CERN-TH.5722/90

**Collective Phenomena and Instabilities of
Perturbative Quantum Gravity
at Non-Zero Temperature**

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ABSTRACT

The leading temperature corrections to the graviton self-energy at small momenta and at temperatures well below the Planck scale are calculated, and gauge invariance of the underlying effective action is verified, generalizing and correcting previous investigations which were restricted to the static limit. By this the full dispersion relations of a graviton plasma in thermal equilibrium are obtained, which comprise a spatially longitudinal, a spatially transverse, and a partially spatially transverse mode. These are characterized by a common plasma frequency $\omega_{pl.}^2 = 22/45 \times 16\pi G\rho$ (ρ being the energy density $\propto T^4$) above which there are gravitational plasma waves and below which dynamical screening occurs. In the static case only the spatially transverse mode is shielded with screening mass $m_s^2 = 2/9 \times 16\pi G\rho$, whereas the spatially longitudinal mode develops an imaginary mass $m_J^2 = -58/27 \times 16\pi G\rho$, which is the analogue of the classical Jeans instability. For spatial inhomogeneities with wavelengths larger than $1/m_J$ exponential anti-damping occurs in the spatially longitudinal mode, and for sufficiently large wavelengths also in the partially spatially transverse mode.

CERN-TH.5722/90

May 1990

1. Introduction

Besides potential cosmological implications for the very early universe, the study of quantum gravity at non-zero temperature is interesting in its own right and might offer new insight into the structure of a still to be developed quantum theory of gravitation.

The purely attractive nature of gravity distinguishes it from the other fundamental interactions and is the root of many instabilities. Instabilities in semi-classical gravity at finite temperature were first studied in Ref. 1, where a tachyonic behaviour of the perturbative graviton propagator as well as instabilities about the Schwarzschild instanton were found.

In this work we shall investigate only the former type of instabilities. These have been the subject of some controversy in the literature which has appeared on this issue in the meantime [2, 3, 4], as each of these papers has been contradicting all the previous ones. The present paper will be no exception, but claims to be able to settle this issue.

The previous works have focused on a calculation of the analogue of the classical Jeans mass [5, 6], which appears as an imaginary mass in the static limit of the thermal graviton self-energy. This “Jeans mass” was expected to mark the onset of instabilities for spatial inhomogeneities with wavelengths greater than the inverse of this mass, in analogy to the classical Jeans instability which was originally derived in 1902 within the Newtonian theory.

In the present work we shall calculate the full leading temperature corrections of the graviton self-energy for momenta small compared to temperature (since there collective phenomena will be most pronounced) and for temperatures much smaller than the Planck scale, so that higher loop corrections can be neglected. By this we aim at finding the full dispersion relations of a graviton plasma in thermal equilibrium, and also at resolving the controversy on the correct definition of the Jeans mass in this setting. While we shall find that the strategy outlined in Ref. 4 was largely correct, this as well as the older works fell short because they did not take into account properly the complicated tensorial structure of the graviton self-energy. In particular, it has been overlooked that the physical information is contained only in the transverse-traceless part of the self-energy, and also that at finite temperature

there are three independent transverse-traceless modes in contrast to only one at zero temperature.

This paper is organized as follows. In Sect. 2 and 3 the leading temperature corrections to the 1- and 2-point functions for gravitons are calculated. While the former, the energy-momentum tensor, has been studied extensively before, the results for the 2-point function, in the general, dynamical case considered here, are new. They are calculated for the self-interacting thermal gravitons as well as for three types of additional massless matter, massless scalars, Weyl fermions, and gauge bosons. All these contributions are found to have the same form, differing only by an overall factor in correspondence to their different energy densities. In Sect. 4 we show that these results derive from a gauge invariant effective action by checking the corresponding Ward identities. In Sect. 5 we obtain the dispersion relations of the three transverse-traceless modes of the graviton plasma. These are derived by analysing the equations for small disturbances following from the Einstein equations, and in the course of this we explain the various shortcomings and mistakes of the previous works [1–4]. In Sect. 6 the dispersion relations are then solved numerically in a local momentum space representation and three types of collective phenomena are found: propagating waves above a universal plasma frequency and dynamical screening below it, and finally Jeans-like instabilities in two of the three modes beyond a critical wavelength determined by the Jeans mass. Sect. 7 contains the conclusions.

2. Temperature contributions to the energy-momentum tensor

In flat space-time the temperature contributions to the effective action or to Green’s functions [7] can be most efficiently calculated by analytic continuation of the corresponding Euclidean quantities which are subject to periodic or anti-periodic boundary conditions in the imaginary time at $t = 0$ and $t = i/T$. A more general framework is provided by the so-called real-time formalism, which is a more adequate starting point for curved spaces where a globally defined time may not exist. Furthermore, the real-time formalism also seems to be more promising if one is interested in non-equilibrium processes. However, when considering the flat space-time limit and thermal equilibrium, we will employ the techniques developed

in the imaginary-time formalism, in particular the so-called Dzyaloshinski algorithm [8].

In thermal equilibrium and with zero chemical potential, which we are assuming throughout this work, the thermal weights are given by

$$n(|u \cdot K|) = \left(e^{|u \cdot K|/T_0} \pm 1 \right)^{-1} \quad (2.1)$$

in a (local) momentum space $K_\mu = (k_0, \vec{k})$, where u^μ is a time-like, normalized vector field representing the local rest frame of the plasma, and the $\pm$ sign is $+$ for fermionic and $-$ for bosonic fields.

In a co-moving coordinate system, $u^i \equiv 0$, the 0 component of u is given by $u^0 = (g_{00})^{-1/2}$, hence the scalar (Tolman) temperature T_0 is related to the locally measured temperature T through [6] $T = (g_{00})^{-1/2} T_0$. In the following we will often assume $u^i \equiv 0$ to simplify otherwise unwieldy expressions.

In gravity, one of the most important objects to be considered is the energy-momentum tensor. In flat space-time, the energy-momentum tensor produced by a neutral, massless scalar field at one-loop order is given by

$$\begin{aligned} \sqrt{-g} \langle T_{\mu\nu} \rangle &\equiv 2 \frac{\delta \Gamma}{\delta g^{\mu\nu}} \Big|_{g=\eta} = \int \frac{d^4 K}{(2\pi)^3} u(|u \cdot K|) \delta(K^2) K_\mu K_\nu \\ &= \rho_S u_\mu u_\nu + \frac{\rho_S}{3} (u_\mu u_\nu - \eta_{\mu\nu}) \end{aligned} \quad (2.2)$$

where

$$\rho_S = \frac{\Gamma(4)\zeta(4)}{2} T^4 = \frac{\pi^2}{30} T^4 \quad (2.3a)$$

is the energy density, the pressure assuming the maximal value $\rho_S/3$.

In eq. (2.2) we have disregarded the (ill-defined) $T = 0$ contribution to (2.2) which we are assuming to be taken care of by renormalization. In the following we will always restrict our attention only to the (leading) temperature contributions.

Other massless fields, in particular the gravitons themselves, give contributions of the same form as eq. (2.2) with energy densities

$$\rho_G = \rho_\gamma = 2\rho_S \quad (2.3b)$$

for gravitons and photons, respectively,* in correspondence to the two helicities which make up their physical degrees of freedom. For massless Weyl fermions one obtains

$$\rho_F = \frac{\Gamma(4)\eta(4)}{2} T^4 = \frac{7\pi^2}{240} T^4, \quad (2.3c)$$

where $\eta(x) = (1 - 2^{1-x})\zeta(x)$.

The result (2.2) can be identified as resulting from a contribution to the effective action which has the form of a cosmological constant,

$$\int d^4 x \sqrt{-g} \frac{\rho}{3}, \quad (2.4)$$

without being a cosmological constant, since the metric dependence of the temperature T implicit in ρ must be taken into account when varying with respect to $g_{\mu\nu}$ — in co-moving coordinates $\rho = (g_{00})^{-2} \rho_0$.

Eq. (2.2) is exact only for vanishing curvature tensor $R^\alpha_{\mu\beta\nu}$. The Einstein equations determine

$$R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} = -8\pi G T^{\mu\nu} \quad (2.5)$$

hence $R^{\mu\nu} \sim G T^4$. If the temperature is well below the Planck scale, $T^2 \ll G^{-1}$, this implies $R^{\mu\nu} \ll T^2$, and curvature corrections to eq. (2.2) will be subleading (cf. [2, 3, 4]). This assumption is also necessary for perturbation theory to make sense — only with $G T^2 \ll 1$ will higher loop contributions become negligible.

*See Appendix A for details like the gauge fixing Lagrangians employed as well as for our conventions.

Tab. 1: 14 independent tensors built from $\eta^{\mu\nu}$, Q^μ and u^μ

$$\begin{aligned}
T_1^{\alpha\beta\mu\nu} &= \eta^{\alpha\mu}\eta^{\beta\nu} + \eta^{\alpha\nu}\eta^{\beta\mu} \\
T_2^{\alpha\beta\mu\nu} &= \eta^{\alpha\mu}u^\beta u^\nu + \eta^{\beta\mu}u^\alpha u^\nu + \eta^{\alpha\nu}u^\beta u^\mu + \eta^{\beta\nu}u^\alpha u^\mu \\
T_3^{\alpha\beta\mu\nu} &= u^\alpha u^\beta u^\mu u^\nu \\
T_4^{\alpha\beta\mu\nu} &= \eta^{\alpha\beta}\eta^{\mu\nu} \\
T_5^{\alpha\beta\mu\nu} &= \eta^{\alpha\beta}u^\mu u^\nu + \eta^{\mu\nu}u^\alpha u^\beta \\
T_6^{\alpha\beta\mu\nu} &= (\eta^{\mu\alpha}Q^\nu + \eta^{\nu\alpha}Q^\mu)u^\beta + (\eta^{\mu\beta}Q^\nu + \eta^{\nu\beta}Q^\mu)u^\alpha + \\
&\quad (\eta^{\mu\alpha}u^\nu + \eta^{\nu\alpha}u^\mu)Q^\beta + (\eta^{\mu\beta}u^\nu + \eta^{\nu\beta}u^\mu)Q^\alpha \\
T_7^{\alpha\beta\mu\nu} &= Q^\alpha u^\beta u^\mu u^\nu + u^\alpha Q^\beta u^\mu u^\nu + u^\alpha u^\beta Q^\mu u^\nu + u^\alpha u^\beta u^\mu Q^\nu \\
T_8^{\alpha\beta\mu\nu} &= \eta^{\alpha\mu}Q^\beta Q^\nu + \eta^{\beta\mu}Q^\alpha Q^\nu + \eta^{\alpha\nu}Q^\beta Q^\mu + \eta^{\beta\nu}Q^\alpha Q^\mu \\
T_9^{\alpha\beta\mu\nu} &= u^\alpha u^\beta Q^\mu Q^\nu + Q^\alpha Q^\beta u^\mu u^\nu \\
T_{10}^{\alpha\beta\mu\nu} &= (Q^\alpha u^\beta + u^\alpha Q^\beta)(Q^\mu u^\nu + u^\mu Q^\nu) \\
T_{11}^{\alpha\beta\mu\nu} &= u^\alpha Q^\beta Q^\mu Q^\nu + Q^\alpha u^\beta Q^\mu Q^\nu + Q^\alpha Q^\beta u^\mu Q^\nu + Q^\alpha Q^\beta Q^\mu u^\nu \\
T_{12}^{\alpha\beta\mu\nu} &= Q^\alpha Q^\beta Q^\mu Q^\nu \\
T_{13}^{\alpha\beta\mu\nu} &= \eta^{\alpha\beta}Q^\mu Q^\nu + \eta^{\mu\nu}Q^\alpha Q^\beta \\
T_{14}^{\alpha\beta\mu\nu} &= \eta^{\alpha\beta}(Q^\mu u^\nu + u^\mu Q^\nu) + \eta^{\mu\nu}(Q^\alpha u^\beta + u^\alpha Q^\beta)
\end{aligned}$$

3. Graviton self-energy

We now turn to the leading temperature corrections to the graviton self-energy

$$\frac{\delta^2 \Gamma}{\delta g_{\alpha\beta}(x) \delta g_{\mu\nu}(y)} \Big|_{g=\eta} = \frac{\delta}{\delta g_{\alpha\beta}(x)} \left(-\frac{1}{2} \sqrt{-g} (\Gamma^{\mu\nu}(y)) \right) \Big|_{g=\eta} \equiv -\Pi^{\alpha\beta\mu\nu}(x-y) \quad (3.1)$$

for which we can restrict ourselves to the flat space limit again. The latter makes it possible to evaluate (3.1) by means of ordinary momentum space techniques.

Because of the existence of a preferred coordinate system, the tensorial structure of (3.1) will be much more complicated than at zero temperature. Due to the presence of the additional time-like vector u , which we assume as $u^\mu = \delta_0^\mu$ in the following, there are now 14 (instead of 5) independent tensors which can appear in (3.1). They are listed in Tab. 1.

So far only a zero-momentum limit of (3.1) has been considered in the literature for which only the first 5 tensors of Tab. 1 need to be taken into account.

For momenta $q_0, \vec{q} \ll T$ to which we shall restrict our attention, because there the temperature effects give rise to collective behaviour, the complete evaluation of eq. (3.1) becomes technically a high-temperature calculation, while the temperature may actually be arbitrarily low, and we are in fact assuming that the temperature is small compared to the Planck scale so that loop orders higher than one may be neglected.

This “high-temperature” limit is governed by the parts of Feynman integrals which are superficially most divergent, in our case by quartically divergent parts. Integrals not containing external momenta are readily calculated, and integrals which do can all be reduced to the prototype integral

$$I^{\mu\nu\rho\sigma}(q_0, \vec{q}) = \int_T \frac{d^4 K}{(2\pi)^4} \frac{K^\mu K^\nu K^\rho K^\sigma}{K^2(K-Q)^2}. \quad (3.2)$$

This integral is rather tedious to evaluate using conventional summation techniques [9]. In the high temperature limit the so-called Dzyaloshinski algorithm turns out to be more convenient [8]. The outcome of this calculation is given in Appendix B.

We shall not reproduce here the momentum algebra involved in computing eq. (3.1) for the various massless fields, which becomes particularly unwieldy for the graviton loop (cf. [10]). The end result for the leading contributions for $q_0, \vec{q} \ll T \ll G^{-1/2}$, however, turns out to be the same for all, the scalar, photon, graviton and fermion fields, only the overall factor differs corresponding to the differing energy-densities produced by these fields, see eq. (2.3). In terms of the integral I this universal result reads

$$\Pi^{\alpha\beta\mu\nu} = \frac{1}{2} I^{\alpha\beta\mu\nu} - \frac{1}{4} (\delta_\rho^\alpha \delta_\rho^\mu \eta^{\nu\beta} + \delta_\rho^\alpha \delta_\rho^\nu \eta^{\mu\beta} + \delta_\rho^\beta \delta_\rho^\mu \eta^{\nu\alpha} + \delta_\rho^\beta \delta_\rho^\nu \eta^{\mu\alpha}) \eta_{\gamma\delta} \Gamma^{\delta\sigma\rho}, \quad (3.3)$$

omitting contractions of I which give zero (cf. App. B).

$\Pi^{\alpha\beta\mu\nu}(q_0, \vec{q})$ in the basis of tensors listed in Tab. 1,

$$\Pi^{\alpha\beta\mu\nu}(q_0, \vec{q}) = \sum_{i=1}^{14} c_i(q_0, q) T_i^{\alpha\beta\mu\nu}, \quad (3.4)$$

is finally given by the following 14 structure functions:

$$c_1(q_0, q)/\rho = \frac{1}{8} \frac{Q^4}{q^4} L(q_0, q) - \frac{1}{24} \frac{Q^2}{q^2} + \frac{1}{6} \quad (3.5a)$$

Only the static limit of $\Pi^{\alpha\beta\mu\nu}$ has been considered so far in the literature, which is given by

$$c_1 = \frac{\rho}{12}, \quad c_2 = 0, \quad c_3 = -2\rho, \quad c_4 = -\frac{\rho}{12}, \quad c_5 = \frac{\rho}{3}, \quad (3.8)$$

$$c_i = 0 \quad \text{for } i \geq 6.$$

These results agree with refs. [2, 3, 4] as far as their respective calculations go.

Later we shall need also the limit of long wavelengths approached from above the light-cone. There the structure functions c_i reduce to

$$c_1 = \frac{3\rho}{20}, \quad c_2 = -\frac{2\rho}{7}, \quad c_3 = -\frac{2\rho}{9}, \quad c_4 = -\frac{\rho}{60}, \quad c_5 = \frac{\rho}{21}, \quad (3.9)$$

$$c_i = 0 \quad \text{for } i \geq 6.$$

It is interesting to note that the static limit (3.8), rather than the long-wavelength limit (3.9), which both are momentum-independent, can be identified as resulting from the effective action (2.4). To leading order the latter can therefore be regarded as zero-momentum limit $Q \rightarrow 0$ with $Q^2 < 0$ of the full, dynamical effective action.

4. Ward identities

In nonabelian gauge theories like QCD the leading high-temperature contributions have been found [11, 12] to satisfy QED-type Ward identities, i.e. the leading contributions to the effective action are gauge invariant functionals.

In the case of gravity, however, the gauge group is entangled with space-time, and the existence of preferred (co-moving) coordinate systems could let one expect at most invariance under restricted diffeomorphisms which leave $u^i \equiv 0$ intact. Such diffeomorphisms are generated by static vector fields, for, when $u^i \equiv 0$, the Lie derivative of u^i is

$$\mathcal{L}_\xi u^i = -u^0 \partial_0 \xi^i, \quad (4.1)$$

and the deviation from a scalar transformation law for u^0 is

$$\mathcal{L}_\xi u^0 - \xi^\lambda \partial_\lambda u^0 = -u^0 \partial_0 \xi^0. \quad (4.2)$$

However, it turns out that the leading temperature contributions to the 1- and 2-point-functions calculated in the previous sections in fact derive from a fully gauge

$$c_2(q_0, q)/\rho = \frac{5Q^6}{8\bar{q}^6} L(q_0, q) - \frac{5Q^4}{24\bar{q}^4} + \frac{1Q^2}{12\bar{q}^2} - \frac{1}{3} \quad (3.5b)$$

$$c_3(q_0, q)/\rho = \frac{35Q^8}{8\bar{q}^8} L(q_0, q) - \frac{35Q^6}{24\bar{q}^6} + \frac{7Q^4}{12\bar{q}^4} - \frac{1Q^2}{3\bar{q}^2} \quad (3.5c)$$

$$c_4(q_0, q)/\rho = \frac{1Q^4}{8\bar{q}^4} L(q_0, q) - \frac{1Q^2}{24\bar{q}^2} \quad (3.5d)$$

$$c_5(q_0, q)/\rho = \frac{5Q^6}{8\bar{q}^6} L(q_0, q) - \frac{5Q^4}{24\bar{q}^4} + \frac{1Q^2}{12\bar{q}^2} \quad (3.5e)$$

$$c_6(q_0, q)/\rho = -\frac{q_0}{\bar{q}^2} \left(\frac{5Q^4}{8\bar{q}^4} L(q_0, q) - \frac{5Q^2}{24\bar{q}^2} + \frac{1}{12} \right) \quad (3.5f)$$

$$c_7(q_0, q)/\rho = -\frac{q_0}{\bar{q}^2} \left(\frac{35Q^6}{8\bar{q}^6} L(q_0, q) - \frac{35Q^4}{24\bar{q}^4} + \frac{7Q^2}{12\bar{q}^2} - \frac{1}{3} \right) \quad (3.5g)$$

$$c_8(q_0, q)/\rho = \frac{1}{\bar{q}^2} \left(\left[\frac{5Q^2}{8\bar{q}^2} + \frac{1}{2} \right] \frac{Q^2}{\bar{q}^2} L(q_0, q) - \frac{5Q^2}{24\bar{q}^2} - \frac{1}{12} \right) \quad (3.5h)$$

$$c_9(q_0, q)/\rho = \frac{1}{\bar{q}^2} \left(\left[\frac{35Q^2}{8\bar{q}^2} + \frac{15}{4} \right] \frac{Q^4}{\bar{q}^4} L(q_0, q) - \frac{35Q^4}{24\bar{q}^4} - \frac{2Q^2}{3\bar{q}^2} + \frac{1}{6} \right) \quad (3.5i)$$

$$c_{10}(q_0, q) \equiv c_9(q_0, q) \quad (3.5j)$$

$$c_{11}(q_0, q)/\rho = -\frac{q_0}{\bar{q}^4} \left(\left[\frac{35Q^2}{8\bar{q}^2} + \frac{5}{2} \right] \frac{Q^2}{\bar{q}^2} L(q_0, q) - \frac{35Q^2}{24\bar{q}^2} - \frac{1}{4} \right) \quad (3.5k)$$

$$c_{12}(q_0, q)/\rho = \frac{1}{\bar{q}^4} \left(\left[\frac{35Q^4}{8\bar{q}^4} + 5\frac{Q^2}{\bar{q}^2} + 1 \right] L(q_0, q) - \frac{35Q^2}{24\bar{q}^2} - \frac{13}{12} \right) \quad (3.5l)$$

$$c_{13}(q_0, q)/\rho = \frac{1}{\bar{q}^2} \left(\left[\frac{5Q^2}{8\bar{q}^2} + \frac{1}{2} \right] \frac{Q^2}{\bar{q}^2} L(q_0, q) - \frac{5Q^2}{24\bar{q}^2} - \frac{1}{12} \right) \quad (3.5m)$$

$$c_{14}(q_0, q) \equiv c_6(q_0, q), \quad (3.5n)$$

where we have defined

$$L(q_0, q) = \frac{q_0}{2q} \ln \frac{q_0 + q}{q_0 - q} - 1, \quad q = |\vec{q}|. \quad (3.6)$$

The static and the long-wavelength limits of $L(q_0, q)$ are

$$L(q_0, q) \rightarrow -1 \quad \text{for } q_0 \rightarrow 0, \quad (3.7)$$

$$\rightarrow \frac{1}{3} \left(\frac{q}{q_0} \right)^2 + \frac{1}{5} \left(\frac{q}{q_0} \right)^4 + \dots \quad \text{for } q \rightarrow 0.$$

Note that the limits $q_0 \rightarrow 0$ and $q \rightarrow 0$ do not commute.

(i.e. diffeomorphism) invariant effective action. One important difference from the case of Yang-Mills theories is that, since the 1-point function is non-vanishing, the graviton self-energy is not transverse but its divergence is related to the energy-momentum tensor according to

$$\Pi_{\alpha\beta\mu\nu}(Q)(\eta^{\mu\epsilon}Q^\nu + \eta^{\nu\epsilon}Q^\mu) = -\frac{1}{2}T_{\mu\nu}(\delta_{\alpha\beta}^{\nu\epsilon}Q^\mu + \delta_{\alpha\beta}^{\mu\epsilon}Q^\nu - \delta_{\alpha\beta}^{\mu\nu}Q^\epsilon) \quad (4.3)$$

where $\delta_{\alpha\beta}^{\mu\nu} = \frac{1}{2}(\delta_\alpha^\mu \delta_\beta^\nu + \delta_\alpha^\nu \delta_\beta^\mu)$, which is just the functional equation for gauge invariance of the effective action

$$\frac{\delta\Gamma}{\delta g_{\mu\nu}} \frac{\delta g_{\mu\nu}}{\delta \xi^\epsilon} \equiv 0 \quad (4.4)$$

differentiated once with respect to $g_{\alpha\beta}$ at $g = \eta$. (For restricted covariance only the static version of eq. (4.3) would have to hold.) Eq. (4.3) relates the 14 structure functions in eq. (3.5) in a highly non-trivial way, and thus constitutes a powerful check on the correctness of the results given in the previous section.

At the 1-loop level the leading temperature contributions of $\Pi_{\alpha\beta\mu\nu}$ and $T_{\mu\nu}$ are expressible in terms of contractions of the integral (3.2) and one can show that the combination given in eq. (3.3) is in fact the only one which is covariant in the sense of eq. (4.3), and this is true even if the invariance requirement is loosened to static diffeomorphisms. Hence for the leading T^4 -contributions, which are entirely 1-loop, restricted covariance implies full covariance. This is also the reason why scalars, fermions, photons and even the rather complicated graviton loop have yielded (apart from an overall factor) the same T^4 -contributions to the graviton self-energy.

5. Derivation of dispersion relations

Let us begin with recapitulating the situation in high-temperature quantum chromodynamics [9, 13]. The plasma dispersion relations, which describe the response to small disturbances and also the propagating normal modes, can be obtained from the poles in the transverse part of the gluon propagator [9]. Since by the presence of the plasma Lorentz symmetry is broken down to rotational symmetry, there are now two independent tensors in the gluon propagator which are transverse with respect to 4 dimensions, one which is also spatially transverse (ST) and another one which is spatially longitudinal (SL). The location of the poles in the two structure functions pertaining to these two tensors is reproduced in Fig. 1.

Above a certain minimal frequency ω_{pl} the plasma can support propagating waves ("plasmons"), where the SL-mode corresponds to a propagating longitudinal (chromo-)electric field produced by coherent density waves of charged particles, and the ST-mode corresponds to plane waves with transverse electric and magnetic fields. Below the plasma frequency there is dynamical screening with exponential spatial falloff, which in the static limit approaches a non-zero value for the screening mass of the SL-mode (the so-called electric Debye screening mass) and zero for the ST-mode so that static magnetic fields are not screened.*

In all of the existing work on quantum gravity at non-zero temperature only the static limit of the graviton self-energy has been studied so far. Because of the purely attractive nature of gravitation one would not expect something like an electric screening mass. Indeed, instead an imaginary mass has been reported in the literature, which is interpreted as the analogue of the classical Jeans instability [5, 6] and which would mean anti-damping of spatial inhomogeneities. Correspondingly, perturbative quantum gravity itself appears to be inherently unstable at non-zero temperature, hence to some degree ill-defined.

However, the existing literature on the so-called** Jeans mass in thermal quantum gravity is in disagreement concerning the correct numerical value. Besides a detailed study of the dispersion relations in the dynamical case, we shall explain the respective shortcomings of the previous works and resolve this controversy. It will turn out that only consideration of the general dynamical situation allows an unambiguous definition of the static limit.

In the seminal work of Gross, Perry and Yaffe [1] only the structure function c_3 of the graviton self-energy was expected to be non-vanishing in the static limit, which, however, is disproved by our results (3.8). Consequently they only considered Π_{0000} and interpreted its (negative) static limit as the Jeans mass through $m_J^2 = 32\pi G \Pi_{0000}$. The mistake of Ref. 1 can be traced back to their assumption of

*A non-vanishing magnetic screening mass is expected to be generated non-perturbatively at order $g^2 T$, and is necessary to cure the infrared problems of perturbation theory at higher loop orders.

**In cosmology, the term "Jeans mass" is usually defined in a different manner [6]; here it will be used simply as a synonym for Jeans wave number.

transversality of the graviton self-energy. But, as we have seen in eq. (4.3), the Ward identities expressing gauge invariance imply a longitudinal contribution connected to the non-vanishing one-point function, the energy-momentum tensor.

In Ref. 2 it was later shown that Π_{0000} alone would be dependent on the parametrization of the gravitational field (in the present case for instance we have used $g_{\mu\nu} - \eta_{\mu\nu}$). The reason for this is of course again that the 1-point function does not vanish. The authors of Ref. 2 therefore proposed to focus on the 0000-component of the graviton propagator instead, which usually is given as iterative self-energy insertions in the classical propagator $D_{\alpha\beta\mu\nu}^{(0)}$:

$$D = D^{(0)} + D^{(0)}\Pi D^{(0)} + D^{(0)}\Pi D^{(0)}\Pi D^{(0)} + \dots \quad (5.1)$$

With $D^{(0)}$ in a Feynman-de Donder gauge (see App. A)

$$D_{\alpha\beta\mu\nu}^{(0)}(Q) = \frac{1}{2} (\eta_{\alpha\gamma}\eta_{\beta\delta} + \eta_{\alpha\delta}\eta_{\beta\gamma} - \eta_{\alpha\beta}\eta_{\gamma\delta}) \frac{1}{Q^2}, \quad (5.2)$$

the 0000-component of the first order term in eq. (5.1) is given by

$$(D^{(0)}\Pi D^{(0)})_{0000} = \Pi_{0000} - \Pi_{\mu 00}^{\mu} + \frac{1}{4}\Pi_{\mu}^{\mu}{}_{\nu}{}^{\nu} \quad (5.3)$$

which was interpreted as the Jeans mass, resulting in $m_J^2 = -3 \times 16\pi G\rho$.

The interpretation of (5.3) as the Jeans mass was again a mistake for two reasons: First, since

$$(D^{(0)}\Pi D^{(0)}\Pi D^{(0)})_{0000} \neq (D^{(0)}\Pi D^{(0)})_{0000}^2 / D_{0000}^{(0)} \quad (5.4)$$

this definition of the Jeans mass does not sum up to a propagator

$$D_{0000} \sim \frac{1}{q^2 + m_J^2}$$

in the static limit, and second, the Dyson equation (5.1) holds only for vanishing 1-point functions. Hence, although the result (5.3) happened to be insensitive to changes of the parametrizations, it cannot be the correct answer.

Subsequently in refs. [3, 4] it was pointed out that the equations for small disturbances obtained from the Einstein equations are well defined in contrast to the graviton propagator at non-zero temperature. We shall study these equations for

the general dynamical case, and it will turn out that because of the constraints arising in the process of gauge fixing the complete, dynamical tensor structure of the graviton self-energy has to be taken into account. In Ref. 3 these constraints were not considered at all, and in Ref. 4 only part of them could be covered due to their restriction to the zero momentum limit.

5.1. EINSTEIN EQUATIONS AND SMALL DISTURBANCES

Starting from the effective action $\Gamma = S_G + \Gamma^{(1)}$, where S_G is the Einstein action (cf. App. A) and $\Gamma^{(1)}$ comprises the leading T^4 -contributions generated by graviton and matter loops, the Einstein equations read

$$\frac{\delta\Gamma}{\delta g_{\mu\nu}} = 0 \Rightarrow R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} + \Lambda g^{\mu\nu} = -8\pi GT^{\mu\nu}. \quad (5.5)$$

Because the leading temperature contributions fulfil $T^{\mu}{}_{\mu} = 0$, they imply for cosmological constant $\Lambda = 0$

$$R = 0, \quad R^{\mu\nu} = -8\pi GT^{\mu\nu}. \quad (5.6)$$

The so-called (inhomogeneous) Jacobi equation for small disturbances $\delta g_{\mu\nu}$ around a solution $g_{\mu\nu}$ of the Einstein equation induced by a small disturbance in the energy-momentum tensor is given by

$$\begin{aligned} \frac{\delta^2\Gamma}{\delta g_{\mu\nu}\delta g_{\alpha\beta}} \delta g_{\alpha\beta} = -\delta \frac{\delta\Gamma}{\delta g_{\mu\nu}} \Rightarrow \\ \left\{ \frac{1}{2} \left(g^{\alpha\beta} g^{\mu\nu} - g^{\alpha(\mu} g^{\nu)\beta} \right) \nabla^2 - \frac{1}{2} \left(g^{\alpha\beta} \nabla^{(\mu} \nabla^{\nu)} + g^{\mu\nu} \nabla^{\alpha} \nabla^{\beta} \right) \right. \\ \left. + g^{\alpha(\mu} \nabla^{\nu)} \nabla^{\beta} + \Lambda \left(g^{\alpha(\mu} g^{\nu)\beta} - \frac{1}{2} g^{\alpha\beta} g^{\mu\nu} \right) \right. \\ \left. - R^{\alpha(\mu\nu)\beta} - \frac{1}{2} g^{\alpha\beta} R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R^{\alpha\beta} + g^{\alpha(\mu} R^{\nu)\beta} \right. \\ \left. + \frac{1}{4} g^{\alpha\beta} g^{\mu\nu} R - \frac{1}{2} g^{\alpha(\mu} g^{\nu)\beta} R - 16\pi G \Pi^{\alpha\beta\mu\nu}(\nabla^0, \vec{\nabla}) \right\} \delta g_{\alpha\beta} = 8\pi G \delta T^{\mu\nu}. \end{aligned} \quad (5.7)$$

(Brackets around indices denote symmetrization including normalization so that e.g. $a^{(\mu} a^{\nu)} = a^{\mu} a^{\nu}$.) This equation is fundamental for the following analysis, and its form is in agreement with Refs. [14, 15].*

*It is, however, derived incorrectly in Ref. 4, which additionally invalidates their calculation of the Jeans mass.

Rewriting $R^{\alpha\mu\nu\beta}$ in terms of its contractions and the contraction-free conformal curvature (Weyl) tensor $C^{\alpha\mu\beta}$

$$R^{\alpha(\mu\nu)\beta} = C^{\alpha(\mu\nu)\beta} + g^{\alpha(\mu} R^{\nu)\beta} - \frac{1}{2} (g^{\alpha\beta} R^{\mu\nu} + g^{\mu\nu} R^{\alpha\beta}) + \frac{1}{6} R (g^{\alpha(\mu} g^{\nu)\beta} - g^{\alpha\beta} g^{\mu\nu}), \quad (5.8)$$

eq. (5.7) simplifies to

$$\begin{aligned} & \left\{ \frac{1}{2} (g^{\alpha\beta} g^{\mu\nu} - g^{\alpha(\mu} g^{\nu)\beta}) \nabla^2 - \frac{1}{2} (g^{\alpha\beta} \nabla^{\mu} \nabla^{\nu}) + g^{\mu\nu} \nabla^{\alpha} \nabla^{\beta} \right\} \\ & + g^{\alpha(\mu} \nabla^{\nu)} \nabla^{\beta} + \Lambda (g^{\alpha(\mu} g^{\nu)\beta} - \frac{1}{2} g^{\alpha\beta} g^{\mu\nu}) \\ & - C^{\alpha(\mu\nu)\beta} + \frac{1}{12} g^{\alpha\beta} g^{\mu\nu} R - \frac{1}{3} g^{\alpha(\mu} g^{\nu)\beta} R \\ & - 16\pi G \Pi^{\alpha\beta\mu\nu} (\nabla^0, \vec{\nabla}) \Big\} \delta g_{\alpha\beta} = 8\pi G \delta T^{\mu\nu}. \end{aligned} \quad (5.9)$$

In order to make the Jacobi equations invertible, we need to fix the diffeomorphism invariance. Since the leading temperature correction $\Gamma^{(1)}$ is gauge invariant, as we have shown in sect. 4, we can, exactly as in the classical theory, require that $\delta g_{\alpha\beta}$ be transverse and traceless,

$$\nabla^{\alpha} \delta g_{\alpha\beta} = 0, \quad g^{\alpha\beta} \delta g_{\alpha\beta} = 0. \quad (5.10)$$

By this the Jacobi equation further simplifies to

$$\left\{ \frac{1}{2} g^{\alpha(\mu} g^{\nu)\beta} (\nabla^2 - 2\Lambda + \frac{2}{3} R) + C^{\alpha(\mu\nu)\beta} + 16\pi G \Pi^{\alpha\beta\mu\nu} \right\} \delta g_{\alpha\beta} = -8\pi G \delta T^{\mu\nu}. \quad (5.11)$$

With vanishing cosmological constant eq. (5.6) implied $R = 0$, so that if we assume conformally flat space-time, i.e. vanishing Weyl tensor C which is undetermined by (5.5), then only the graviton self-energy appears in addition to the derivative terms in the Jacobi equation (5.9).

$\Pi^{\alpha\beta\mu\nu}$ thus acts as a (non-constant) mass matrix for small perturbations $\delta g_{\alpha\beta}$. However, because of the constraints (5.10), $\Pi^{\alpha\beta\mu\nu}$ must be first decomposed into transverse-traceless parts and parts dropping out from (5.11), before any information can be extracted from eq. (5.11). This obviously holds true even if one is looking only at the static limit of eq. (5.11).

5.2. TRANSVERSE-TRACELESS TENSORS AT FINITE TEMPERATURE

In the following we shall employ a local momentum space representation to determine the structure functions of $\Pi^{\alpha\beta\mu\nu}$ which correspond to the transverse-traceless parts. Such a momentum representation can be introduced even in curved spaces through the use of geodesic normal coordinates [16], however at the expense of not having control over the global behaviour of Green's functions. Our simple usage of this momentum representation in fact will restrict everything that follows to the local tangent space.

With the flat-space evaluation of $\Pi^{\alpha\beta\mu\nu} \sim GT^4$ of sect. 3 we already have the leading contributions for the curved space since $R^{\alpha\mu\nu\beta} \sim GT^4 \ll T^2$ and curvature corrections to Π will be of the same order as two-loop contributions.*

However, in eq. (5.11) ∇^2 being a tensorial rather than a scalar operator, contains in normal coordinates additional curvature contributions $\sim GT^4$, which cannot be neglected and which also have to be decomposed in possible transverse-traceless parts. Since in the origin of a normal coordinate system

$$\partial^{\nu} \Gamma_{\nu\mu}^{\alpha} \Big|_0 = \frac{1}{3} R_{\mu}^{\alpha}, \quad (5.12)$$

∇^2 reads

$$g^{\alpha(\mu} g^{\nu)\beta} \nabla^2 \Big|_0 = \eta^{\alpha(\mu} \eta^{\nu)\beta} \partial^2 - \frac{1}{3} (\eta^{\alpha(\mu} R^{\nu)\beta} + \eta^{\beta(\mu} R^{\nu)\alpha}). \quad (5.13)$$

At zero temperature there is only one transverse-traceless (TT) tensor available,

$$T_o^{\alpha\beta\mu\nu} = P^{\alpha(\mu} P^{\nu)\beta} - \frac{1}{3} P^{\alpha\beta} P^{\mu\nu}, \quad (5.14)$$

where $P^{\alpha\beta} = \eta^{\alpha\beta} - Q^{\alpha} Q^{\beta} / Q^2$, but at $T \neq 0$ T_o can be decomposed into 3 independent TT-tensors due to the presence of the additional time-like vector u^{μ} ,

$$T_o^{\alpha\beta\mu\nu} = T_A^{\alpha\beta\mu\nu} + T_B^{\alpha\beta\mu\nu} + T_C^{\alpha\beta\mu\nu}. \quad (5.15)$$

These TT-tensors can be represented in the tensorial basis of Tab. 1, and the coefficients in

$$T_I = \sum_{j=1}^{14} c_{Ij} T_j, \quad I = A, B, C \quad (5.16)$$

*Some curvature corrections have been discussed in refs. [2, 3, 4].

Tab. 2: Components of the transverse-traceless tensors T_A , T_B and T_C in the basis T_i given in Tab. 1.

i	c_A	c_B	c_C
1	$1/2$	0	0
2	$Q^2/2\bar{q}^2$	$-Q^2/2\bar{q}^2$	0
3	$Q^4/2\bar{q}^4$	$-2Q^4/\bar{q}^4$	$3Q^4/2\bar{q}^4$
4	$-1/2$	0	$1/6$
5	$-Q^2/2\bar{q}^2$	0	$Q^2/2\bar{q}^2$
6	$-q_0/2\bar{q}^2$	$q_0/2\bar{q}^2$	0
7	$-q_0Q^2/2\bar{q}^4$	$2q_0Q^2/\bar{q}^4$	$-3q_0Q^2/2\bar{q}^4$
8	$1/2\bar{q}^2$	$-q_0^2/\bar{q}^2Q^2$	0
9	$(q_0^2 + \bar{q}^2)/2\bar{q}^4$	$-2q_0^2/\bar{q}^4$	$(3q_0^2 - \bar{q}^2)/2\bar{q}^4$
10	$Q^2/2\bar{q}^4$	$-(4q_0^2 - \bar{q}^2)/2\bar{q}^4$	$3q_0^2/2\bar{q}^4$
11	$-q_0/2\bar{q}^4$	$(2q_0^2 - \bar{q}^2)q_0/Q^2\bar{q}^4$	$-(3q_0^2 - \bar{q}^2)q_0/2Q^2\bar{q}^4$
12	$1/2\bar{q}^4$	$-2q_0^2/Q^2\bar{q}^4$	$2/3Q^4 + (3q_0^2 + \bar{q}^2)/2Q^2\bar{q}^4$
13	$-1/2\bar{q}^2$	0	$1/2\bar{q}^2 + 1/3Q^2$
14	$q_0/2\bar{q}^2$	0	$-q_0/2\bar{q}^2$

are listed in Tab. 2. The TT-tensors T_I thus defined are idempotent and orthogonal to each other.

Both $\Pi^{\alpha\beta\mu\nu}$ and ∇^2 need to be decomposed into these TT-tensors. Projecting out the TT components of the graviton self-energy

$$\Pi^{\mu\nu\alpha\beta}|_{TT} = (c_A T_A + c_B T_B + c_C T_C)^{\mu\nu\alpha\beta}, \quad (5.17)$$

one finds that only the structure functions c_1 , c_2 , and c_3 contribute:

$$c_A(q_0, q) = 2c_1 = \left(\frac{1}{4\bar{q}^4} L(q_0, q) - \frac{1}{12\bar{q}^2} + \frac{1}{3} \right) \rho, \quad (5.18a)$$

$$c_B(q_0, q) = 2 \left(c_1 - \frac{\bar{q}^2}{Q^2} c_2 \right) \quad (5.18b)$$

$$= \left(-\frac{Q^4}{\bar{q}^4} L(q_0, q) + \frac{1}{3\bar{q}^2} + \frac{1}{6} + \frac{2}{3\bar{q}^2} \right) \rho, \quad (5.18c)$$

$$c_C(q_0, q) = 2 \left(c_1 - \frac{4}{3\bar{q}^2} c_2 + \frac{1}{3\bar{q}^4} c_3 \right) \\ = \left(\frac{3Q^4}{2\bar{q}^4} L(q_0, q) - \frac{1}{2\bar{q}^2} + \frac{1}{2} + \frac{2}{3\bar{q}^2} \right) \rho. \quad (5.18c)$$

With

$$\eta^{\alpha(\mu} \eta^{\nu)\beta} \nabla^2|_{TT} = (\square_A T_A + \square_B T_B + \square_C T_C)^{\mu\nu\alpha\beta}, \quad (5.19)$$

and using eq. (5.13) together with the Einstein equations (5.6) and the result for the energy-momentum tensor (2.2), we obtain in the local momentum space

$$\square_A = - \left(Q^2 + \frac{16\pi G\rho}{9} \right), \quad (5.20a)$$

$$\square_B = - \left(Q^2 + \frac{16\pi G\rho}{9} \left[1 + 2\frac{\bar{q}^2}{Q^2} \right] \right), \quad (5.20b)$$

$$\square_C = - \left(Q^2 + \frac{16\pi G\rho}{9} \left[1 + \frac{8}{3}\frac{\bar{q}^2}{Q^2} \right] \right). \quad (5.20c)$$

Inverting the Jacobi equation (5.11) (with vanishing cosmological constant and vanishing Weyl tensor) in the subspace of TT-tensors we thus obtain

$$\delta g_{\alpha\beta} = -16\pi G \sum_{I=A,B,C} (\square_I + 32\pi G c_I)^{-1} T_{I\alpha\beta\mu\nu} \delta T^{\mu\nu}. \quad (5.21)$$

The physical meaning of the A,B,C-modes can now be understood by noting that

$$T_A^{\mu\nu\alpha\beta} u_\alpha \equiv 0, \quad (5.22a)$$

$$T_B^{\mu\nu\alpha\beta} u_\alpha \neq 0, \quad T_B^{\mu\nu\alpha\beta} u_\alpha u_\beta \equiv 0, \quad (5.22b)$$

$$T_C^{\mu\nu\alpha\beta} u_\alpha \neq 0, \quad T_C^{\mu\nu\alpha\beta} u_\alpha u_\beta \neq 0. \quad (5.22c)$$

Evidently, the three modes differ in that mode A neither couples to the energy density δT^{00} nor to the energy flux δT^{0i} , mode B does not couple to energy density but couples to energy flux, and only mode C couples to energy density.

Because eqs. (5.22) imply that T_A and T_B are also spatially traceless, the modes A and B moreover couple only to the anisotropic part of the stress tensor δT^{ij} .

Eqs. (5.22) furthermore imply that in addition to transversality and tracelessness with respect to 4 dimensions, mode A is also spatially transverse and traceless,

$$q_i \delta g_{ij}^A = 0, \quad \delta g_{ii}^A = 0, \quad (5.23a),$$

mode B is also spatially traceless but spatially transverse only if the second index corresponds to the direction u^β or Q^β ,

$$q_i \delta g_{i0}^B = 0, \quad q_i (\delta g_{ij}^B Q^\beta) = 0, \quad \delta g_{ii}^B = 0, \quad (5.23b)$$

whereas no such relations hold for mode C. The three TT-modes A, B, and C could therefore be loosely called purely spatially transverse, partially spatially transverse and spatially longitudinal, respectively.

Further properties are described in Appendix C in terms of the “electric” and “magnetic” components of the Riemann tensor; here we only mention that the purely spatially transverse mode A is similar to ordinary gravitational plane waves, whereas mode C contains the gravitational potential in the Newtonian limit and is thus the analogue of the spatially longitudinal electric mode of the QCD plasma. The appearance of the “anti-screening” Jeans mass (in place of the electric Debye screening mass) would therefore be expected in mode C.

6. Collective phenomena and instabilities of the graviton plasma

The structure functions in eq. (5.21) describe the response of a graviton plasma in thermal equilibrium to weak external perturbations, and their poles which in the local momentum space are given by

$$\square_I(q_0, q) + 32\pi G c_I(q_0, q) = 0, \quad I = A, B, C, \quad (6.1)$$

determine the dispersion relations associated with the modes A, B, C, for $|q_0|, |q| \ll T$. Actual normal modes of the graviton plasma will not be simple plane waves with dispersion relation (6.1) since the full background geometry has to be taken into account, in particular as concerns modes with low momenta. However, here we shall be interested only in determining the effective dynamical masses of possible modes rather than the global form of these modes, for which eq. (6.1) should be sufficient.

These effective masses are given by the solutions of

$$Q^2/(16\pi G\rho) = \frac{1}{2} \frac{Q^4}{q^4} L(q_0, q) - \frac{1}{6} \frac{Q^2}{q^2} + \frac{5}{9} \quad (6.2a)$$

$$Q^2/(16\pi G\rho) = -2 \frac{Q^4}{q^4} L(q_0, q) + \frac{2}{3} \frac{Q^2}{q^2} + \frac{2}{9} + \frac{10}{9} \frac{q^2}{Q^2}, \quad (6.2b)$$

$$Q^2/(16\pi G\rho) = +3 \frac{Q^4}{q^4} L(q_0, q) - \frac{Q^2}{q^2} + \frac{8}{9} + \frac{28}{27} \frac{q^2}{Q^2}, \quad (6.2c)$$

for the modes A, B, C, respectively.

In the following we shall determine the solutions of Eqs. (6.2) for real values of q_0^2 and q^2 . By this we can cover three types of collective phenomena: (I) For real q_0 and q poles in eq. (5.21) correspond to propagation of waves which the graviton plasma can support in the absence of any external sources; (II) poles at real q_0 and imaginary q correspond to spatial screening with screening length $|q|^{-1}$ of perturbations with frequency q_0 ; and (III) potential poles at imaginary q_0 and real q indicate (depending on the sign of $i q_0$) damping or anti-damping of (Fourier components of) spatial inhomogeneities with wavelength q^{-1} .*

The corresponding analytical continuations of eqs. (6.2) is trivial apart from the function $L(q_0, q)$ contained in the c_I 's. For the three cases of interest, $L(q_0, q)$, which has been introduced in eq. (3.6) reads as follows.

I) Real $q_0 = \omega$, real q :

$$L = \frac{\omega}{2q} \ln \frac{\omega + q}{\omega - q} - 1, \quad (6.3a)$$

which is real above the light-cone $\omega > q$, and develops an imaginary part below the light-cone which corresponds to Landau damping [17] and which excludes solutions of eqs. (6.2) for real $\omega \neq 0, q$.

II) Real $q_0 = \omega$, imaginary $q = i\kappa$:

$$L = \frac{\omega}{2\kappa} \arctan \frac{2\omega\kappa}{\omega^2 - \kappa^2} - 1, \quad (6.3b)$$

where the arctangent is on the principal branch for $\omega^2 > \kappa^2$.

*The case of both q_0 and q being imaginary does not seem to be particularly interesting and will not be considered in the following analysis.

III) Imaginary $q_0 = i\gamma$, real q : Damping and anti-damping correspond to $\gamma < 0$ and $\gamma > 0$, respectively, and they differ in that a different branch of L has to be taken:

$$L = -\frac{\gamma}{2q} \left(\arctan \frac{2\gamma q}{q^2 - \gamma^2} + \pi \right) - 1 \quad \text{for } \gamma < 0, \quad (6.3c)$$

where the arctangent is on the principal branch for $q^2 > \gamma^2$.

The numerical solutions of eqs. (6.2) for these three regions are plotted in figs. 2.1-2.3, with some limiting cases given analytically below.

6.1. WAVE PROPAGATION

In all three TT-modes A, B, and C, propagating waves appear above a minimal, common plasma frequency

$$\omega_{pl}^2 = \frac{22}{45} \times 16\pi G\rho \quad (6.4)$$

(see fig. 2.1).

For small q the analytic forms of the three dispersion relations are

$$\omega_A^2 = \omega_{pl}^2 + \frac{166}{315} q^2 + \dots, \quad (6.5a)$$

$$\omega_B^2 = \omega_{pl}^2 + \frac{152}{105} q^2 + \dots, \quad (6.5b)$$

$$\omega_C^2 = \omega_{pl}^2 + \frac{1658}{945} q^2 + \dots, \quad (6.5c)$$

whereas for high frequencies, $\omega^2, q^2 \gg 16\pi G\rho$ the dispersion relation for the A-mode approaches a hyperboloid with dynamical mass

$$m_A^2 \rightarrow \frac{5}{9} \times 16\pi G\rho, \quad (6.6)$$

and those for the modes B and C go over into

$$\omega_{B,C} = q + \mu_{B,C} + \dots, \quad (6.7)$$

where the constants μ are

$$\mu_B^2 = \frac{5}{18} \times 16\pi G\rho, \quad \mu_C^2 = \frac{7}{27} \times 16\pi G\rho. \quad (6.8)$$

The asymptotic behaviour (6.7) for the modes B and C means that their effective mass increases without bound according to

$$m_{B,C}^2 \rightarrow 2\omega \mu_{B,C} \approx 2q \mu_{B,C}. \quad (6.9)$$

6.2. DYNAMICAL SCREENING

When the graviton plasma is driven below the plasma frequency, spatial screening occurs (see fig. 2.2). In the static limit a non-vanishing screening mass

$$m_s^2 = \frac{2}{9} \times 16\pi G\rho \quad (6.10)$$

appears in the purely spatially transverse A-mode (whose analogue in the QCD plasma was unshielded at leading order), while the dynamical screening of modes B and C disappears for $\omega \rightarrow 0$. A peculiarity appears in mode B in that at very small frequencies this mode is in fact a superposition of modes, one with vanishing screening mass which of course dominates and a further one whose screening mass approaches

$$m_s'^2 = \frac{4}{9} \times 16\pi G\rho. \quad (6.11)$$

Mode C which corresponds to spatially longitudinal electrical fields (s. eqs. (C.2), (C.3)) cannot be expected to develop static screening masses in view of the attractive nature of gravitation and indeed does not, but instead picks up an additional pole in region III corresponding to an imaginary mass, the so-called Jeans mass which is the only piece of the dispersion relations studied so far in the literature, although incorrectly as argued above.

6.3. JEANS-LIKE INSTABILITIES

In region III (see fig. 2.3), mode C develops poles which change the Riemann sheets corresponding to damping and anti-damping at $q_0 = 0$ and wave numbers q corresponding to the imaginary mass

$$m_J^2 = -\frac{58}{27} \times 16\pi G\rho, \quad (6.12)$$

so that anti-damping of spatial inhomogeneities occurs for $|q| < |m_J|$. Hence if spatial inhomogeneities have Fourier components with wavelengths larger than the critical wavelength $\sim |m_J|^{-1}$, they are unstable and will provoke gravitational collapse.

Contrary to the classical (Newtonian) Jeans instability which is given by a dispersion relation of the form [6]

$$q_0^2 = v_s^2 q^2 - 4\pi G\rho, \quad (6.13)$$

the anti-damping rate γ reaches a maximum for some $q < |m_J|$ and drops to zero again for $q \rightarrow 0$. Moreover, propagating modes exist in the relativistic graviton plasma only above the light-cone, as we have seen above,* and also the value (6.12) differs from the classical one in eq. (6.13).

In Ref. 1 a simple argument has been given that the generally relativistic value of the Jeans mass should be twice the Newtonian value. However, in this argument only the contributions of energy-density have been taken into account, while in general relativity also pressure can act as a source of gravitation, and furthermore a flat background has been assumed in spite of the presence of a plasma which necessarily produces curvature. In fact, our result (6.12) is larger than that expected in [1].

Curiously enough, an instability occurs also in mode B, but starting at much smaller values of q , i.e. larger wavelengths. Here a superposition of two anti-damped modes occurs, where the one with larger γ is the dominant one. No such instabilities occur in the purely spatially transverse mode A.

In Ref. 4 it has been suggested that the Jeans instabilities of thermal gravitons might be balanced by adjusting the cosmological constant to the negative value $\Lambda = \frac{1}{2}m_J^2$, for in the Jacobi equation the cosmological term appears as sort of a mass term with reversed sign. This is, however, deceptive [18]. A non-vanishing cosmological constant also implies non-vanishing $R = 4\Lambda$ (recall that $T^\mu{}_\mu = 0$) in the first term of eq. (5.11), which gives a total apparent mass $-2\Lambda + \frac{2}{3}R = \frac{2}{3}\Lambda$, and this is exactly cancelled by the change $\nabla^2|_0 \rightarrow \nabla^2|_0 - \frac{2}{3}\Lambda$ induced by $R_{\mu\nu} \rightarrow R_{\mu\nu} + \Lambda g_{\mu\nu}$ in (5.13). One thus cannot use the cosmological constant to make the gravitons massive so that they could ‘outweigh’ the negative m_J^2 .

*However, there exist poles on the unphysical sheets covering region I which start at $q_0 = 0$, $q = |m_J|$ (not included in figs. 2).

7. Summary and conclusion

We have calculated the leading temperature contributions to the graviton self-energy with $Q \ll T \ll T_{\text{Planck}}$, and have used them to obtain the dispersion relations for small disturbances of gravitational fields in thermal equilibrium.

The effective dynamical masses of the collective modes of the perturbative graviton plasma have been found to be to some extent similar to those known from the relativistic electromagnetic (or QCD) plasma [9]. In both, there is a minimal plasma frequency above which the plasma can support propagating normal modes. However, the additional collective modes B and C of the graviton plasma which have no analogue in the $T = 0$ case differ from the extra spatially longitudinal (plasmon) mode of the QCD plasma in that their dynamical masses grow without bound with increasing frequency, whereas the QCD plasmon has exponentially vanishing mass. Below the plasma frequency the main differences are that in QCD the spatially transverse (plane wave) is not screened to leading order in the static limit, whereas the corresponding mode of the graviton plasma is. The opposite holds for the spatially longitudinal (electric) modes: in the QCD plasma there is a non-zero Debye screening mass, whereas in the graviton plasma this mode is not screened, but instead develops instabilities beyond a critical wavelength of spatial inhomogeneities. The appearance of the tachyonic Jeans mass only marks the onset of these instabilities, but does not give rise to waves propagating below the light-cone as we have seen in 6.1.

This instability seems to be the most significant consequence of our investigations. Although its appearance was already expected in Ref. 1, we believe to have presented its first consistent determination within perturbative quantum gravity at non-zero temperature, correcting the mutually disagreeing approaches of refs. [1–4].

Admittedly, practical (e.g. cosmological) implications beyond the known ones [6] are probably sparse. First, we had to assume temperatures well below the Planck scale in order for a loop expansion to make sense, but in the early universe only at temperatures beyond the Planck temperature ($\sim 10^{32} K$) could gravitons be expected to have been in thermal equilibrium (if Einstein gravity is at all the true theory at this mass scale). Second, the collective phenomena we have found appear at extremely small momenta which are sensitive to the large-scale structure of

the hot curved space disregarded in the above analysis. For instance in a standard Robertson-Walker-Friedmann model one would find that the characteristic times are of the same order of magnitude as the age of the universe at a given temperature. In the conventional treatment of cosmological perturbations (where gravitons are not assumed to be in thermal equilibrium) [6, 19], the expansion of the universe modifies significantly the form of the Jeans instability and gives rise to e.g. power-like instead of exponential growth.

Therefore we expect our results to be of interest more with respect to the fundamental (and formidable) issue of understanding the structure of quantized Einstein gravity. Since the instabilities we have found appear in principle for any non-vanishing temperature, as low as it may be, they point to a fundamental deficiency of perturbative quantum gravity in addition to the more widely recognized ones like non-renormalizability.

ACKNOWLEDGEMENTS

I would like to thank Gabor Kunstatter for drawing my attention to the topic of quantum gravity at finite temperature, and John Donoghue for useful discussions.

Appendix A. Conventions and notations

We shall follow the conventions of Ref. 20, that is: metric signature $(+ - - -)$, and curvature tensors

$$R_{\beta\gamma\delta}^{\alpha} = \Gamma_{\beta\gamma,\delta}^{\alpha} - \dots, \quad R_{\mu\nu} = R_{\mu\alpha\nu}^{\alpha}. \quad (A.1)$$

Further, we use units in which $\hbar = c = k = 1$.

The energy-momentum tensor is defined by

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = g_{\mu\alpha} g_{\nu\beta} \left(\frac{-2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\alpha\beta}} \right), \quad (A.2)$$

which, when fermionic fields have to be included, must be replaced by

$$T_{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta e^{\alpha\mu}_\nu} e^{\alpha}_\mu, \quad (A.3)$$

where $g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$.

The Einstein action (with a cosmological term) is given by

$$S_G = \frac{1}{16\pi G} \int d^4x \sqrt{-g} (R - 2\Lambda), \quad (A.4)$$

and for the calculation of the graviton self-energy (3.3) we have used the following gauge fixing and Faddeev-Popov Lagrangian densities (Feynman-de Donder gauge):

$$\mathcal{L}_{G-g.f.} = \frac{1}{32\pi G} \eta_{\mu\nu} (\partial_\rho \sqrt{-g} g^{\rho\mu}) (\partial_\sigma \sqrt{-g} g^{\sigma\nu}), \quad (A.5)$$

$$\mathcal{L}_{G-F.P.} = (\partial_\mu \chi_\nu) \frac{\partial(\sqrt{-g} g^{\mu\nu})}{\delta \xi^\lambda} \eta. \quad (A.6)$$

The Lagrange densities for the various massless fields considered in this work are as follows.

Scalar fields:

$$\mathcal{L}_S = \frac{1}{2} \sqrt{-g} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \quad (A.7)$$

(a possible coupling of ϕ^2 to the Ricci scalar does not change the leading temperature contributions);

Weyl fermions:

$$\mathcal{L}_F = \frac{i}{4} \sqrt{-g} [\bar{\psi} (1 - \gamma_5) \gamma^\mu \nabla_\mu \psi - (\nabla_\mu \bar{\psi}) (1 - \gamma_5) \gamma^\mu \psi], \quad (A.8)$$

where $\gamma^\mu = e^\mu_a \gamma^a$, $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$, and ∇ is the covariant derivative containing the spin connection;

photons:

$$\mathcal{L}_\gamma = -\frac{1}{4} \sqrt{-g} g^{\mu\nu} g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} \quad (A.9)$$

with the following gauge fixing and Faddeev-Popov Lagrangian (which here is essential even in the abelian case)

$$\mathcal{L}_{\gamma-g.f.} = -\frac{1}{2} (\nabla_\mu A^\mu)^2, \quad (A.10)$$

$$\mathcal{L}_{\gamma-F.P.} = g^{\mu\nu} (\partial_\mu c) \delta_{\nu c}. \quad (A.11)$$

(The self-interactions of Yang-Mills fields do not make any difference as concerns the leading temperature contributions to the graviton self-energy.)

Appendix B. High-temperature expansion of the graviton self-energy

The high-temperature limit of the integral (3.2), to which the leading contributions of the graviton self-energy at momenta small compared to temperature can be reduced, is most efficiently calculated by means of the so-called Dzyaloshinski algorithm described in Refs. [8, 7, 12], where one employs a momentum representation in space and a coordinate representation in time. A bosonic propagator, for example, is represented by

$$\frac{1}{K^2} = \int_0^{1/T} d\tau e^{iK\tau} \frac{1}{2k} \left[(1 + n(k)) e^{-k\tau} + n(k) e^{k\tau} \right], \quad (B.1)$$

and the sum over the discrete Matsubara frequencies can then be used to reduce the number of integrals of the auxiliary proper times. Energy factors in the numerator of a momentum space integral are thereby transformed into differentiations with respect to the proper times, and can be simplified by partial integrations, unless they occur in higher powers. A useful trick is therefore to rewrite for example $k_0^2 = K^2 + \vec{k}^2$.

The bosonic thermal integral

$$J \equiv T \sum_{k_0/2\pi T = -\infty}^{\infty} \int \frac{d^3k}{(2\pi)^3} \frac{f(k_0, p_0; \vec{k}, \vec{p})}{K^2 P^2}, \quad (B.2)$$

where $P^\mu = (K - Q)^\mu$ and f depends at most linearly on k_0 and p_0 , is thus reduced to

$$J = \int \frac{d^3k}{(2\pi)^3} \frac{1}{4kp} \left\{ [n(k) - n(p)] \left(\frac{f(k, p)}{q_0 - k + p} - \frac{f(-k, -p)}{q_0 + k - p} \right) + [n(k) + n(p) + 1] \left(\frac{f(-k, p)}{q_0 + k + p} - \frac{f(k, -p)}{q_0 - k - p} \right) \right\}, \quad (B.3)$$

where analytic continuation to real q_0 has been performed after using $e^{iq_0/T} = 1$, and only the first two arguments of f are given explicitly. In contrast to other methods, a high-temperature expansion, i.e. $q_0, q \ll T$, is now readily performed. For this only the integration region where $k, p \gg q_0$ contributes, and approximating

$$n(k) - n(p) \approx n'(k)(k - p), \quad k - p \approx \vec{k} \cdot \vec{q}/k, \quad (B.4)$$

one obtains

$$J \approx \int \frac{d^3k}{(2\pi)^3} \frac{1}{4k^3} \left\{ n(k) [f(-k, k) + f(k, -k)] + n'(k) \vec{k} \cdot \vec{q} \left[\frac{f(k, k)}{q_0 - \vec{k} \cdot \vec{q}/k} - \frac{f(-k, -k)}{q_0 + \vec{k} \cdot \vec{q}/k} \right] \right\}. \quad (B.5)$$

The integral (3.2) can now be evaluated by taking all possible contractions with the tensors of Tab. 1. The results are

$$\begin{aligned} I_{0000} &= \left(\frac{q_0}{q} \ln \frac{q_0 + q}{q_0 - q} - \frac{1}{2} \right) \rho, \\ I_{\mu\nu}{}^{\mu\nu} &= I_{\mu}{}^{\mu}{}_{\nu}{}^{\nu} = 0, \\ I_{\mu 0}{}^{\mu}{}_0 &= I_{00}{}^{\mu}{}_{\mu} = \rho, \\ I_{\mu\lambda 0}{}^{\mu}{}_{\lambda} Q^\lambda &= I_{\lambda 0}{}^{\mu}{}_{\mu} Q^\lambda = q_0 \rho, \\ I_{000\lambda} Q^\lambda &= \frac{3}{2} q_0 \rho, \\ I_{\mu\lambda}{}^{\mu}{}_{\gamma} Q^\lambda Q^\gamma &= I_{\lambda\gamma}{}^{\mu}{}_{\mu} Q^\lambda Q^\gamma = \left(q_0^2 + \frac{1}{3} \vec{q}^2 \right) \rho, \\ I_{00\lambda\gamma} Q^\lambda Q^\gamma &= I_{\lambda\gamma 0} Q^\lambda Q^\gamma = \frac{1}{2} \left(3q_0^2 - \vec{q}^2 \right) \rho, \\ I_{0\lambda\gamma\sigma} Q^\lambda Q^\gamma Q^\sigma &= q_0 \left(\frac{3}{2} q_0^2 - \frac{5}{6} \vec{q}^2 \right) \rho, \\ I_{\kappa\lambda\gamma\sigma} Q^\kappa Q^\lambda Q^\gamma Q^\sigma &= \frac{1}{2} \left(3q_0^2 + \vec{q}^2 \right) Q^2 \rho, \end{aligned} \quad (B.6)$$

where $\rho = \pi^2 T^4/30$ for the bosonic case and $\rho = 7\pi^2 T^4/240$ for the fermionic one. A translation of these results into the components of I in the tensorial basis of Tab. 1 is only an algebraical problem, for which we have used the computer program REDUCE [21].

Appendix C. Spatial decomposition of the Riemann tensor and the finite temperature TT-modes

In the local plasma rest frame given by comoving coordinates where $u^\mu = \delta_0^\mu$, a natural spatial decomposition of the Riemann tensor is given by the following 3-dimensional 2-tensors

$$E_{ij} = R_{0i0j}, \quad (C.1a)$$

$$H_{ij} = \frac{1}{2}\epsilon_{ilm}R_{0jlm}, \quad (C.1b)$$

$$K_{ij} = \frac{1}{4}\epsilon_{ilm}\epsilon_{jnp}R_{lmnp}, \quad (C.1c)$$

where E and K are symmetric, while H is generally not symmetric but traceless, in correspondence to the $6 + 6 + 8 = 20$ components of the Riemann tensor. The fields E and H will be called electric and magnetic because they are to some extent analogous to the ordinary electromagnetic field strengths [22].

Considering the TT-modes $\delta g_{\mu\nu}^{A,B,C}$ introduced in sect. 5.2, they can be characterized in terms of the tensors (C.1) and their projections onto the momentum of the TT-mode.

In the Newtonian limit only $R_{00} = R_{0i0i} = E_{ii}$ survives, and there only mode C contributes,

$$\delta E_{ii} = -\frac{1}{2}Q^2\delta g_{00}^C. \quad (C.2)$$

In the relativistic (linearized) theory at zero temperature the only propagating mode is given by transversally polarized plane waves, and one can easily show that the purely spatially transverse mode δg^A corresponds to equally spatially transverse fields δE , δH and δK .

A phenomenon peculiar to the plasma is the possibility of propagating spatially longitudinal fields, which are produced by the modes B and C:

$$q_i\delta E_{ij} = \frac{1}{2}Q^2\left(q_j\delta g_{00}^C - q_0\delta g_{0j}^{B,C}\right), \quad (C.3a)$$

$$q_i\delta E_{ij}q_j = \frac{1}{2}Q^4\delta g_{00}^C, \quad (C.3b)$$

$$\delta H_{ij}q_j = -\frac{1}{2}Q^2\epsilon_{ilm}q_l\delta g_{0m}^{B,C}, \quad (C.3c)$$

where in contrast to the usual electromagnetic fields there can exist also spatially longitudinal magnetic fields — H is automatically transverse only with respect to

left projection, $q_i\delta H_{ij} \equiv 0$. The fields K , on the other hand, are always spatially transverse: $q_i\delta K_{ij} \equiv 0 \equiv \delta K_{ij}q_j$.

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FIGURE CAPTIONS

1. Dispersion relations of a (colour-) electromagnetic plasma for frequencies ω and wave vectors q (when real) and $i\kappa$ (when imaginary) in units $(N + N_f/2)^{1/2}gT$, where N_f is the number of flavours with colour group $SU(N)$. The full and the dashed lines correspond to the spatially longitudinal and the spatially transverse modes, respectively.
- 2.1. Dispersion relations of the graviton plasma in units $(16\pi G\rho)^{1/2}$ for real frequencies and wave vectors. The dashed, dotted and full lines correspond to modes A, B, and C, respectively. The diagonal just marks the light-cone.
- 2.2. Dispersion relations for real frequencies and imaginary wave vectors. Only mode A (dashed) approaches a finite screening length κ^{-1} in the static limit, while for modes B and C κ goes to zero. Note that for very small frequencies the dispersion relation of mode B (dotted) possesses more than one solution.
- 2.3. Dispersion relations for imaginary frequencies and real wave vectors on the Riemann sheet corresponding to anti-damping. The solution appearing for mode C on the sheet which corresponds to damping is given by the dotted line. The main instability appears in mode C (full line). For very small wave vectors an instability also occurs in mode B (dotted) with two solutions for one wave vector.

Fig. 1

Fig. 2.1

Fig. 2.2

Fig. 2.3