ABELIANIZATION OF GAUGE ALGEBRAS IN THE HAMILTONIAN FORMALISM

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ABSTRACT

The Abelianization of gauge algebras for general models with first-class constraints is considered at the classical level. An explicit ansatz for the Abelian constraints is proposed. It is shown that under comparatively weak conditions, this ansatz will solve the problem of Abelianization. The finite canonical transformation, which relates the corresponding BRST charges, is discussed and is explicitly constructed for the case of a general Lie algebra. As examples, we treat the models of a scale-invariant point particle, bosonic string and pure Yang-Mills.

1. Introduction

Our understanding of the quantization of theories with first-class constraints has greatly improved through the use of the BRST formalism.^[1] In a series of papers,^[2] we have examined the use of the BRST and antiBRST symmetry in a quantum theory and have proved general theorems for the decoupling of the Faddeev-Popov ghosts. We are then left with a set of operator equations whose solutions determine the physical BRST-invariant matter state space and thus the question of unitarity. The main assumption underlying the treatment given in these papers is that solutions to the BRST (and antiBRST, if included) equation

$$Q_B |phys\rangle = 0 \quad (1.1)$$

would at least contain a state of the form

$$|phys\rangle = |matter\rangle |0\rangle_{ghost}. \quad (1.2)$$

This is clearly a reasonable assumption, since we would believe that for a consistent theory, only some unique ghost state, "vacuum" state, would not decouple. It is, for example, the case in string theory. Unfortunately, unlike in string theory, we do not know in general how to find a solution of the form eq.(1.2). One has to find suitable state spaces for both the ghost variables and the matter variables, which admit these solutions. We are, therefore, faced with the questions of how we should find these state spaces in which we can find solutions of the form (1.2) and then how to solve the ensuing operatorial constraint equations. It is finding answers to these questions that motivated this work. Our considerations here, however, will be purely classical, but it is our belief that in specific theories, one may use these techniques at the quantum level. We will briefly discuss some quantum aspects in the examples we give.

There is another issue, which is related to the one above. When one uses the path integral approach, it is crucial that one fixes the gauge, to get a well-defined quantum theory. Consequently, the statement analogous to eq.(1.1) has no meaning in the path integral without a gauge fixing. This is the reason why the BRST symmetry was at first thought to be intimately connected to the gauge fixing procedure. After all, the BRST symmetry was

discovered in a path integral formalism. In the canonical approach this is not so. There the eq.(1.1) has no reference to a gauge fixing. If the BRST charge commutes with the Hamiltonian, which may be determined independently of any gauge fixing, then we can choose to solve eq.(1.1) at a fixed time. This means that it can be solved without any knowledge of the dynamics of the theory. It seems, therefore, that the two quantization procedures are different, as far as the BRST symmetry is concerned. This is, however, a slightly simplified picture. Although one may state the BRST equation (1.1) without reference to a gauge fixing, it is in actually solving the equation that one may encounter analogous restrictions. In particular, if we solve them by using the approach to Abelianization considered in this paper, then there is, as we will see, a very close connection between a gauge fixing procedure and the problem of finding Abelian constraints.

The basic idea is to take a general set of first-class constraints, $\psi_a, a = 1, \dots, m$, which for simplicity we take to be bosonic, satisfying the equal time Poisson bracket (PB)

$$\{\psi_a, \psi_b\} = U_{ab}^c \psi_c \quad (1.3)$$

and then substituting these for another set of constraints ϕ_a

$$\phi_a \equiv S_a^b \psi_b \quad (1.4)$$

where S_a^b is an invertible coordinate dependent matrix, such that

$$\{\phi_a, \phi_b\} = 0. \quad (1.5)$$

i.e. the constraints form an Abelian algebra. Then it is trivial to solve the constraints, since we can identify them locally as some momenta p_a . For a set of constraints forming an arbitrary algebra, this identification can only be true on the constraint surface. That one indeed can always locally perform the substitution eq.(1.4) is a relatively recent result^[3] and, more importantly, in the BRST approach, it may always be realized as a canonical transformation in the extended phase space including the ghosts. Thus the Abelianization seems to provide

us with an answer to our basic question of how to solve the BRST condition. This is, however, not the whole story. One could object that the Abelian constraints may break *e.g.* manifest Lorentz covariance and locality and are non-polynomial. In many cases these are probably valid objections, but far from always. The breaking of manifest Lorentz covariance is more a problem in the Lagrangian formalism. After all, in the Hamiltonian formalism, manifest covariance is lost anyhow in many theories. The non-locality is also not such a problem in the Hamiltonian formalism, since it is only space-like and therefore causality is not lost. Whether or not the above objections are valid, the study of the Abelian form of the constraints may still prove helpful in solving the *original* constraints, since we have identified, through the Abelian generators, the unphysical momenta in the theory. In many cases we will also directly be able to find the corresponding coordinates and thus we have completely determined the unphysical space. The physical space is then simply given by the orthogonal complement to these phase space coordinates.

If we insert eq.(1.4) into eq.(1.5), we find an equation for S_b^a

$$\{[S_a^c, S_b^d]\psi_d + S_b^d\{S_a^c, \psi_d\} + S_a^d\{\psi_d, S_b^c\} + S_a^c S_b^d U_{ed}^e\}\psi_c = 0 \quad (1.6)$$

This set of $\frac{1}{2}m(m-1)$ equations determine all possible solutions for S_b^a . There is of course a large amount of redundancy in these equations. If we find one solution yielding a set of Abelian constraints, then we can always perform a reversible canonical transformation to obtain another set. In fact we must locally obtain all possible solutions this way. This follows from the definition of a canonical transformation and the fact that we can locally identify the Abelian generators with a set of momenta. Given this large freedom in S_b^a , it is still far from obvious how one can obtain an explicit expression for S_b^a from eq.(1.6). The main result of this paper is to give an ansatz for S_b^a , which reduces eq.(1.6) to equations which are in many cases far easier to handle. We will also consider the explicit finite canonical transformation, which will transform the original BRST charge into the one for the Abelian case. Although we know that it can always be found, the general equations given in ref.3, which determine the form of the transformation, are just like eq.(1.6), not easy to solve directly. For our solution to the Abelianization, we will be able to derive the explicit form of this transformation in the case of a general Lie algebra. To illustrate the formalism, we apply our construction to a scale-invariant point particle, bosonic string and pure Yang-Mills theory.

Our approach is related to work done in connection with the OSp symmetry in string theory.^[4] There, in the course of constructing a framework for the OSp symmetry applicable to string theory, it was found that one could solve the problem of Abelianization in string theory.^[4] Their framework is, however, rather special to string theory and does not usually generalize to other theories. We will report elsewhere on a more general construction of the OSp symmetry.^[4] Some examples of Abelianization at the level of gauge transformations in the Lagrangian formalism, have also been given previously for Yang-Mills theory,^[7] using an approach which has features in common with our treatment.

2. Abelianization

Let us begin by giving an intuitive argument for the choice of our ansatz. We know, by the theorem of Batalin and Fradkin, that locally there exists an Abelian set of constraint generators ϕ_a . We can identify them with the momenta p_a . Darboux's theorem assures us that we can find the corresponding coordinates q^a , $\{q^a, p_b\} = \delta_b^a$. We can then rewrite the original constraints by inverting eq.(1.4)

$$\psi_a = (S^{-1})^b_a (q^a, p^a, q^c, p_c) p_b, \quad (2.1)$$

where q^a and p^a are all the remaining phase space variables. Taking the PB with q^c we find

$$\{q^c, \psi_a\} = (S^{-1})^c_a + \{q^c, (S^{-1})^b_a\} p_b. \quad (2.2)$$

If we assume that S_b^a is independent of p_c , then the matrix S_b^a is trivially solved for in eq.(2.2)

$$S_b^a = \{q^a, \psi_b\}^{-1} \quad (2.3)$$

and thus the Abelian constraints are known. It is, therefore, by a clever choice of the coordinates q^a , possible to perform the Abelianization. To find such a choice of coordinates is, however, essentially equivalent to finding a suitable set of gauge fixing functions χ^a , $a = 1, \dots, m$. Then the matrix $\{\chi^a, \psi_b\}$ is invertible and we are led by the analogy to eq.(2.3), that S_b^a is given by $\{\chi^a, \psi_b\}^{-1}$.

We will now turn to a more rigorous treatment. We assume a set of linearly independent constraint generators $\psi_a, a = 1, \dots, m$ satisfying the algebra eq.(1.3), where the structure functions $U_{ab}{}^c$ may depend on the phase space variables. We take for simplicity all generators to be of even Grassmann type. Generalizing to odd type is straightforward. Introduce a set of functions $\chi^a, a = 1, \dots, m$ such that the matrix

$$M_b^a \equiv \{\chi^a, \psi_b\} \quad (2.4)$$

is invertible in some finite region of the constraint surface $\psi_a = 0$. We can then define a new set of generators

$$\phi_a \equiv (M^{-1})^b{}_a \psi_b. \quad (2.5)$$

We have

$$\{\psi_c, M_a^b\} - \{\psi_a, M_c^b\} + \{\chi^b, U_{ca}{}^d\} \psi_d = 0 \quad (2.6)$$

by using the algebra eq.(1.3) and the Jacobi identities from eq.(2.4). Then

$$\{\phi_a, \phi_b\} = [((M^{-1})^c{}_a, (M^{-1})^d{}_b) - (M^{-1})^c{}_a (M^{-1})^d{}_b] \{\chi^e, U_{ef}\} \psi_e \psi_d$$

Consequently the generators ϕ_a will satisfy an Abelian algebra provided

$$[\{M_a^c, M_b^d\} - \{\chi^d, U_{ab}{}^e\} M_e^c] (M^{-1})^g{}_c (M^{-1})^h{}_d \psi_g \psi_h = 0. \quad (2.7)$$

In general, the terms in the brackets may depend on ψ_a and eq.(2.7) will be a set of conditions for each order of ψ_a . In many cases, however, they are independent of ψ_a and then eq.(2.7) reduces to

$$\{M_a^c, M_b^d\} + \{M_a^d, M_b^c\} - \{\chi^d, U_{ab}{}^e\} M_e^c - \{\chi^c, U_{ab}{}^e\} M_e^d = 0. \quad (2.8)$$

The problem of Abelianization is, therefore, reduced to finding functions χ^a such that eq.(2.8), or in the general case eq.(2.7), are satisfied. We will show in several examples that these equations are not difficult to solve.

Let us make some comments about the ansatz, first of all about the invertibility of the matrix M_b^a . If we take the PB of M_b^a with ψ_c we find, if we assume M_b^a to be constant

$$U_{ab}{}^c = -(M^{-1})^c{}_d \{\chi^d, U_{ab}{}^e\} \psi_e. \quad (2.9)$$

We see, in particular, that this requires the structure functions to vanish on the constraint surface. Usually this is not the case and then M_b^a cannot be a constant matrix. The equation

$$\det(M_b^a) = 0 \quad (2.10)$$

will then specify a hypersurface, on which M_b^a is not invertible and this hypersurface will prevent a global construction with this particular ansatz. To find an Abelianization valid on this surface, we must choose another set of functions χ^a , for which the corresponding matrix is invertible on the surface. Note that we do not have to assume that M_b^a is invertible on the hypersurface $\chi^a = 0$ and therefore this is a slightly different situation than if we were to fix the gauge by imposing the additional constraints $\chi^a = 0$. One may also note that in the special case of χ^a having zero PB among themselves and with M_b^a , then

$$\{\chi^a, \phi_b\} = \delta_b^a. \quad (2.11)$$

Thus if we identify ϕ_a with the momenta p_a , then χ^a are the corresponding coordinates q^a .

3. The BRST formalism

A particularly important aspect connected to the Abelianization is that if we extend the phase space by including ghost variables, then the form of the BRST charge corresponding to the original constraints ψ_a and the one corresponding to the Abelian set ϕ_a , are related by a canonical transformation.^[9] We will specialize to the case when the structure functions $U_{ab}{}^c$

are constants i.e. we have a Lie algebra. The BRST charge is then given by

$$Q_B = \psi_a \eta^a + \frac{1}{2} U_{ab}{}^c \mathcal{P}_c \eta^b \eta^a. \quad (3.1)$$

The form of the BRST charge for the Abelian constraints is of course

$$Q_B = \phi_a^i \eta^a. \quad (3.2)$$

Then generally the non-Abelian form of the charge is related to the Abelian one by a canonical transformation

$$\begin{aligned} \eta^a &= \sum_{k=1}^m A_{b_1 \dots b_k}^{(k)} \mathcal{P}_{c_1} \dots \mathcal{P}_{c_{k-1}} \mathcal{P}_{c_k} \eta^{b_1} \dots \eta^{b_k} \\ \mathcal{P}_a' &= \sum_{k=1}^m B_{a b_1 \dots b_k}^{(k)} \mathcal{P}_{c_1} \dots \mathcal{P}_{c_k} \eta^{b_1} \dots \eta^{b_{k-1}} \\ \phi_a' &= \sum_{k=0}^m C_{a b_1 \dots b_k}^{(k)} \mathcal{P}_{c_1} \dots \mathcal{P}_{c_k} \eta^{b_1} \dots \eta^{b_k}. \end{aligned} \quad (3.3)$$

Each coefficient is determined by the requirement that eq.(3.3) is a canonical transformation and that the Abelian constraints are given by eq.(1.4), with S_b^a satisfying eq.(1.6). The general theorem of Batalin and Fradkin ensures that a solution always exists locally. We have found the first terms in the expansion to be

$$\begin{aligned} \eta^a &= (S^{-1})_b^a \eta^b + \frac{1}{2} S_c^e \{ (S^{-1})_b^e, (S^{-1})_d^c \} \mathcal{P}_e \eta^d \eta^b + \dots \\ \mathcal{P}_a' &= S_b^a \mathcal{P}_b - \frac{1}{2} (S^{-1})_c^e \{ S_b^e, S_d^a \} \mathcal{P}_b \mathcal{P}_d \eta^c + \dots \\ \phi_a' &= \phi_a + (S^{-1})_b^c \{ S_c^d, \phi_a \} \mathcal{P}_d \eta^b + \dots \end{aligned} \quad (3.4)$$

provided the bracket in eq.(1.6) vanishes. One should perhaps emphasize that the above expansion is in the number of ghost variables and represents a *finite* canonical transformation. In a theory with finite number of constraints, the expansion will always stop after a finite number of terms. In the case of interest here, namely when $S_b^a = (M^{-1})_b^a$, there are considerable simplifications if M_a^a satisfies eq.(2.8). In this equation the last two terms are absent,

since $U_{ab}{}^c$ is assumed to be constant. Consequently, the second term in the expansion of η^a above is zero. In fact, this is true for all the following terms in the same expansion. We will show this by explicitly constructing the complete canonical transformation. Let us therefore make the following ansatz

$$\begin{aligned} \eta^a &= M_b^a \eta^b \\ \mathcal{P}_a' &= \sum_{k=1}^m B_{a b_1 \dots b_{k-1}}^{(k)} \mathcal{P}_{c_1} \dots \mathcal{P}_{c_{k-1}} \eta^{b_1} \dots \eta^{b_{k-1}} \\ \phi_a' &= \sum_{k=0}^m C_{a b_1 \dots b_k}^{(k)} \mathcal{P}_{c_1} \dots \mathcal{P}_{c_k} \eta^{b_1} \dots \eta^{b_k} \end{aligned} \quad (3.5)$$

To determine the coefficients we now only have to require

$$\{\eta^a, \mathcal{P}_b'\} = \delta_b^a, \quad \{\eta^a, \phi_b'\} = 0 \quad (3.6)$$

Then by Jacobi identities we have e.g.

$$\{\eta^a, \{\mathcal{P}_b', \mathcal{P}_c'\}\} = 0,$$

which means that $\{\mathcal{P}_b', \mathcal{P}_c'\}$ cannot depend on $\mathcal{P}_d'$. But this in turn means that it cannot depend on $\mathcal{P}_d$, since we can invert eq.(3.5). Consequently it must be zero. In the same fashion, one proves that all the other PB's follow from eq.(3.6). Using eq.(3.5) in eq.(3.6) one finds the following recurrence relation

$$\begin{aligned} B_{a b_1 \dots b_k}^{(k+1)} \mathcal{P}_{c_1} \dots \mathcal{P}_{c_{k+1}} \eta^{b_1} \dots \eta^{b_k} \\ = -\frac{1}{k+1} M_{b_k}^d \{ (M^{-1})_{c_1}^d, B_{a b_1 \dots b_{k-1}}^{(k)} \} \mathcal{P}_{c_1} \dots \mathcal{P}_{c_{k+1}} \eta^{b_1} \dots \eta^{b_k} \\ C_{a b_1 \dots b_{k+1}}^{(k+1)} \mathcal{P}_{c_1} \dots \mathcal{P}_{c_{k+1}} \eta^{b_1} \dots \eta^{b_{k+1}} \\ = \frac{1}{k+1} M_{b_{k+1}}^d \{ (M^{-1})_{c_1}^d, C_{a b_1 \dots b_k}^{(k)} \} \mathcal{P}_{c_1} \dots \mathcal{P}_{c_{k+1}} \eta^{b_1} \dots \eta^{b_{k+1}}. \end{aligned} \quad (3.7)$$

The coefficients are now trivially obtained from the above expressions and the fact that

$B_b^{(1)a} = (M^{-1})_b^a$ and $C_a^{(0)} = \phi_a$. By using eq.(2.8), they can be written

$$B_{ab_1 \dots b_{k-1}}^{(k)} \quad c_1 \dots c_k \mathcal{P}_{c_1} \dots \mathcal{P}_{c_k} \eta^{b_1} \dots \eta^{b_{k-1}} = \frac{(-1)^{k-1}}{k!} M_{b_{k-1}}^{d_1} M_{b_{k-2}}^{d_2} \dots M_{b_1}^{d_{k-1}} \{ (M^{-1})_{d_1}^{c_1} \{ (M^{-1})_{d_2}^{c_2} \{ \dots \{ (M^{-1})_{d_{k-1}}^{c_{k-1}} (M^{-1})_{a_1}^{c_k} \} \dots \} \mathcal{P}_{c_1} \dots \mathcal{P}_{c_k} \eta^{b_1} \dots \eta^{b_{k-1}} \} \} \quad (3.8)$$

It remains to check that this canonical transformation relates the two BRST charges eq.(3.1) and eq.(3.2). Inserting eq.(3.5) into eq.(3.2), one finds that it gives the non-Abelian charge provided

$$C_{ab_1 \dots b_k}^{(k)} \quad c_1 \dots c_k M_{b_{k+1}}^a \mathcal{P}_{c_1} \dots \mathcal{P}_{c_k} \eta^{b_1} \dots \eta^{b_{k+1}} = 0 \quad (3.9)$$

for $k \geq 2$. It is straightforward to verify this relation for eq.(3.8), using the Jacobi identities and eq.(2.8). Eq. (3.9) implies that only the first two terms in the expansion of ϕ'_a in eq.(3.5) contribute when inserted into the BRST charge. Thus, in conclusion, the finite canonical transformation has been constructed and is given by eq.(3.5) with coefficients eq.(3.8). Note that the above transformation may be inverted by interchanging primed with unprimed variables and M with M^{-1} . In this case it is more useful to have an expression for ψ in terms of ψ' and the primed ghost variables. If we make an expansion similar to (3.5) for ψ in terms of primed variables, then we find that the corresponding coefficients are of the same form as $C^{(k)}$ in eq (3.8), but with ϕ' changed into ψ' .

4. Examples

Let us now consider some explicit examples. A simple model, where the constraint generators form a non-Abelian algebra, is a scale and reparametrization invariant point particle.^[4] This theory is so simple that not much is gained by studying the Abelianization. We use this example merely to illustrate the formulas derived in the previous sections. The constraints

are given by

$$\begin{aligned} \psi_1 &\equiv \frac{1}{2} p^2 = 0 \\ \psi_2 &\equiv p \cdot y = 0, \end{aligned} \quad (4.1)$$

where y^μ and p_μ , $\mu = 0, 1, \dots, d-1$, are the coordinates and momenta of the particle and satisfy the fundamental PB

$$\{y^\mu, p^\nu\} = \eta^{\mu\nu} = \text{diag}(1, -1, \dots, -1). \quad (4.2)$$

The algebra is

$$\{\psi_1, \psi_2\} = -2\psi_1. \quad (4.3)$$

To proceed with the Abelianization, we must find a set of functions χ^a . We can take as an ansatz $\chi^a = (n_1 y, n_2 p)$, where n_1^μ and n_2^μ are constants. We have

$$M_b^a \equiv \{\chi^a, \psi_b\} = \begin{pmatrix} n_1 \cdot p & n_1 \cdot y \\ 0 & -n_2 \cdot p \end{pmatrix}. \quad (4.4)$$

Then eq.(2.8) will imply

$$n_1^2 = 0. \quad (4.5)$$

In addition we must require M_b^a to be invertible, which implies

$$n_1 p \cdot n_2 p \neq 0. \quad (4.6)$$

With the conditions eq.(4.5) and eq.(4.6) fulfilled, it is straightforward to invert M_b^a and find the Abelian generators

$$\begin{aligned} \phi_1 &\equiv \frac{1}{2} \frac{1}{n_1 p} p^2 \\ \phi_2 &\equiv \frac{1}{n_1 p \cdot n_2 p} \left(\frac{1}{2} n_1 y \cdot p^2 - n_1 p \cdot p \cdot y \right). \end{aligned} \quad (4.7)$$

If we in addition choose n_2^μ to be orthogonal to n_1^μ , $n_1 \cdot n_2 = 0$, then we have

$$\{\chi^a, \chi^b\} = 0, \quad \{\chi^a, \phi_b\} = \delta_b^a, \quad (4.8)$$

so that we have found a complete decomposition of the unphysical phase space in coordinates χ^a and momenta ϕ_a . Denoting these by Q^a and P_a respectively, we can rewrite the original constraint generators ψ_a in the form

$$\begin{aligned} \psi_1 &= n_1 p P_1 \\ \psi_2 &= Q^1 P_1 - Q^2 P_2 \end{aligned} \quad (4.9)$$

Since these are the original generators they are, of course, Lorentz invariant, although the manifest invariance has been broken by the introduction of the variables Q^a and P_a . In this form it is equally evident that the constraints $\psi_1 = \psi_2 = 0$ are solved by taking $P_1 = P_2 = 0$, which implies *e.g.* that the timelike components of y^μ and p_μ are unphysical. The BRST charge is

$$Q_B = \psi_a \eta^a + 2P_1 \eta^1 \eta^2. \quad (4.10)$$

For the choice $n_1 \cdot n_2 = 0$ all components of M_b^a have zero PB among themselves. The canonical transformation to the Abelian BRST charge may therefore be easily constructed. Using eq.(3.5) one finds explicitly

$$\eta'^a = \begin{pmatrix} n_1 p \eta^1 + n_1 y \eta^2 \\ -n_2 p \eta^2 \end{pmatrix}, \quad P'_a = \begin{pmatrix} \frac{1}{n_1 p} P_1 \\ \frac{n_1 y}{n_1 p n_2 p} P_1 - \frac{1}{n_2 p} P_2 \end{pmatrix} \quad (4.11)$$

$$\phi'_a = \begin{pmatrix} \phi_1 - \frac{1}{n_1 p} P_1 \eta^2 \\ \phi_2 - \frac{1}{n_2 p} (P_1 \eta^1 + P_2 \eta^2) \end{pmatrix}$$

The Abelian constraints are valid as long as eq.(4.6) holds *i.e.* we have to exclude the hypersurface Γ , $n_1 p n_2 p = 0$. The intersection of Γ with the constraint surface $\psi_a = 0$ depends on the choice of n_2^μ . One can minimize the intersection by choosing n_2^μ to be lightlike and furthermore

take $n_1 \cdot n_2 = 0$. Then $n_2^\mu \sim n_1^\mu$ and the intersection is determined by $p^\mu = n_1 p = n_1 y n_3 p = 0$, where n_3^μ satisfies $n_1 n_3 = 1$. In general, there is no way to avoid a singularity in $\det M$ for $p^\mu = 0$. This is not surprising, since we must also exclude this point in a gauge fixing procedure.

Our next example is slightly more non-trivial. We consider the bosonic string theory. As we remarked in the introduction, an Abelianization has been obtained previously in this case.^[4] The constraints are

$$\psi_\pm(\tau, \sigma) \equiv \frac{1}{4}(p \pm \partial_1 x)^2(\tau, \sigma), \quad (4.12)$$

where $\partial_1 \equiv \partial_\sigma$. They satisfy the algebra

$$\{\psi_\pm(\sigma), \psi_\pm(\sigma')\} = \pm(\psi_\pm(\sigma) + \psi_\pm(\sigma'))\partial_1 \delta(\sigma - \sigma'). \quad (4.13)$$

We take as an ansatz for χ^a

$$\chi^\pm \equiv n^\pm \cdot (p \pm \partial_1 x), \quad (4.14)$$

where $n^{\pm\mu}$ are constant vectors. This gives

$$M_b^a(\sigma, \sigma') \equiv \{\chi^a(\sigma), \psi_b(\sigma')\} = \begin{pmatrix} n^+ \cdot (p + \partial_1 x)(\sigma') & 0 \\ 0 & -n^- \cdot (p - \partial_1 x)(\sigma') \end{pmatrix} \partial_1 \delta(\sigma - \sigma'). \quad (4.15)$$

The requirements eq.(2.8) read here

$$\{M_b^a(\sigma, \sigma'), M_d^c(\sigma'', \sigma''')\} + \{M_b^c(\sigma'', \sigma'), M_d^a(\sigma, \sigma''')\} = 0 \quad (4.16)$$

which implies

$$(n^\pm)^2 = (n^\mp)^2 = 0. \quad (4.17)$$

The inverse of M_b^a does not exist on the hypersurface Γ

$$n^+ \cdot (p + \partial_1 x) \quad n^- \cdot (p - \partial_1 x) = 0. \quad (4.18)$$

Excluding this hypersurface, we find from eq.(2.5) the Abelian constraints

$$\phi_{\pm}(\sigma) \equiv \pm \int d\sigma' \frac{1}{n^{\pm} \cdot (p \pm \partial_1 x)(\sigma')} G(\sigma', \sigma) \psi_{\pm}(\sigma'), \quad (4.19)$$

where $G(\sigma, \sigma')$ is the Green's function

$$\partial_1 G(\sigma, \sigma') = -\partial_1' G(\sigma, \sigma') = \delta(\sigma - \sigma'). \quad (4.20)$$

We can form new Abelian constraints by taking the σ -derivative of eq.(4.19)

$$\phi'_{\pm} \equiv -\partial_1 \phi_{\pm} = \pm \frac{1}{n^{\pm} \cdot (p \pm \partial_1 x)(\sigma)} \psi_{\pm}(\sigma). \quad (4.21)$$

The conditions eq.(4.17) imply that $\chi^{\pm}$ have zero PB among themselves and with M_b^a . Consequently $\chi^{\pm}$ and $\phi_{\pm}$ form a set of canonical coordinates and momenta. Furthermore, since all components of M_b^a have zero PB among themselves, it is a simple exercise to apply eqs.(3.5) and (3.8) and construct the explicit canonical transformation, which makes the BRST operator Abelian.

The construction above is limited by the requirement that we do not include points on Γ . The intersection of Γ with the constraint surface is simple to evaluate. Take e.g. $n^+ \cdot (p - \partial_1 x) = 0$, solving eq.(4.18). Then $\psi_- = 0$ implies that $p^{\mu} - \partial_1 x^{\mu} = 0$, since $n^+ \cdot$ is lightlike. Inserting this into $\psi_+ = 0$, we find $p^2 = 0$. Thus all points satisfying these conditions are excluded in this case.

Let us also briefly discuss the quantization. Using eq.(4.21), we can write the original constraints as

$$\psi_{\pm} = \pm \chi^{\pm} \phi'_{\pm}. \quad (4.22)$$

We would like to use $\chi^{\pm}$ and $\phi'_{\pm}$ as our basic operators in a quantum theory. Due to the ordering problems, an expression like eq.(4.21) is, however, not particularly useful to work with. We can, therefore, choose to consider $\chi^{\pm}$ and $\psi_{\pm}$ instead. Now, there is an old theorem

in string theory due to Goddard and Thorn,^[9] which says that the on-shell states spanned by the modes of $\chi^{\pm}$ and $\psi_{\pm}$ will in, and only in, $d = 26$ form a complete set of unphysical non-zero modes. Thus the relevance of the fields $\chi^{\pm}$ has been known for a long time. Our treatment here simply gives a more intuitive argument why this is so, even at the classical level. It serves to further motivate the study of the Abelianization.

Our final example is that of pure Yang-Mills theory. We assume, for simplicity, four dimensions and a compact gauge group. The non-Abelian constraints are

$$\psi_a(x, t) \equiv D_t E_a^i \equiv \partial_t E_a^i + g f_{ab}^c A_b^j E_c^i. \quad (4.23)$$

We take as our basic equal time PB (all time dependence will be suppressed from now on)

$$\{A^{ai}(x), E_{bj}(x')\} = \delta_b^a \delta_j^i \delta^3(x - x'). \quad (4.24)$$

The PB algebra then reads

$$\{\psi_a(x), \psi_b(x')\} = g f_{ab}^c \psi_c(x) \delta^3(x - x'). \quad (4.25)$$

In the case of Yang-Mills theory, it is simple to find a set of functions χ^a , such that the corresponding matrix M_b^a satisfies eq.(2.8). In fact any set χ^a depending only on either A_b^i or E_a^i will suffice. The condition of invertibility will of course restrict this choice to some extent, but there is essentially a large freedom in constructing the Abelian constraints. We will here only give two examples. One will be a choice of χ^a , which is suitable for perturbation theory, while the other is more unusual and is distinctly separated from conventional perturbation theory.

In the first case we take

$$\chi^a \equiv \partial_i A^{ai}. \quad (4.26)$$

As we remarked above, this trivially solves eq.(2.8), since M_b^a depends only on A_b^i . One finds

explicitly

$$M_{\frac{1}{2}}^a(z, x') \equiv \{\chi^a(z), \psi_b(x')\} = -(\nabla_z^2 \delta_{\frac{1}{2}}^a + g f_{ab}^c A^c(x') \partial_z) \delta^3(z - x'), \quad (4.27)$$

where $\nabla_z^2 \equiv \frac{\partial}{\partial z^2} \frac{\partial}{\partial z}$. The condition that $M_{\frac{1}{2}}^a$ is invertible will define hypersurfaces, Gribov surfaces,^[10]

$$\det[\nabla_z^2 \delta_{\frac{1}{2}}^a + g f_{ab}^c A^c(x') \partial_z] \delta^3(z - x') = 0, \quad (4.28)$$

which have to be excluded. In a perturbation around $g = 0$ and for weak vector fields, this condition will be fulfilled. We can proceed and construct the Abelian constraints. If $N_{\frac{1}{2}}^a(x, x')$ is the inverse of $M_{\frac{1}{2}}^a(x, x')$,

$$\int d^3 x'' N_{\frac{1}{2}}^a(x, x'') M_{\frac{1}{2}}^c(x'', x') = \delta_{\frac{1}{2}}^a \delta^3(x - x'), \quad (4.29)$$

then they are given by

$$\phi_a(x) = \int d^3 x' N_{\frac{1}{2}}^b(x', x) D_1 E_{\frac{1}{2}}^b(x'). \quad (4.30)$$

The inverse matrix $N_{\frac{1}{2}}^a$ is non-local in space and non-polynomial in A^{ai} . However, if we consider the weak-coupling limit, then we can make a perturbative expansion in the coupling constant, which will be polynomial in A^{ai} . One finds to first order in g

$$\Phi_a(x) \equiv \nabla_z^2 \phi_a = D_1 E_{\frac{1}{2}}^a(x) - g f_{ab}^c \int d^3 x' \partial_z G(x', x) A^{bi}(x) D_1 E_{\frac{1}{2}}^j(x') + \mathcal{O}(g^2), \quad (4.31)$$

where $G(x, x')$ is the Green's function of the Laplacian,

$$\nabla_z^2 G(x, x') = \nabla_z^2 G(x, x') = \delta^3(x - x'), \quad (4.32)$$

Since χ^a have zero PB among themselves and with M_c^b , the set (χ^a, ϕ_a) will be a set of canonical coordinates for the unphysical degrees of freedom, corresponding to the longitudinal polarization.

The canonical transformation, which relates the two BRST charges, is simple to read off from eqs.(3.5) and (3.8). The non-Abelian BRST charge is given by

$$Q_B = \int d^3 x (D_i E_{\frac{1}{2}}^i(x) \eta^a(x) + \frac{1}{2} f_{ab}^c \mathcal{P}_c(x) \eta^b(x) \eta^a(x) + E_{\frac{1}{2}}^0(x) \bar{\mathcal{P}}^0(x)), \quad (4.33)$$

where we have added the part corresponding to the trivial constraint $E_{\frac{1}{2}}^0(x) = 0$. By performing the finite canonical transformation

$$\begin{aligned} \eta^{ia}(x) &= \int d^3 x' M_{\frac{1}{2}}^a(x, x') \eta^i(x') \\ \mathcal{P}_a^i(x) &= \int d^3 x' N_{\frac{1}{2}}^b(x', x) \mathcal{P}_b^i(x') \\ \phi_a^i(x) &= \phi_a(x) + \int d^3 x' d^3 x'' d^3 x''' M_{\frac{1}{2}}^d(x', x'') \{N_{\frac{1}{2}}^c(x''', x'), \phi_a(x)\} \mathcal{P}_c(x''') \eta^b(x'''), \end{aligned} \quad (4.34)$$

this BRST charge will become

$$Q_B = \int d^3 x (\phi_a^i(x) \eta^{ia}(x) + E_{\frac{1}{2}}^0(x) \bar{\mathcal{P}}^0(x)). \quad (4.35)$$

The Abelian constraints constructed above are particularly well suited for a perturbation expansion for weak couplings, since for $g = 0$, the Abelian constraints are simply copies of those of electrodynamics. There exist many other possibilities. One may, for example, find completely local Abelian constraints by choosing the set of functions χ^a to depend on the electric field $E_{\frac{1}{2}}^i$ only. They are of the following general form

$$\chi^a \equiv K^{abi} E_{\frac{1}{2}}^i, \quad (4.36)$$

where K^{abi} are constants. The corresponding gauge fixing constraints, $\chi^a = 0$, have been introduced previously.^[11] A distinctive feature about the choice eq.(4.36) is that it is clearly

non-perturbative in the coupling constant g . The matrix M_g^a is given by

$$M_g^a(x, x') \equiv \{\chi^a(x), \psi_b(x')\} = -g f_{bc}^a K^{cdi} E_{ci}(x) \delta^3(x - x'), \quad (4.37)$$

which vanishes when $g = 0$. One may easily show^[11] that there exists a choice for K^{abi} such that the determinant of eq.(4.37) does not vanish identically and hence its zeros will specify hypersurfaces, which have to be excluded. This choice is given by

$$K^{abi} = f^{abi}, \quad (4.38)$$

where the space index $i, i = 1, 2, 3$, corresponds to a minimally embedded $SU(2)$ algebra. The corresponding Abelian constraints, eq.(4.30), where $N_b^a(x, x')$ is now the inverse of the matrix in eq.(4.37), will be local in space, but non-polynomial in E_a^i .

We would like to end by some remarks on possible quantum applications. We will consider the first case with χ^a given by eq.(4.26). Introduce the new phase space coordinates α_a, β_a, c_a and b_a

$$\begin{aligned} \alpha_a(x) &\equiv \frac{1}{2}(\phi'_a - i\chi'_a + A_a^0 + iE_a^0)(x) \\ \beta_a(x) &\equiv \frac{1}{2}(\phi'_a + i\chi'_a - A_a^0 + iE_a^0)(x) \\ c^a(x) &\equiv \frac{1}{2}(\eta'^a + i\tilde{P}'^a)(x) \\ b_a(x) &\equiv (\tilde{\eta}'_a - i\tilde{P}'_a)(x) \end{aligned} \quad (4.39)$$

They satisfy the fundamental PB

$$-\{\alpha^a(x), \alpha_b^i(x')\} = \{\beta^a(x), \beta_b^i(x')\} = \{c^a(x), b_b^i(x')\} = i\delta_b^a \delta^3(x - x'). \quad (4.40)$$

We can then rewrite the BRST charge in eq.(4.35) as

$$Q_B = \int d^3x (\alpha_a^* c^a + \beta_a^* c^a + c^{*a} \alpha_a + c^{*a} \beta_a). \quad (4.41)$$

This form is more suggestive for applications in a quantum theory. The above coordinates

will then be operators in a Fock space. The solution to the equation

$$Q_B |phys\rangle = 0 \quad (4.42)$$

will be obtained trivially if Q_B is given by eq.(4.41) as a state of the form

$$|phys\rangle = |T\rangle |0\rangle_{\alpha\beta} |0\rangle_{bc}, \quad (4.43)$$

where $|0\rangle_{\alpha\beta}$ and $|0\rangle_{bc}$ are the Fock vacua of the $\alpha\beta$ - and bc -operators and $|T\rangle$ is any state spanned by the remaining operators. Eq.(4.43) implies, therefore, that eq.(4.42) is equivalent, for non-trivial states, to the equations

$$\alpha_b |phys\rangle = \beta_b |phys\rangle = c_b |phys\rangle = b_b |phys\rangle = 0 \quad (4.44)$$

for all b . We can now re-express these equations, using eq.(4.34), in terms of the original operators

$$\alpha_b(A_a^i, E_a^i, \eta^a, \dots) |phys\rangle = \dots = 0. \quad (4.45)$$

These equations now determine a specific decomposition of the BRST equation, eq.(4.42), in terms of the original variables. The point is that if we consider Q_B expressed in the original operators, there is no obvious way to see how one should solve eq.(4.42). The procedure of Abelianization seems to give at least a possible answer to this question. Note that the eq.(4.45) mixes the matter and ghost variables non-trivially, so that it does not directly correspond to a Dirac-type quantization, i.e. one in which the constraints are transformed into operator equations.

Of course, our remarks above for the quantum theory are very heuristic, since we have not at all dealt with the problem of regularizing products of operators at the same space point. We believe, however, that one can, within perturbation theory, make these statements more precise and rigorous. We hope to be able to do this in a future publication.

ACKNOWLEDGEMENTS

I am indebted to R. Marnelius for suggestions and illuminating remarks. I would also like to thank C. Bachas and B.-S. Skagerstam for discussions.

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