x) = h_n \delta_{nl}.$$

The general decomposition:

$$x P_n(x) = P_{n+1}(x) + S_n P_n(x) + R_n P_{n-1}(x) \quad (2)$$

defines the basic quantities of interest: R_n and S_n^* . In terms of them one can compute all correlation-functions of the matrix model. In particular, the free-energy is given by:

$$\ln Z = \sum_{i=1}^N (N - i) \ln R_i + N \ln h_0$$

and hence depends only on the quantities R_n .

To derive equations for the R_n , S_n one uses the facts that: $n h_{n-1} = \int d\mu P_n' P_{n-1}$ and $0 = \int d\mu P_n' P_n$ and integrates by part to obtain:

$$\frac{n}{N} = \sum_{k=2}^K g_k \sum_{\substack{\text{walks from} \\ n \text{ to } n-1 \text{ in} \\ k-1 \text{ steps}}} (\text{weight}) \equiv \mathcal{A}(R_i, S_i), \quad (3.a)$$

$$0 = \sum_{k=2}^K g_k \sum_{\substack{\text{walks from} \\ n \text{ to } n \text{ in} \\ k-1 \text{ steps}}} (\text{weight}) \equiv \mathcal{B}(R_i, S_i) \quad (3.b)$$

where the walks are on a one-dimensional ladder, and the weight of each "time step" is: 1 for going up, S_i for staying in place at height l and R_l for going one step down starting from height l . For the sake of concreteness we give the first few terms of the r.h.s. of eqs. (3.a,b):

$$\begin{aligned}\mathcal{A} &= g_2 R_n + g_3 (R_n S_n + R_n S_{n-1}) + \\ &\quad + g_4 (R_n S_n^2 + R_n S_n S_{n-1} + R_n S_{n-1}^2 + R_n R_{n+1} + R_n^2 + R_n R_{n-1}) + \dots, \\ \mathcal{B} &= g_2 S_n + g_3 (S_n^2 + R_n + R_{n+1}) + \\ &\quad + g_4 (S_n^3 + 2 S_n R_{n+1} + 2 S_n R_n + S_{n+1} R_{n+1} + S_{n-1} R_n) + \dots.\end{aligned}$$

* Lower order polynomials do not appear in this decomposition because $x P_l$ is orthogonal to P_n for $l \leq n-2$; we are of course assuming that all moments of the measure are defined.

For even potentials one has $S_n = 0$, in which case eq. (3.b) is automatically satisfied: the reason is that one can not come back to one's point of departure in an odd number of steps, when "staying in place" is not an allowed elementary move. For arbitrary potentials, on the other hand, we must keep track of both eqs. (3.a,b).

To analyse these equations, we set $n/N = x$, $R_n = R(x)$, $S_n = S(x)$ and Taylor expand:

$$R_{n+k} = R(x) + \frac{k}{N} R'(x) + \frac{1}{2} \left(\frac{k}{N} \right)^2 R''(x) + \dots$$

and similarly for S , where k is a finite integer, smaller or equal to the order of the potential V . Now A and B are functions of R , S and all of their derivatives at the point x . We will use the notation:

$$A = A + \frac{1}{N} A_{(1)} R' + \frac{1}{N} A^{(1)} S' + \dots + \left(\frac{1}{N} \right)^{\sum_{j=1}^r p_j + \sum_{j=1}^s q_j} \times \\ \times \prod_{i=1}^r (p_i!)^{-1} \prod_{j=1}^s (q_j!)^{-1} A_{(p_1 \dots p_r, q_1 \dots q_s)}^{(q_1 \dots q_s)} R^{(p_1)} \dots R^{(p_r)} S^{(q_1)} \dots S^{(q_s)} + \dots$$

where $R^{(p_i)}$ stands for the p_i th derivative of $R(x)$, and the $A_{(p_1 \dots p_r)}^{(q_1 \dots q_s)}$ are functions only of R and S at the point x , but not of their derivatives. A similar expansion also holds for the function B .

Consider first the naïve $N \rightarrow \infty$ limit, corresponding to surfaces of spherical topology, in which case one can neglect all derivative-terms in the above expansions. Eqs. (3.a,b) then take the elegant form:

$$xT = A = \frac{\partial \Omega}{\partial S}, \quad (4.a)$$

$$0 = B = \frac{\partial \Omega}{\partial R} \quad (4.b)$$

where:

$$\Omega(R, S) = \frac{1}{2\pi i} \oint dz V\left(z + \frac{R}{z} + S\right). \quad (5)$$

The function Ω is a polynomial of degree: $\deg S + 2 \deg R \leq K + 1$, characterized by the relation:

$$R \frac{\partial^2 \Omega}{\partial R^2} = \frac{\partial^2 \Omega}{\partial S^2}. \quad (6)$$

This relation is easily established if one notes that to every walk ending one step down from its point of departure, there is a "reflected" walk ending one step up, with weight differing in the large N limit by one factor of R (see fig. 1).

Now let us assume that the potential V is such that, at some specific values R_c , S_c one has: $\partial \Omega / \partial S|_c = T_c$, while all other derivatives of Ω up to some order m vanish. By successive differentiation of eq. (6) one finds:

$$\alpha m! \equiv \frac{\partial^{m+1} \Omega}{\partial S \partial R^m} \Big|_c = \frac{\partial^{m+1} \Omega}{\partial S^3 \partial R^{m-2}} \Big|_c R_c^{-1} = \frac{\partial^{m+1} \Omega}{\partial S^5 \partial R^{m-4}} \Big|_c R_c^{-2} = \dots, \quad (7.a)$$

$$\beta m! \equiv \frac{\partial^{m+1} \Omega}{\partial R^{m+1}} \Big|_c R_c^{1/2} = \frac{\partial^{m+1} \Omega}{\partial S^2 \partial R^{m-1}} \Big|_c R_c^{-1/2} = \frac{\partial^{m+1} \Omega}{\partial S^4 \partial R^{m-3}} \Big|_c R_c^{-3/2} = \dots \quad (7.b)$$

i.e. all $(m+1)$ th derivatives are functions of three non-universal parameters: α , β and R_c depending on the details of the potential. Note that S_c can always be chosen equal to zero, by changing matrix variables: $M \rightarrow M - 1S_c$, thus shifting the potential: $V(x) \rightarrow V(x - S_c)$. With this choice it is easy to check that $\beta = 0$ if the potential is even, while $\alpha = 0$ if the potential is odd. We will thus refer to α and β as the *even* and *odd* parameters, respectively.

Working in the vicinity of $x = 1 - \delta x$, $T = T_c + \delta T$, and letting: $R(x) = R_c + \delta R(x)$, $S(x) = S_c + \frac{1}{\sqrt{R_c}} \delta S(x)$ we may write eqs. (4.a,b), with the help of (7.a,b), as follows:

$$\delta T - T_c \delta x = \alpha \left\{ \delta R^m + \binom{m}{2} \delta R^{m-2} \delta S^2 + \binom{m}{4} \delta R^{m-4} \delta S^4 + \dots \right\} \\ + \beta \left\{ \binom{m}{1} \delta R^{m-1} \delta S + \binom{m}{3} \delta R^{m-3} \delta S^3 + \dots \right\} \\ + \text{higher order terms}, \quad (8.a)$$

$$0 = \beta \left\{ \delta R^m + \binom{m}{2} \delta R^{m-2} \delta S^2 + \binom{m}{4} \delta R^{m-4} \delta S^4 + \dots \right\} \\ + \alpha \left\{ \binom{m}{1} \delta R^{m-1} \delta S + \binom{m}{3} \delta R^{m-3} \delta S^3 + \dots \right\} \\ + \text{higher order terms} \quad (8.b)$$

or, taking the sum and difference:

$$\delta T - T_c \delta x = (\alpha \pm \beta) (\delta R \pm \delta S)^m + \text{higher order terms}.$$

One possible consistent solution is:

$$\delta R \sim \delta S \sim (\delta T - T_c \delta x)^{\frac{1}{m}} \quad (9)$$

where "higher orders" can indeed be neglected. We will come back to the possibility of other critical behaviours later.

For the time being we will assume behaviour (8) and go beyond the spherical topology to the double scaling limit [1]: $N \rightarrow \infty$, $\delta x \rightarrow 0$ and $\delta T \rightarrow 0$ with:

$$z \equiv \frac{1}{\sqrt{R_c}} \left(\frac{\delta T}{T_c} - \delta x \right) N^{\frac{2-m}{2m+1}} \quad (10.a)$$

$$f(z) \equiv \delta R N^{\frac{2}{2m+1}} \quad (10.b)$$

$$g(z) \equiv \delta S N^{\frac{2}{2m+1}} \quad (10.c)$$

held fixed. Eqs. (8.a,b) receive in this case derivative corrections. To analyse them we will use an integral representation for the functions $A_{(P)}^{(Q)}$ and $B_{(P)}^{(Q)}$ by slightly modifying a trick of Douglas and Shenker [1]. In analogy to eqs. (4,5) one can write:

$$\begin{aligned} A_{(P)}^{(Q)} &= \frac{\partial}{\partial S} \Omega_{(P)}^{(Q)} \\ B_{(P)}^{(Q)} &= \frac{\partial}{\partial R} \Omega_{(P)}^{(Q)} \end{aligned} \quad (11)$$

with:

$$\begin{aligned} \Omega_{(P)}^{(Q)} &= -\frac{1}{4\pi^2} \oint dz \oint \frac{dw}{w} V(w) \times \\ &\times \sum_{\text{perms } i=1}^r \prod_{i=1}^r \left(D \frac{1}{wz} \left(\frac{\partial}{\partial \ln z} \right)^{p_i} \right) \prod_{j=1}^s \left(D \frac{1}{w} \left(\frac{\partial}{\partial \ln z} \right)^{q_j} \right) D. \end{aligned} \quad (12)$$

Here

$$D = \frac{1}{1 - w^{-1} \left(z + \frac{R}{z} + S \right)} \quad (13)$$

is a generating function whose $w^{-t} z^j$ term gives the weight of walks ending t steps higher in t elementary moves, and the summation is over all *distinguishable permutations* of the $r + s$ insertions in eq. (12). The reader can easily verify that the above expression reduces to our function Ω for empty sequences of integers $P \equiv p_1 \dots p_r$ and $Q \equiv q_1 \dots q_s$, i.e. in the absence of insertions. We will in fact find it convenient to extend our notation and allow some of the integers in the sequences P and Q to be zero, by defining:

$$A_{(0 \dots 0 \tilde{P})}^{(0 \dots 0 \tilde{Q})} \equiv \frac{1}{l! k!} \frac{\partial^{l+k}}{\partial R^l \partial S^k} A_{(\tilde{P})}^{(\tilde{Q})}$$

$\begin{matrix} \text{\scriptsize k times} \\ \text{\scriptsize l times} \end{matrix}$

with $\tilde{P}, \tilde{Q}$ sequences of positive integers. Expressions (11) and (12) stay valid with this extended notation.

Returning now to our basic equations (3.a,b), we may write a generic term as:

$$\left(\frac{1}{N} \right)^{\sum_{i=1}^r p_i + \sum_{j=1}^s q_j} \prod_{i=1}^r (p_i!)^{-1} \prod_{j=1}^s (q_j!)^{-1} A_{(P)}^{(Q)}|_c \delta R^{(p_r)} \dots \delta R^{(p_1)} \frac{\delta S^{(q_1)}}{\sqrt{R_c}} \dots \frac{\delta S^{(q_s)}}{\sqrt{R_c}}$$

with the understanding that $\delta R^{(0)} = \delta R$, $\delta S^{(0)} = \delta S$. This term is characterized by an index:

$$\text{index}(P, Q) = \sum_{i=1}^r \left(\frac{1}{2} p_i + 1 \right) + \sum_{j=1}^s \left(\frac{1}{2} q_j + 1 \right) \quad (14)$$

so that it is of lower, equal or higher order than the terms we kept in eqs. (8.a,b), according to whether the index is less than, equal to or bigger than m respectively. Note that according to the scaling ansatz (10), a derivative counts as "half the occurrence" of a δR or δS factor. Now the proof of universality is a consequence of the following:

LEMMA: $\Omega_{(P)}^{(Q)}$ is a linear combination of derivatives of Ω , of order less than or equal to $\text{index}(P, Q)$.

Since by our criticality assumption all derivatives of Ω of order at most m vanish, it follows that all potentially lower order additions to eqs. (8.a,b) vanish, while all terms of the same order as those kept have coefficients depending on the $(m+1)$ th derivatives of Ω at R_c, S_c , and hence are functions of only three non-universal parameters: α, β and R_c . Furthermore there exists a generalization of the equation (6) relating terms in which two occurrences of δR are replaced by two occurrences of δS . Letting: $P' \equiv p_1 \dots p_i \dots \hat{p}_i \dots p_r, Q' \equiv q_1 \dots q_s p_i p_j$ and assuming that $\text{index}(P, Q) = \text{index}(P', Q')$ is integer, this relation reads:

$$R C_{(P)}^{(Q)} \Omega_{(P)}^{(Q)} = C_{(P')}^{(Q')} \Omega_{(P')}^{(Q')} + \text{lower order terms} \quad (15)$$

where the omitted terms are derivatives of Ω of order $< \text{index}(P, Q)$, and $C_{(P)}^{(Q)}$ is a combinatorial factor counting the number of inequivalent arrangements of the corresponding $r + s$ insertions. With the help of this relation one can show that the sum and difference of the r.h.s. of eqs. (3.a,b) combine precisely, as in the naïve $N \rightarrow \infty$ limit, to yield:

$$\epsilon_{\pm} z = \mathcal{D}^{(m)}(f \pm g) \quad (16)$$

where:

$$\epsilon_{\pm} = \frac{T_c \sqrt{R_c}}{\alpha \pm \beta} \quad (17)$$

and $\mathcal{D}^{(m)}(h(z))$ is the universal expression of h and its derivatives up to order $2m - 2$, found in refs. [1], i.e.

$$\begin{aligned}\mathcal{D}^{(2)}(h) &= h^2 + \frac{1}{3}h'', \\ \mathcal{D}^{(3)}(h) &= h^3 + hh'' + \frac{1}{2}(h')^2 + \frac{1}{10}h^{(4)}, \\ &\dots\end{aligned}$$

Before sketching the proof of these statements for arbitrary m , let us first illustrate them in the simplest case $m = 2$. In this case we need only look at terms with at most two derivatives and a single occurrence of δR or δS ; we are thus interested in the functions $\Omega_{(1)}$, $\Omega_{(2)}$ and $\Omega^{(2)}$. In order to check the lemma one must first carry out the differentiations in expression (12), then perform the w -contour integral around the point:

$$y = z + \frac{R}{z} + S$$

and finally use integrations by part. For instance:

$$\begin{aligned}\Omega_{(1)} &= -\frac{1}{4\pi^2} \oint dz \oint \frac{dw}{w} V(w) D \frac{1}{wz} \frac{\partial}{\partial \ln z} D \\ &= \frac{1}{2\pi i} \oint \frac{dz}{z} \frac{\partial y}{\partial \ln z} \frac{1}{2} V''(y) = \frac{1}{4\pi i} \oint \frac{dz}{z} V'(y) \\ &= \frac{1}{2} \frac{\partial \Omega}{\partial R}.\end{aligned}$$

Thus $A_{(1)}|_c = \partial \Omega_{(1)} / \partial S|_c$ and $B_{(1)}|_c = \partial \Omega_{(1)} / \partial R|_c$ both vanish by our criticality assumption, as they should since they multiply terms of index $3/2 < m$; the same is true for $A^{(1)}|_c$ and $B^{(1)}|_c$. Next consider:

$$\begin{aligned}\Omega_{(2)} &= -\frac{1}{4\pi^2} \oint dz \oint \frac{dw}{w} V(w) D \frac{1}{wz} \frac{\partial^2}{\partial \ln z^2} D \\ &= \frac{1}{2\pi i} \oint \frac{dz}{z} \left(\frac{1}{3} \left(\frac{\partial y}{\partial \ln z} \right)^2 V'''(y) + \frac{1}{2} \frac{\partial^2 y}{\partial \ln z^2} V''(y) \right) \\ &= \frac{1}{2} \frac{\partial \Omega}{\partial R} + \frac{1}{3} \frac{\partial^2 \Omega}{\partial R^2}.\end{aligned}$$

Thus, using also eqs. (7), we find: $A_{(2)}|_c = \frac{2}{3} R_c \alpha$ and $B_{(2)}|_c = \frac{2}{3} \sqrt{R_c} \beta$; similarly one can compute $A^{(2)}|_c = \frac{2}{3} R_c^{3/2} \beta$ and $B^{(2)}|_c = \frac{2}{3} R_c \alpha$. Putting all this

together, plugging in the definitions (10.a-c) and taking the sum and difference of eqs. (3.a,b), one arrives precisely at the two independent Painlevé eqs. (1) as advertised, with $\epsilon_{\pm}$ given by eq. (17) and $\lambda = 1/3$. Note that this is not a trivial statement since eqs. (3.a,b) could a priori be characterized by *ten* non-universal parameters in the $m = 2$ continuum limit! Note also that the (non-universal) normalization of the string coupling, (10a), has absorbed the parameter λ which was left arbitrary in eq. (1) in order to make the counting of non-universal constants manifest. Note finally that the normalization of the scaling function $g \sim (S - S_c)$ is not an independent parameter.

Let us now proceed to the general

PROOF OF THE LEMMA: carrying out the differentiations in expression (12), and using the fact that: $\partial^t y / \partial \ln z^t = (z + (-)^t z / R)$, we can expand:

$$\begin{aligned}\Omega_{(P)}^{(Q)} &= \sum d_{lk} \oint \frac{dz}{z^l} \left(z + \frac{R}{z} \right)^l \left(z - \frac{R}{z} \right)^k \times \\ &\quad \times \oint dw \left(\frac{D}{w} \right)^{l+r+s+t+k} V(w).\end{aligned}\quad (18)$$

Now we can perform the w -integration, and at least $\left\lfloor \frac{k+l}{2} \right\rfloor$ integrations by parts using the fact that: $(z - R/z) V^{(n)}(y) = \partial V^{(n-1)}(y) / \partial \ln z$. We obtain:

$$\Omega_{(P)}^{(Q)} = \sum c_{nl} R^{l-n} \oint \frac{dz}{z^l} V^{(n)}(y) \quad (19)$$

where $n \leq \text{index}(P, Q)$ and $|l| \leq n$. The second inequality follows trivially from eq. (18); the first follows from the fact that each $\partial / \partial \ln z$ in expression (12) can either increase the number of $w^{-1} D$ and $(z - R/z)$ -factors by one, or "destroy" one $(z - R/z)$ -factor, or do neither. Finally using:

$$\oint dz z^p V^{(n)} = (1 - \rho) \oint dz z^{\rho-1} V^{(n-1)} + 2R \oint dz z^{\rho-2} V^{(n)}$$

we can express the r.h.s. of (19) as a sum of derivatives of Ω , of order at most $\text{index}(P, Q)$,

QED.

Relation (15) is now easy to establish; the point is that modulo the combinatorial factor, a δR -insertion differs from a δS -insertion in eq. (12) by one extra power of z^{-1} , which can be brought through all derivative operators, modulo lower order terms. The difference between the two sides of (15), has thus an extra power of $(z - R/z)$ and by our previous arguments is also lower order. Similarly one can relate $A_{(P)}^{(Q)}$ to $B_{(P')}^{(Q')}$ where (P', Q') is obtained from (P, Q) by replacing a single

δR -occurrence by a δS -one. Finally note that for an even (odd) potential, all odd (even) terms in δS in eq. (3.a) as well as all even (odd) terms in (3.b) vanish. Consequently these terms are all proportional to $\beta(c)$. To finally derive eqs. (16), (17) one need only keep carefully track of the combinatorial and R_c -factors. This is straightforward and totally unilluminating, so we will skip it.

Let us briefly discuss the solutions of the pair of Painlevé equations (1). In the weak coupling limit ($z \rightarrow \infty$) one finds:

$$\left. \begin{aligned} f \\ g \end{aligned} \right\} = \sum_{l=0}^{\infty} C_l z^{\frac{l-1}{2}} \left(\epsilon_+^{\frac{l-1}{2}} \pm \epsilon_-^{\frac{l-1}{2}} \right) \\ + z^{-\frac{1}{2}} \left\{ \theta_+ \exp \left(-\frac{4\sqrt{6}}{5} \epsilon_+^{\frac{1}{2}} z \right) \pm \theta_- \exp \left(-\frac{4\sqrt{6}}{5} \epsilon_-^{\frac{1}{2}} z \right) \right\} + \dots$$

with l the number of loops, C_l universal numbers [1] and $\theta_{\pm}$ the two non-perturbative parameters. Note that to fix all non-universal constants one must know the partition function up to and including two loops. Note also that the scaling function g vanishes to all orders in the loop expansion for even potentials ($\epsilon_+ = \epsilon_-$), as already advertized; the breaking of the parity-symmetry could however occur spontaneously at the non-perturbative level in this case.

We conclude with an important remark: in this paper we have assumed that the extension of Kazakov's multicriticality condition [6] to arbitrary potentials is the condition that all derivatives of Ω up to order m vanish. Derivatives of order $m+1$ are generically non-zero, and one finds the scaling: $\delta R \sim \delta S \sim \delta T^{1/m}$, with $\delta S = 0$ as a special case, for $\beta = 0$ in eq. (7.b). Different scaling behaviours are, however, allowed at least on the sphere. For instance, for $\beta = 0$ and m even, and assuming non-vanishing $(m+2)$ th derivatives, one has a consistent solution: $\delta R \sim \delta S^2 \sim \delta T^{2/m}$ with eqs. (3.a,b) reading:

$$\begin{aligned} \delta T - T_c \delta x &= a \delta S^m + \text{higher order terms} \\ 0 &= a \delta R \delta S^{m-1} + b \delta S^{m+1} + \text{higher order terms} . \end{aligned}$$

By fine-tuning enough parameters one can, more generally find: $\delta R \sim \delta S^n \sim \delta T^{n/m}$. The possibility of defining double scaling limits for such multicritical points is currently under investigation.

References

- [1] E. Brézin and V. A. Kazakov, Phys. Lett. **236B** (1990) 144;
M. R. Douglas and S. Shenker, RU-89/34 Rutgers preprint, (October 1989);
D. J. Gross and A. A. Migdal, Phys. Rev. Lett. **64** (1990) 127.
- [2] F. David, Nucl. Phys. **B257** [FS14] (1985) 45;
V. A. Kazakov, Phys. Lett. **150B** (1985) 282;
J. Ambjørn, B. Durhuus and J. Fröhlich, Nucl. Phys. **B257** [FS14] (1985) 433.
- [3] F. Dyson, J. Math. Phys. **3** (1962) 140-175;
E. Brézin, C. Itzykson, G. Parisi and J. B. Zuber, Commun. Math. Phys. **59** (1978) 35.
- [4] D. Bessis, Commun. Math. Phys. **69** (1979) 147;
C. Itzykson and J. B. Zuber, J. Math. Phys. **21** (1980) 411;
D. Bessis, C. Itzykson and J. B. Zuber, Adv. Appl. Math. **1** (1980) 109.
- [5] O. Alvarez and P. Windey, PAR-LPTHE 90-12 preprint, (February 1990).
- [6] V. A. Kazakov, Mod. Phys. Lett. **A4** (1989) 2125.

Figure Captions

FIG. 1 An example of a 14-step walk ending one step down, and its reflected image ending one step up; their weights differ in the "spherical" $N \rightarrow \infty$ limit, by one factor of R .

Figure

Fig.1