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CHERN-SIMONS THEORY FOR COMPACT NON-SEMISIMPLE LIE GROUPS

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ABSTRACT

We define Chern-Simons theory for the compact non-semisimple group $U(n)$. Quantum Chern-Simons theory for a $U(n)$ gauge bundle of degree r is related to the representation theory of the r^{th} component of the loop group of $U(n)$. The representations of the loop group of $U(n)$ which feature in the quantum Chern-Simons theory are also considered from the viewpoint of geometric quantization of the $U(n)$ WZW model.

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1. Introduction

In the last few years it has emerged that there is a remarkable connection between invariants of knots and links, two-dimensional conformal field theory and three-dimensional topological gauge theory. In three dimensions there is a natural topological quantum field theory defined by the Chern-Simons action. It was shown in [1] that the Jones polynomial of knot theory can be interpreted as the expectation value of Wilson lines in the Chern-Simons theory. It was also observed in [1] that when Chern-Simons theory is canonically quantized, the quantum Hilbert space associated to a two-dimensional surface may be interpreted as the space of conformal blocks of a two-dimensional conformal field theory. The quantization of Chern-Simons theory and its relationship to conformal field theory was further developed in [2,3].

Chern-Simons theory for a compact simple simply-connected Lie group G is now well understood; in this paper we consider the case of a compact non-semisimple non-simply-connected Lie group. The fundamental example of such a group is the unitary group $U(n)$, and this is the case we shall discuss here. In Section 2 we define the Chern-Simons action for the gauge group $U(n)$ and show that the path integral is well-defined. In Section 3 we consider the canonical quantization of Chern-Simons theory with gauge group $U(n)$. It is argued that the canonical phase space is the space of projectively flat $U(n)$ connections on a surface Σ . This phase space is then quantized using the methods of geometric quantization. The connection between $U(n)$ Chern-Simons theory and the representation theory of the loop group of $U(n)$ is discussed in Section 4. The quantum Chern-Simons theory for a $U(n)$ gauge bundle of first Chern class r is related to the basic representation of the r^{th} component of the loop group $LU(n)$ of $U(n)$. Finally, in Section 5, the connection between $U(n)$ Chern-Simons theory and the representations of $LU(n)$ is discussed from the viewpoint of the WZW model with group $U(n)$. It is shown that the representations of the components of $LU(n)$ which occur in the Chern-Simons theory arise from the geometric quantization of the $U(n)$ WZW model.

2. The Chern-Simons Action for the Gauge Group $U(n)$

Let M be a compact three-manifold and G a compact Lie group. In addition let $P \rightarrow M$ be a principal G -bundle over M . We denote by $\text{ad } P$ the bundle associated to P by the adjoint action of G on its Lie algebra $\mathfrak{g}$, i.e., $\text{ad } P = P \times_G \mathfrak{g}$. The space of $\text{ad } P$ -valued p -forms on M is denoted by $\Omega^p(M; \text{ad } P)$. The space $\mathcal{A}$ of all connections on P is an affine space with associated vector space $\Omega^1(M; \text{ad } P)$. If P is the trivial bundle $P = M \times G$ then the gauge connection A is globally defined on M and the Chern-Simons action is given by

$$\tilde{S}[A] = \frac{k}{4\pi} \int_M \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \quad (1)$$

where Tr represents an adjoint invariant bilinear form on the Lie algebra $\mathfrak{g}$ of G .

If the gauge group G is simply connected [for example $G = SU(n)$] then any G -bundle on M will be trivial and the Chern-Simons action can always be written as (1). For the trivial bundle $P = M \times G$, the group of gauge transformations $\mathcal{G} = \text{Map}(M, G)$, the group of smooth maps from M to G . For G simple and simply connected, it may be shown [4] that $\mathcal{G}$ is not connected; the group of components of G is $\pi_0(G) \simeq \pi_3(G) \simeq \mathbb{Z}$. Under a gauge transformation $g : M \rightarrow G$ the action (1) changes according to

$$\tilde{S}[A^g] = \tilde{S}[A] + 2\pi k w(g) \quad (2)$$

where $w(g)$ is the degree of the map g under the isomorphism $\pi_0(G) \simeq \mathbb{Z}$. Thus for $e^{\tilde{S}[A]}$ to be well-defined we require that k should be an integer and then the path integral

$$Z = \int_{\mathcal{A}} \mathcal{D}A \, e^{\tilde{S}[A]} \quad (3)$$

makes sense. As the action (1) is independent of any choice of metric on M it is clear that the partition function Z will be a topological invariant of M - in fact it is an invariant closely related to the Ray-Singer analytic torsion of M . The remarkable discovery made in [1] was that Chern-Simons theory is also related to knot invariants. If $C : S^1 \rightarrow M$ is an embedded curve in

M and R is an irreducible representation of G , then the Wilson line $W_R(C) = \text{Tr}_R P \int_C A_i dx^i$ associated to C and R gives a gauge-invariant observable which is independent of any metric on M . The expectation value of $W_R(C)$

$$\langle W_R(C) \rangle = \int_A \mathcal{D}A e^{iS[A]} W_R(C) \quad (4)$$

gives an invariant of the embedded knot C , which for $M = S^3$, $G = SU(n)$ and R the defining representation is the two-variable Jones polynomial [1].

We now wish to consider the definition of the Chern-Simons action when the group G is not simply-connected; the case we have in mind is $G = U(n)$. A related discussion of this problem for simple groups has been given in [6]. If G is not simply-connected then there will in general exist non-trivial G -bundles P over the three-manifold M . For a non-trivial bundle $P \rightarrow M$ the connection one-form A is not globally defined on M and the action (1) cannot be written down directly. However, for the trivial bundle [with gauge group $SU(n)$, say], it is well-known that the Chern-Simons action (1) may be written in a different, but equivalent, form. We introduce a compact oriented four-manifold Y whose boundary is M . The trivial gauge bundle $V \rightarrow M$ clearly extends as the trivial bundle over Y . Any connection on $V \rightarrow M$ automatically extends to a connection over Y . The Chern-Simons functional satisfies

$$d \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) = \text{Tr}(F \wedge F) \quad (5)$$

and thus the action (1) may be written as

$$S[A] = \frac{k}{4\pi} \int_Y \text{Tr}(F \wedge F) \quad (6)$$

where F denotes the curvature of the connection over Y . In fact, as will be discussed below, any compact oriented three-manifold bounds a compact oriented four-manifold and thus the action (1) may always be written in the form (6). The only problem with this definition is that (6) depends upon a choice of coboundary Y , which in general will not be unique. We require

that the quantum theory should be independent of the choice of coboundary. Thus if Y_1 and Y_2 are two four-manifolds which M bounds (M bounds Y_1 with the opposite orientation to Y_2) and $\hat{S}_1[A]$ and $\hat{S}_2[A]$ are the corresponding actions $\hat{S}_1[A] = \frac{k}{4\pi} \int_{Y_1} \text{Tr}(F \wedge F)$ and $\hat{S}_2[A] = -\frac{k}{4\pi} \int_{Y_2} \text{Tr}(F \wedge F)$, then we require that $e^{i\hat{S}_1} = e^{i\hat{S}_2}$. This implies that

$$\frac{k}{4\pi} \int_{Y_1+Y_2} \text{Tr}(F \wedge F) = 2\pi N \quad (7)$$

where N is an integer. Now $X = Y_1 + Y_2$ is a closed compact four-manifold with a non-trivial $SU(n)$ -bundle $\hat{V}$ defined over it. The second Chern class $c_2(\hat{V})$ is defined by $c_2(\hat{V}) = \frac{1}{8\pi^2} \text{Tr}(F \wedge F)$. Thus the second Chern number

$$n = \frac{1}{8\pi^2} \int_X \text{Tr}(F \wedge F) \quad (8)$$

is an integer and we obtain

$$\hat{S}_1 = \hat{S}_2 + 2\pi kn \quad (9)$$

Thus, $e^{i\hat{S}_1} = e^{i\hat{S}_2}$ requires that k should be an integer, as is well known.

We have seen that for the trivial G -bundle over M the Chern-Simons action (1) may be defined equivalently by (6). For the case $G = U(n)$, that we are interested in here, we will adopt (6) as the definition of the Chern-Simons action for general non-trivial bundles over M . Before showing that (6) is well-defined for $G = U(n)$ we will briefly discuss the classification of G -bundles over a three-manifold M ; these ideas will be needed below.

A G -bundle V over a manifold M is classified by a map from M into a certain space associated to G [6]. To any Lie group G there is associated its universal classifying space BG , which is characterized, up to homotopy type, by the requirement that BG is the base space of a G -bundle

$$G \rightarrow EG \rightarrow BG \quad (10)$$

in which EG is a contractible space. It is a fundamental result in topology [6] that given any G -bundle $V \rightarrow M$ there exists a map $f : M \rightarrow BG$ such that $V \simeq f^*EG$. If two maps

$f_1, f_2 : M \rightarrow BG$ are homotopic then the bundles f_1^*EG and f_2^*EG are isomorphic. Thus, the space of isomorphism classes of G -bundles over M is isomorphic to the set $[M; BG]$ of homotopy classes of maps from M to BG .

Now let us consider the case of interest here in which M is a three-manifold. It is well known [6] that the homotopy groups $\pi_i(BG)$ of BG are related to those of G by

$$\pi_i(BG) \simeq \pi_{i-1}(G) \quad (11)$$

for $i \geq 1$. Thus, for any connected Lie group G , the first three homotopy groups of BG are: $\pi_1(BG) = \pi_0(G) = 0$, $\pi_2(BG) = \pi_1(G) \equiv \pi$, $\pi_3(BG) = \pi_2(G) = 0$, where we have used the well-known fact that $\pi_2(G) = 0$ for any Lie group G . It follows, therefore, that the space BG is approximated homotopically up to dimension three by the Eilenberg-MacLane space $K(\pi, 2)$, i.e., by the space whose only non-vanishing homotopy group is in dimension two and is isomorphic to π . The G -bundles over a three-manifold M are therefore classified by $[M; K(\pi, 2)]$, the set of homotopy classes of maps from M to $K(\pi, 2)$. It is a standard result in topology [7] that

$$[M; K(\pi, 2)] \simeq H^2(M; \pi) \quad (12)$$

Hence a G -bundle $V \rightarrow M$ is classified by a cohomology class $\alpha(V) \in H^2(M; \pi)$, where $\pi = \pi_1(G)$. If $\pi_1(G) = 0$ (e.g., if $G = SU(n)$) then $H^2(M; \pi) = 0$ and any G -bundle over M is trivial. If $G = SO(n)$ then $\pi_1(G) = \mathbb{Z}_2$ and an $SO(n)$ -bundle $V \rightarrow M$ is classified by a class $\alpha(V) \in H^2(M; \mathbb{Z}_2)$, which may be identified with the second Stiefel-Whitney class $w_2(V)$ of V . If $G = U(n)$ then $\pi_1(G) = \mathbb{Z}$ and a $U(n)$ -bundle $V \rightarrow M$ is classified by a class $\alpha(V) \in H^2(M; \mathbb{Z})$, which may be identified with the first Chern class $c_1(V)$ of V .

We can now define the Chern-Simons action for a general $U(n)$ -bundle over a three-manifold M . We wish to model the definition on (6), however, for a general (possibly non-trivial) $U(n)$ -bundle over M there is a new problem. It was shown above that the Chern-Simons action (1) defined for a connection on the trivial bundle V over M could be rewritten in the form of (6). This new form of the action required that the bundle V be extended over a coboundary Y of M . If V is the trivial bundle it can clearly always be extended over Y , however, for a general

non-trivial $U(n)$ -bundle it is not clear that it is always possible to extend the bundle over some coboundary Y . The problem of extending bundles in this way is an aspect of cobordism theory [8], which we will now briefly review. First recall that two oriented n -manifolds M_1 and M_2 are (oriented) if there exists an oriented $(n+1)$ -dimensional manifold Y such that the disjoint union of M_1 and M_2 is the boundary of Y . The set of all equivalence classes of cobordant n -manifolds forms a group Ω_n^{SO} , the n -dimensional oriented cobordism group [8]. The identity element of Ω_n^{SO} consists of those oriented n -manifolds which bound an $(n+1)$ -manifold. Thus if $\Omega_n^{SO} = 0$ then every n -manifold bounds an $(n+1)$ -dimensional manifold. It is a standard result in cobordism theory [8] that $\Omega_3^{SO} = 0$, thus every oriented three-manifold M bounds an oriented four-manifold Y . This result was used earlier in defining (6).

In the case at hand we have, in addition to a manifold M , a G -bundle $V \rightarrow M$, and we wish to determine whether V extends over some coboundary of M . There is a further development of cobordism theory which deals with this problem. Let M_1 and M_2 be two oriented n -manifolds as before and let $h_1 : M_1 \rightarrow W$ and $h_2 : M_2 \rightarrow W$ be maps from M_1 and M_2 to some additional space W . We define $h_1 : M_1 \rightarrow W$ and $h_2 : M_2 \rightarrow W$ to be (oriented) if there exists a map $\hat{h}$ from a $(n+1)$ -dimensional oriented manifold Y to W , such that the disjoint union of M_1 and M_2 bounds Y and $\hat{h}|_{M_i} = h_i$, $\hat{h}|_{M_2} = h_2$. The corresponding n -dimensional bordism group is denoted by $\Omega_n^{SO}(W)$. If $\Omega_n^{SO}(W) = 0$ then for any n -manifold M and map $h : M \rightarrow W$, there exists a coboundary Y of M such that h extends to a map $\hat{h} : Y \rightarrow W$.

To apply bordism theory to a G -bundle over a manifold M we simply take the space W to be the classifying space BG . If $\Omega_n^{SO}(BG) = 0$ then for any n -manifold M and map $h : M \rightarrow BG$ there exists a coboundary Y of M such that h extends to a map $\hat{h} : Y \rightarrow BG$. Hence the G -bundle over M classified by $h : M \rightarrow BG$ extends to the G -bundle over Y classified by $\hat{h} : Y \rightarrow BG$.

We are interested here in the case in which M is a three-manifold and $G = U(n)$. It follows from general results of cobordism theory [9] that for any space W , whose homology is free Abelian in all dimensions (as holds for $BU(n)$), the cobordism ring $\Omega_*^{SO}(W)$ is given by

$$\Omega_*^{SO}(W) \simeq \Omega_*^{SO} \bigotimes_{\mathbb{Z}} H_*(W; \mathbb{Z}), \quad (13)$$

where Ω_3^{SO} denote the cobordism ring of manifolds with no maps, i.e., Ω_3^{SO} (point), and $H_*(W; \mathbb{Z})$ is the homology ring of W . Taking $W = BU(n)$ we have

$$\Omega_3^{SO}(BU(n)) = \Omega_3^{SO} \bigotimes_{\mathbb{Z}} H_3(BU(n); \mathbb{Z}) = 0, \quad (14)$$

as $BU(n)$ has zero homology in odd dimensions. It follows from (14) that for any $U(n)$ -bundle V over a three-manifold M there exists a coboundary Y such that V extends to a $U(n)$ -bundle $\hat{V}$ over Y . Note that any connection A on V extends to a connection $\hat{A}$ on $\hat{V}$. We can now define the Chern-Simons action to be

$$S[A] = \frac{k}{2\pi} \int_Y \text{Tr}(F \wedge F) \quad (15)$$

where F denotes the curvature of the connection $\hat{A}$ on $\hat{V} \rightarrow Y$. If Y_1 and Y_2 are two coboundaries of M then the $U(n)$ -bundles $\hat{V}_1 \rightarrow Y_1$ and $\hat{V}_2 \rightarrow Y_2$ define a $U(n)$ -bundle $\hat{V}$ over the closed compact four-manifold $X = Y_1 + Y_2$. The second Chern class of the $U(n)$ -bundle $\hat{V} \rightarrow X$ is given by

$$c_2(\hat{V}) = \frac{1}{8\pi^2} [\text{Tr}(F \wedge F) - \text{Tr}(F) \wedge \text{Tr}(F)] \quad (16)$$

It follows that

$$\frac{1}{4\pi^2} \text{Tr}(F \wedge F) = 2c_2 - c_1^2, \quad (17)$$

where $c_1(\hat{V}) = -\frac{1}{2\pi i} \text{Tr}(F)$ is the first Chern class of $\hat{V} \rightarrow X$. The number

$$n = \frac{1}{4\pi^2} \int_X \text{Tr}(F \wedge F) \quad (18)$$

is therefore an integer. It follows, as before, that if S_1 and S_2 are actions defined using (15) with the coboundaries Y_1 and Y_2 , respectively, then

$$S_1 = S_2 + 2\pi kn \quad (19)$$

Thus, for the quantum theory to be well-defined, k must be an integer. It follows from this

discussion that given any $U(n)$ -bundle $V \rightarrow M$ the path integral

$$Z = \int_A \mathcal{D}A e^{iS[A]} \quad (20)$$

is well-defined. Similarly, if R is any irreducible representation of $U(n)$ and $C: S^1 \rightarrow M$ is an embedded curve in M then the expectation value of the Wilson line $W_R(C)$

$$\langle W_R(C) \rangle = \int_A \mathcal{D}A e^{iS[A]} W_R(C) \quad (21)$$

is well-defined. In principle (21) yields an invariant of the knot C for each $U(n)$ -bundle V over M .

3. Canonical Quantization of the $U(n)$ Chern-Simons Theory

The canonical quantization of the Chern-Simons action (1) for a trivial bundle over a three-manifold M is by now well known [1-3]. We are interested here in the quantization of the action defined by (15) for an arbitrary $U(n)$ -bundle over M . Recall that to canonically quantize the action (1) one considers the three-manifold M to be of the form $M = \mathbb{R} \times \Sigma$, where Σ is a compact two-dimensional surface, and chooses the gauge $A_0 = 0$ (where A_0 is the component of the connection in the $\mathbb{R}$ direction). The action (1) then becomes

$$S = \frac{k}{8\pi} \int_{\Sigma} dt \int_{\Sigma} d^2x \cdot \epsilon^{ij} \text{Tr}(A_i \dot{A}_j) \quad (22)$$

and the Gauss' law constraint $\delta S / \delta A_0 = 0$ is

$$\epsilon^{ij} F_{ij} = 0 \quad (23)$$

The phase space of the system is given, therefore, by the space of flat G -connections on Σ modulo gauge equivalence.

We wish now to consider the action (15) for an arbitrary $U(n)$ -bundle $V \rightarrow M$. The most obvious procedure to adopt in canonically quantizing (15) is to take the three-manifold M to be $M = \mathbf{R} \times \Sigma$ and to take $Y = \mathbf{R} \times B$ where B is a compact three-manifold which Σ bounds. As $\Omega_2^{SO} = 0$, for any Σ there always exists a B such that $\Sigma = \partial B$. Choosing the gauge $A_0 = 0$ (where A_0 is the component of the connection in the $\mathbf{R}$ direction) the action (15) then becomes

$$S = \frac{k}{\pi} \int_B dt \int d^3x \epsilon^{\alpha\beta\gamma} \text{Tr}(\dot{A}_\alpha F_{\beta\gamma}) \quad (24)$$

and the Gauss' law constraint $\delta S / \delta A_0 = 0$ is

$$\vec{F}_\Sigma = 0 \quad (25)$$

where $\vec{F}_\Sigma$ is the curvature of the connection on Σ . However, we now see that (25) causes a problem when considering a general $U(n)$ -bundle over M . As $M = \mathbf{R} \times \Sigma$, we have a $U(n)$ -bundle $V \rightarrow \Sigma$ with a connection A whose curvature F satisfies (25). However, the first Chern class of a $U(n)$ -bundle $V \rightarrow \Sigma$ is defined by

$$c_1(V) = -\frac{1}{2\pi i} \text{Tr}(F) \quad (26)$$

where F is the curvature of a connection A on V . Thus if V admits a flat connection $c_1(V) = 0$. A $U(n)$ -bundle V over a surface Σ is classified topologically by the integral first Chern class $c_1(V) \in H^2(\Sigma; \mathbf{Z})$. As $H^2(\Sigma; \mathbf{Z}) \simeq \mathbf{Z}$ is torsion free it follows that V is, in fact, completely determined by the rational first Chern class $c_1(V) \in H^2(\Sigma; \mathbf{R})$, where $c_1(V)$ is defined by (26). Therefore, any $U(n)$ -bundle $V \rightarrow \Sigma$ which admits a flat connection must be topologically trivial. The constraint (25) is, therefore, incompatible with any non-trivial $U(n)$ -bundle over Σ . In fact there is a fundamental topological reason for this problem. It was implicitly assumed above that the $U(n)$ -bundle $V \rightarrow \Sigma$ could be extended to a $U(n)$ -bundle $\hat{V} \rightarrow B$, and we must now ask if this is correct. Such an extension of $V \rightarrow \Sigma$ will be possible if $\Omega_2^{SO}(BU(n)) = 0$.

Applying Eq. (13) with $W = BU(n)$ we obtain

$$\begin{aligned} \Omega_2^{SO}(BU(n)) &\simeq \Omega_2^{SO} \bigotimes_{\mathbf{Z}} H_2(BU(n); \mathbf{Z}) \\ &\simeq H_2(BU(n); \mathbf{Z}) \simeq \mathbf{Z} \end{aligned} \quad (27)$$

For a $U(n)$ -bundle $V \rightarrow \Sigma$ the integer one obtains under the isomorphism $\Omega_2^{SO}(BU(n)) \simeq \mathbf{Z}$ is given by

$$\langle c_1(V), [\Sigma] \rangle = \int_{\Sigma} c_1(V) \quad (28)$$

where $c_1(V) = -\frac{1}{2\pi i} \text{Tr}(F)$. It follows from this that the only $U(n)$ -bundle over Σ which can be extended over B is the trivial bundle. For this bundle the constraint $F = 0$ is perfectly consistent.

It follows from this cobordism argument that any complete treatment of the canonical formulation must be reasonably subtle. We will not here attempt to give such a treatment, rather we will simply propose that the constraint equation that follows from a full treatment is the simplest geometrical modification of (25) that is consistent with a $U(n)$ -bundle of arbitrary topology. Thus we propose that the correct constraint equation is

$$\vec{F}_\Sigma = c \omega \quad (29)$$

where ω is a constant two-form and c is a scalar. Equation (29) defines a connection. For a suitable value of c , (29) is compatible with a $U(n)$ -bundle V of arbitrary topological type. If the bundle V is topologically trivial then (29) reduces to (25). Assuming (29) to be the correct constraint equation it follows that the phase space of Chern-Simons theory, with a $U(n)$ -bundle $V \rightarrow \Sigma$ of first Chern number (degree) $r = \int_{\Sigma} c_1(V)$, is given by the space of projectively flat connections on V modulo gauge transformations. We shall denote this moduli space by $\mathcal{M}_{n,r}$. In the remainder of this section we shall discuss the natural symplectic structure on $\mathcal{M}_{n,r}$ and the geometric quantization of $\mathcal{M}_{n,r}$. This generalizes the standard results on the quantization of Chern-Simons theory with a trivial gauge bundle (e.g., when the gauge group is simply connected) in which case the phase space is $\mathcal{M}_{n,0}$. The quantization of

$\mathcal{M}_{n,r}$ has also been discussed in a topological approach to the quantization of two-dimensional Yang-Mills theory [10].

Let us now consider the natural symplectic structure on $\mathcal{M}_{n,r}$. As the action (15) is generally covariant the Hamiltonian of the Chern-Simons theory is identically zero and the dynamics of the theory is completely determined by the symplectic structure (i.e., the Poisson bracket) on the phase space. The phase space $\mathcal{M}_{n,r}$ is the moduli space of projectively flat $U(n)$ -bundles of degree r over Σ . A projectively flat $U(n)$ -bundle $V \rightarrow \Sigma$ can equivalently be viewed as a flat $PU(n)$ -bundle $\tilde{V} \rightarrow \Sigma$, where $PU(n) = U(n)/\{\text{centre}\}$ is the projective unitary group. Thus, $\mathcal{M}_{n,r}$ may be thought of as the moduli space of flat $PU(n)$ -bundles of degree r over Σ . We will now describe the natural symplectic structure on $\mathcal{M}_{n,r}$: in fact, there is a natural symplectic structure on the moduli space of flat G -bundles over Σ [11,12] for any compact Lie group G , which for $G = PU(n)$ yields the symplectic structure on $\mathcal{M}_{n,r}$.

Let us consider first the moduli space $\mathcal{M}$ of flat G -bundles over Σ . It is well known that $\mathcal{M} = \text{Hom}(\pi_1(\Sigma), G)/G$, where G acts on $\text{Hom}(\pi_1(\Sigma), G)$ by conjugation. A symplectic structure on $\mathcal{M}$ is a closed non-degenerate two-form on $\mathcal{M}$. To determine such a two-form on $\mathcal{M}$ we must first consider the tangent space to $\mathcal{M}$. Let $\varphi \in \text{Hom}(\pi_1(\Sigma), G)/G$ and let P_φ be the associated flat principal G -bundle over Σ . We will denote the canonical flat connection on P_φ by D_φ . Associated to P_φ by the adjoint action $G \rightarrow \text{Aut}(\mathfrak{g})$ we have a flat vector bundle $\text{ad } P_\varphi = P_\varphi \times_G \mathfrak{g}$ over Σ . The canonical flat connection on $\text{ad } P_\varphi$ induced by D_φ will be denoted by ∇_φ . Now let $\Omega^p(\Sigma; \text{ad } P_\varphi)$ denote the space of p -forms on Σ with values in the flat vector bundle $\text{ad } P_\varphi$. Locally such a p -form $\omega \in \Omega^p(\Sigma; \text{ad } P_\varphi)$ is of the form $\omega = \eta \otimes \sigma$, where η is a p -form on Σ and σ is a section of $\text{ad } P_\varphi$. We can then define the differential of $\omega = \eta \otimes \sigma$ to be

$$d_\varphi(\eta \otimes \sigma) = d\eta \otimes \sigma + (-1)^p \eta \otimes \nabla_\varphi \sigma, \quad (30)$$

where ∇_φ here represents the covariant derivative associated to the canonical flat connection on $\text{ad } P_\varphi$. The flatness of ∇_φ implies that $d_\varphi^2 = 0$ and thus we may define the de Rham cohomology $H^p(\Sigma; \text{ad } P_\varphi)$ of the complex $(\Omega^p(\Sigma; \text{ad } P_\varphi), d_\varphi)$. It may be shown [12] that the tangent space $T_\varphi \mathcal{M}$ to $\mathcal{M}$ at φ is given by

$$T_\varphi \mathcal{M} = H^1(\Sigma; \text{ad } P_\varphi) \quad (31)$$

Given two tangent vectors $\alpha, \beta \in T_\varphi \mathcal{M} = H^1(\Sigma; \text{ad } P_\varphi)$ the map

$$\omega_\varphi : H^1(\Sigma; \text{ad } P_\varphi) \times H^1(\Sigma; \text{ad } P_\varphi) \rightarrow \mathbb{R} \quad (32)$$

given by

$$\omega_\varphi(\alpha, \beta) = \int_\Sigma \text{Tr}(\alpha \wedge \beta) \quad (33)$$

defines a non-degenerate two-form on $\mathcal{M}$ which may be shown to be closed [11,12]. Thus, ω defines a canonical symplectic structure on $\mathcal{M}$. Taking $G = PU(n)$, and considering those bundles of degree r , ω defines a canonical symplectic structure on $\mathcal{M}_{n,r}$ for $k = 1$; for $k \neq 1$, $\tilde{\omega} = k\omega$ defines a symplectic two-form on $\mathcal{M}_{n,r}$. This symplectic structure defines the natural Poisson bracket on $\mathcal{M}_{n,r}$ which in turn determines the dynamics of Chern-Simons theory.

To proceed further and quantize the phase space $\mathcal{M}_{n,r}$ it is necessary for $\mathcal{M}_{n,r}$ to have some structure in addition to a symplectic structure. In practice the extra structure needed for quantization requires that $\mathcal{M}_{n,r}$ be either a cotangent bundle or a Kähler manifold. In fact, as we will now discuss, $\mathcal{M}_{n,r}$ may be given a Kähler structure and may be quantized using the standard methods of geometric quantization [13].

The phase space $\mathcal{M}_{n,r}$ of projectively flat $U(n)$ -bundles over Σ of degree r modulo gauge equivalence acquires a Kähler structure once a complex structure σ is chosen on Σ . Upon such a choice, $\mathcal{M}_{n,r}$ can be identified with a suitable subspace of the space of holomorphic vector bundles of rank n and degree r over the Riemann surface Σ_σ . Consider a complex vector bundle $E \rightarrow \Sigma_\sigma$ of rank n and degree r over the Riemann surface Σ_σ given by Σ together with the complex structure σ . Let $\mathcal{E}$ denote the space of holomorphic structures on E . The group of automorphisms $\text{Aut}(E)$ of E acts on $\mathcal{E}$ and the quotient space $\mathcal{E}/\text{Aut}(E)$ is the moduli space of holomorphic vector bundles of rank n and degree r over Σ_σ . We will denote this space by $\mathcal{M}_{n,r}$. This space is non-Hausdorff and to obtain a good moduli space it is necessary to consider a suitable subspace of $\mathcal{M}_{n,r}$. The moduli space of semi-stable vector bundles in $\mathcal{M}_{n,r}$ provides such a subspace. If we let $\mu(E) = \text{degree}(E)/\text{rank}(E)$ then a holomorphic vector bundle E is semi-stable if for every holomorphic sub-bundle $F \subset E$, $\mu(F) \leq \mu(E)$. If $\bar{\mathcal{E}}$ denotes the space

of all semi-stable holomorphic vector bundles of rank n and degree r over Σ_σ then the moduli space $\bar{M}_{n,r} = \bar{\mathcal{E}}/\text{Aut}(E)$ is a Hausdorff space. The fundamental connection between the phase space $\mathcal{M}_{n,r}$ and the space $\bar{M}_{n,r}$ is provided by the theorem of Narasimhan and Seshadri [14] as reformulated in [11] and [15]. This theorem states that a holomorphic vector bundle E over a compact Riemann surface is semi-stable if and only if E admits a projectively flat Hermitian structure. Thus, given a complex structure σ on Σ , we can identify the phase space $\bar{M}_{n,r}$ with the moduli space $\bar{M}_{n,r}$ of semi-stable bundles over Σ_σ . For the purposes of quantization the key property of $\bar{M}_{n,r}$ is that it is a compact Kähler manifold [16]. It follows, therefore, that $\mathcal{M}_{n,r}$ may be quantized geometrically.

The general principles of geometric quantization [13] imply that to quantize $\mathcal{M}_{n,r}$ we require a holomorphic line bundle $\mathcal{L} \rightarrow \mathcal{M}_{n,r}$, such that $\mathcal{L}$ admits a connection whose curvature two-form Ω is equal to the Kähler form $\hat{\omega}$ on $\mathcal{M}_{n,r}$. Such a holomorphic line bundle is given by the determinant bundle, which is defined as follows. Given a complex vector bundle $E \rightarrow \Sigma_\sigma$ of rank n and degree r there is a $\bar{\partial}$ operator $\bar{\partial} : \Omega^{0,0}(\Sigma_\sigma; E) \rightarrow \Omega^{0,1}(\Sigma_\sigma; E)$ associated to each holomorphic structure on E . To a holomorphic structure $y \in \mathcal{E}$ corresponding to the operator $\bar{\partial}$ we can define the determinant line

$$\mathcal{L}_y = \bigwedge^{\max} (\ker \bar{\partial})^* \otimes \bigwedge^{\max} (\ker \bar{\partial}^*) \quad (34)$$

As y varies over $\mathcal{E}$ the lines $\mathcal{L}_y$ vary holomorphically forming a holomorphic line bundle $\mathcal{L} = \text{Det } \bar{\partial}$ over $\mathcal{E}$ [17,18]. The determinant line bundle $\text{Det } \bar{\partial}$ descends to $\mathcal{M}_{n,r} = \bar{\mathcal{E}}/\text{Aut}(E)$ and $\text{Det } \bar{\partial}$ has a natural connection whose curvature Ω is (for $k=1$) equal to the Kähler form ω on $\mathcal{M}_{n,r}$ [17,18]. For $k \neq 1$, the curvature $\hat{\Omega}$ of line bundle $\hat{\mathcal{L}} = (\text{Det } \bar{\partial})^{\otimes k}$ satisfies $\hat{\Omega} = k\omega$, where $\hat{\omega} = k\omega$. The quantum Hilbert space is then given by the space of holomorphic sections $\Gamma(\mathcal{L})$ of $\mathcal{L} \rightarrow \mathcal{M}_{n,r}$. The compactness of $\mathcal{M}_{n,r}$ implies that the Hilbert space $\mathcal{H} = \Gamma(\mathcal{L})$ is finite dimensional.

We will now briefly consider how the Hilbert space $\mathcal{H}$ depends on the complex structure σ on Σ . A choice of complex structure σ induces a Kähler structure on $\mathcal{M}_{n,r}$ and let $\mathcal{L}_\sigma$ be the determinant line bundle $\text{Det } \bar{\partial}$ corresponding to this choice of σ . The Hilbert space of the quantum theory is $\mathcal{H}_{\Sigma,\sigma} = \Gamma(\mathcal{L}_\sigma)$. Now let $\mathcal{T}_\Sigma$ be the Teichmüller space of Σ . Associated

to a point $\sigma \in \mathcal{T}_\Sigma$ we have a finite-dimensional vector space $\mathcal{H}_{\Sigma,\sigma}$. As σ varies over $\mathcal{T}_\Sigma$ the vector spaces $\mathcal{H}_{\Sigma,\sigma}$ vary holomorphically to form a holomorphic vector bundle $\mathcal{V} \rightarrow \mathcal{T}_\Sigma$. The quantization of the theory should be independent of the choice of complex structure $\sigma \in \mathcal{T}_\Sigma$. This implies that for any two complex structures $\sigma_1, \sigma_2 \in \mathcal{T}_\Sigma$ the corresponding Hilbert spaces $\mathcal{H}_{\Sigma,\sigma_1}$ and $\mathcal{H}_{\Sigma,\sigma_2}$ must be projectively isomorphic. Such a projective identification of the fibres of the vector bundle $\mathcal{V} \rightarrow \mathcal{T}_\Sigma$ is equivalent to the existence of a canonical projectively flat connection on $\mathcal{V}$. Hence for any $U(n)$ -bundle over Σ of degree r we obtain a natural projectively flat holomorphic vector bundle over $\mathcal{V} \rightarrow \mathcal{T}_\Sigma$; these projectively flat vector bundles are the essential ingredient of the geometrical approach to conformal field theory of [19]. The quantization of Chern-Simons theory for a $U(n)$ -bundle of degree r yields, once a complex structure σ has been chosen, a Hilbert space $\mathcal{H}_{\Sigma,\sigma} = \Gamma(\mathcal{L}_\sigma)$ and the association $\Sigma_\sigma \rightarrow \mathcal{H}_{\Sigma,\sigma}$ is a modular functor in the sense of [20]. It is shown in [20] that any modular functor $\Sigma_\sigma \rightarrow \mathcal{H}_{\Sigma,\sigma}$ yields a projectively flat holomorphic vector bundle over $\mathcal{T}_\Sigma$. For the modular functor defined by the quantization of Chern-Simons theory the associated projectively flat vector bundle over $\mathcal{T}_\Sigma$ is just the bundle $\mathcal{V} \rightarrow \mathcal{T}_\Sigma$ defined above.

4. Representations of the Loop Group of $U(n)$

We have seen in the last section that for any $U(n)$ -bundle the quantization of Chern-Simons theory yields a projectively flat vector bundle $\mathcal{V} \rightarrow \mathcal{T}_\Sigma$, and this is the fundamental geometric object associated to conformal field theory on Σ [19,20]. The conformal field theory associated to Chern-Simons theory with a simple simply-connected gauge group G is known to be the current algebra of G (i.e., the representation theory of the loop group of G) [1-3]. We will show here that Chern-Simons theory with gauge group $U(n)$ is related in the same way to the representation theory of the loop group of $U(n)$. For related discussions see [2] and [10].

The phase space of $U(n)$ Chern-Simons theory is (once a complex structure σ has been fixed on Σ) the space $\mathcal{M}_{n,r}$ of holomorphic vector bundles of rank n and degree r over the Riemann surface Σ . Let $x \in \Sigma$ be a distinguished point, let $D \subset \Sigma$ be the unit disc around x and let $\hat{\Sigma} = \Sigma \setminus D$ be the complement of D . Consider a holomorphic vector bundle $V \rightarrow \Sigma$; V is clearly trivial, topologically and holomorphically, over D . V is also topologically trivial over $\hat{\Sigma}$ and as $\hat{\Sigma}$

is a Stein manifold it follows that V is also holomorphically trivial over Σ . The bundle $V \rightarrow \Sigma$ is therefore completely determined by the patching map $\phi : S^1 \rightarrow \Sigma \cap D \rightarrow U(n)_{\mathbb{C}}$, where $U(n)_{\mathbb{C}} = GL(n, \mathbb{C})$ is the complexification of $U(n)$. Thus, V is determined by an element of the loop group $LU(n)_{\mathbb{C}}$. The loop group of $U(n)$ or $U(n)_{\mathbb{C}}$ is not a connected space. To see this first consider an arbitrary Lie group G . The free loop group LG is topologically $LG = \Omega G \times G$, where ΩG is the subgroup of LG consisting of those loops $\phi : S^1 \rightarrow G$ such that $\phi(1) = 1$. Thus the homotopy groups of LG are given by $\pi_i(LG) = \pi_i(\Omega G) \oplus \pi_i(G)$. It is well known [7] that $\pi_i(\Omega G) = \pi_{i+1}(G)$, which implies that $\pi_i(LG) = \pi_i(G) \oplus \pi_{i+1}(G)$. Thus for any connected Lie group G , the group of components of LG is given by $\pi_0(LG) = \pi_1(G)$. If G is simply connected then LG is connected. In the case $G = U(n)$ or $U(n)_{\mathbb{C}}$, $\pi_0(LG) = \pi_1(G) = \mathbb{Z}$; therefore the components of LG are labelled by a single integer.

Consider a $U(n)$ -bundle $V \rightarrow \Sigma$. Such a bundle is classified topologically by a map $f : \Sigma \rightarrow BU(n)$, and the set of isomorphism classes of $U(n)$ -bundles over Σ is the set of $[\Sigma; BU(n)]$ of homotopy classes of maps from Σ to $BU(n)$. The space $BU(n)$ is approximated homotopically up to dimension two by the space $K(\mathbb{Z}, 2)$ and hence $[\Sigma; BU(n)] \simeq [\Sigma; K(\mathbb{Z}, 2)] \simeq H^2(\Sigma; \mathbb{Z}) \simeq \mathbb{Z}$. The isomorphism $[\Sigma; BU(n)] \simeq \mathbb{Z}$ is given by the map $(\Sigma, V) \rightarrow \int_{\Sigma} c_1(V)$ which assigns to a bundle $V \rightarrow \Sigma$ its degree $r = \int_{\Sigma} c_1(V)$. We also know, however, that a $U(n)$ -bundle $V \rightarrow \Sigma$ is determined by the patching map $\phi : S^1 \rightarrow U(n)$. Thus the set of isomorphism classes of $U(n)$ -bundles over Σ is given by $[S^1; U(n)] = \pi_1(U(n)) = \mathbb{Z}$, where the isomorphism $[S^1; U(n)] \simeq \mathbb{Z}$ is given by assigning to a map $\phi : S^1 \rightarrow U(n)$ its degree $\deg \phi$. It is now clear that these two ways of classifying $U(n)$ -bundles over Σ coincide and that $\deg \phi = \int_{\Sigma} c_1(V)$. For the Abelian case of $U(1)$ -bundles over the 2-sphere S^2 this result is well known from the theory of the Dirac monopole.

The components of the loop group $LU(n)$ are labelled by an integer corresponding to the isomorphism $\pi_0(LU(n)) \simeq \mathbb{Z}$. We will denote by $(LU(n))_r$, ($r \in \mathbb{Z}$) the component of the loop group $LU(n)$ consisting of maps $\phi : S^1 \rightarrow U(n)$ with $\deg \phi = r$. Exactly the same arguments apply to $LU(n)_{\mathbb{C}}$ and the component of this space consisting of loops of degree r will be denoted by $(LU(n)_{\mathbb{C}})_r$. It now follows from the above remarks that a holomorphic vector bundle $V \rightarrow \Sigma$ of rank n and degree r may be obtained by gluing together the trivial bundles over D and $\bar{\Sigma}$ by an element $\phi \in (LU(n)_{\mathbb{C}})_r$. If we let $(L_D U(n)_{\mathbb{C}})_r$ and $(L_{\bar{D}} U(n)_{\mathbb{C}})_r$ denote the subgroups of

$(LU(n)_{\mathbb{C}})_r$, which extend to holomorphic maps $D \rightarrow U(n)_{\mathbb{C}}$ and $\bar{\Sigma} \rightarrow U(n)_{\mathbb{C}}$, respectively, then it is clear that multiplying the map $\phi \in (LU(n)_{\mathbb{C}})_r$ on one side by an element of $(L_D U(n)_{\mathbb{C}})_r$ or on the other side by an element of $(L_{\bar{D}} U(n)_{\mathbb{C}})_r$ does not change the isomorphism class of the holomorphic structure determined by ϕ . Therefore, the moduli space of holomorphic vector bundles of rank n and degree r over Σ is given by the double coset space [21]

$$\mathcal{M}_{n,r} = (L_{\bar{D}} U(n)_{\mathbb{C}})_r \backslash (LU(n)_{\mathbb{C}})_r / (L_D U(n)_{\mathbb{C}})_r \quad (35)$$

There is a natural holomorphic line bundle over $\mathcal{M}_{n,r}$ determined in the following way. The Lie algebra of the loop group $LU(n)$ is the loop algebra $Lu(n)$ of smooth maps from S^1 into the Lie algebra $\mathfrak{u}(n)$ of $U(n)$. The Lie algebra $Lu(n)$ has a central extension $\widehat{Lu}(n)$

$$\mathbb{R} \rightarrow \widehat{Lu}(n) \rightarrow Lu(n). \quad (36)$$

As a vector space $\widehat{Lu}(n) = Lu(n) \oplus \mathbb{R}$ and the Lie bracket is defined by $\{(\xi, \lambda), (\eta, \mu)\} = ([\xi, \eta], \omega(\xi, \eta))$, for $\xi, \eta \in Lu(n)$ and $\lambda, \mu \in \mathbb{R}$, where $\omega : Lu(n) \times Lu(n) \rightarrow \mathbb{R}$ is given by

$$\omega(\xi, \eta) = \frac{1}{2\pi} \int_0^{2\pi} \text{Tr}(\xi(\theta)\eta'(\theta)) d\theta \quad (37)$$

It is a basic fact in the theory of loop groups [21] that there is a canonical central extension $\widehat{LU}(n)$ of $LU(n)$ by the circle group S^1 ,

$$S^1 \rightarrow \widehat{LU}(n) \rightarrow LU(n) \quad (38)$$

such that the Lie algebra of $\widehat{LU}(n)$ is $\widehat{Lu}(n)$. The group $LU(n)_{\mathbb{C}}$ is the complexification of $LU(n)$; the extension $\widehat{LU}(n)$ has a complexification $\widehat{LU}(n)_{\mathbb{C}}$ which is an extension of $LU(n)_{\mathbb{C}}$ by $\mathbb{C}^*$ [21]

$$\mathbb{C}^* \rightarrow \widehat{LU}(n)_{\mathbb{C}} \rightarrow LU(n)_{\mathbb{C}} \quad (39)$$

The loop group $LU(n)_{\mathbb{C}}$ consists of the disjoint union of the components $(LU(n)_{\mathbb{C}})_r$, for $r \in \mathbb{Z}$. We will denote the corresponding component of $\widehat{LU}(n)_{\mathbb{C}}$ by $(\widehat{LU}(n)_{\mathbb{C}})_r$. The double quotient

space

$$\mathcal{P}_{n,r} = (L_2 U(n)_C)_r \setminus (\widehat{LU}(n)_C)_r / (L_D U(n)_C)_r \quad (40)$$

defines a principal $C^\times$ -bundle $\mathcal{P}_{n,r} \rightarrow \mathcal{M}_{n,r}$. Associated to $\mathcal{P}_{n,r}$ by the action of $C^\times$ on C we have a holomorphic line bundle $\mathcal{L} \rightarrow \mathcal{M}_{n,r}$, where

$$\mathcal{L} = \mathcal{P}_{n,r} \times_{C^\times} C \quad (41)$$

This holomorphic line bundle may be identified with the determinant line bundle $\mathcal{L} = \text{Det } \tilde{\partial}$ which was of fundamental importance in the quantization of Chern-Simons theory in Section 3. The space of holomorphic sections of the bundle $\mathcal{L} \rightarrow \mathcal{M}_{n,r}$ defined by (41) is related to the representation theory of loop groups: the space of holomorphic sections of $\mathcal{L}$ is naturally isomorphic to the subspace of the basic representation of $(LU(n)_C)_r$ which is invariant under $(L_2 U(n)_C)_r$ [21]. For $k \neq 1$, we consider the holomorphic line bundle $\mathcal{L}^{\otimes k} \rightarrow \mathcal{M}_{n,r}$. The space of holomorphic sections of $\mathcal{L}^{\otimes k}$ is isomorphic to the subspace of the level k representation of $(LU(n)_C)_r$ which is invariant under $(L_2 U(n)_C)_r$. Hence the quantum Hilbert space of Chern-Simons theory with a $U(n)$ gauge bundle of degree r is directly related to the representation theory of the component $(LU(n)_C)_r$ of the loop group $LU(n)_C$. This result generalizes the familiar relation between Chern-Simons theory for a simply-connected gauge group G and the representation theory of the loop group of G , to the non-simply-connected group $U(n)$.

5. The WZW Model for $U(n)$

We have seen in the last two sections that Chern-Simons theory for a $U(n)$ -bundle of degree r may be quantized geometrically and the resulting theory is closely related to the representation theory of the component $(LU(n)_r)$ of $LU(n)$ of degree r . For a simple simply-connected group G , Chern-Simons theory with gauge group G is related to the current algebra of G (i.e., the representation theory of LG) which may be realized physically by the WZW model [22]. It is natural therefore to ask how the representation theory of $LU(n)$ which arises in connection with $U(n)$ Chern-Simons theory is realized by the $U(n)$ WZW model.

To define the WZW model with group $U(n)$ on a surface Σ , let $g : \Sigma \rightarrow U(n)$ be map from Σ into $U(n)$ and consider the action (defined for simple groups in [22])

$$S_{WZW} = \frac{1}{4\lambda^2} \int_{\Sigma} d^2x \text{Tr}(\partial_\mu g \partial^\mu g^{-1}) + \Gamma \quad (42)$$

where Γ is the Wess-Zumino functional. Γ is defined (see [22]) by extending the map $g : \Sigma \rightarrow U(n)$ to a map $\hat{g} : B \rightarrow U(n)$, where B is an oriented coboundary of Σ (i.e., $\partial B = \Sigma$), and setting

$$\Gamma = \frac{\kappa}{24\pi} \int_B d^3y \epsilon^{\alpha\beta\gamma} \text{Tr}(\hat{g}^{-1} \partial_\alpha \hat{g} \hat{g}^{-1} \partial_\beta \hat{g} \hat{g}^{-1} \partial_\gamma \hat{g}) \quad (43)$$

Such an extension $\hat{g}$ of g always exists because (by (13)) $\Omega_2^{SO}(U(n)) = 0$. Γ is only well-defined modulo 2π and thus κ must be an integer. We expect, in analogy with the case of simple groups, that for suitable values of λ the quantum field theory defined by the action S_{WZW} will be related to $U(n)$ Chern-Simons theory and the representation theory of $LU(n)$.

We note first that in the path integral formulation of Chern-Simons quantum field theory we have to choose a $U(n)$ -bundle V over the three-manifold M . Such a $U(n)$ -bundle over M is classified by an element of $[M, BU(n)] \simeq H^2(M; \mathbb{Z})$. Thus the quantum theory decomposes into distinct topological sectors labelled by the elements of $H^2(M; \mathbb{Z})$. In the canonical formulation $M = \mathbb{R} \times \Sigma$ and an element of $H^2(M; \mathbb{Z}) \simeq H^2(\Sigma; \mathbb{Z}) \simeq \mathbb{Z}$ corresponds to the degree of the $U(n)$ -bundle over Σ . In the $U(n)$ WZW model there is no $U(n)$ -bundle, so how do these topological sectors arise? In the path integral approach the quantum field theory is defined by

$$Z = \int_{\mathcal{C}} \mathcal{D}g e^{iS_{WZW}} \quad (44)$$

where $\mathcal{C} = \text{Map}(\Sigma; U(n))$ is the space of fields. The group of components of $\mathcal{C}$ is $\pi_0(\mathcal{C}) \simeq [\Sigma; U(n)] \simeq H^1(\Sigma; \mathbb{Z})$. For a compact surface Σ of genus g , $H^1(\Sigma; \mathbb{Z}) \simeq \mathbb{Z}^{2g}$. Thus the configuration space $\mathcal{C}$ is disconnected and the quantum theory decomposes into distinct topological sectors labelled by the elements of $H^1(\Sigma; \mathbb{Z}) \simeq \mathbb{Z}^{2g}$. These sectors labelled by $H^1(\Sigma; \mathbb{Z})$ in the $U(n)$ WZW model are the exact analogue of the sectors labelled by $H^2(M; \mathbb{Z})$ in the $U(n)$

Chern-Simons theory. In the canonical formulation of the WZW model we take $\Sigma = \mathbb{R} \times S^1$ and $\pi_0(\mathcal{C}) \simeq [S^1; U(n)] \simeq \pi_1(U(n)) \simeq \mathbb{Z}$. The components of $\mathcal{C}$ are labelled by the degree of the map $S^1 \rightarrow U(n)$. In the canonical formulation there is, therefore, an exact correspondence between the topological sectors of the $U(n)$ Chern-Simons theory and the $U(n)$ WZW model. We note that for a simply connected Lie group G , there are no topological sectors in the Chern-Simons theory because any G -bundle over a three-manifold is trivial; similarly there are no topological sectors in the WZW model because the space $\mathcal{C} = \text{Map}(\Sigma; G)$ is connected.

Let us now consider the geometric quantization of the $U(n)$ WZW model. We will first consider the quantization of the WZW model with an arbitrary compact group G ; this includes the special case of a simple group [22].

The WZW action (42) is first order in time derivatives and is thus in Hamiltonian form. Taking into account the chiral invariance of S_{WZW} it follows, therefore, that the phase space $\mathcal{X}$ of the WZW model is

$$\mathcal{X} = LG/G \quad (45)$$

(In fact the full phase space of the WZW model is $\mathcal{X} \times \mathcal{X}$ corresponding to left- and right-moving modes; however, for our purposes it suffices to consider a single copy $\mathcal{X}$.) The free loop group LG is the semi-direct product of the based loop group ΩG and the group G (where G acts on ΩG by conjugation.) Thus the homogeneous space LG/G may be identified with ΩG . The phase space $\mathcal{X}$ of the WZW theory may, therefore, be identified with ΩG . If Lg denotes the Lie algebra of LG (g is the Lie algebra of G) then there is a skew form ω on Lg defined by (cf. Eq. (37))

$$\omega(\xi, \eta) = \frac{1}{2\pi} \int_0^{2\pi} \text{Tr}(\xi(\theta)\eta(\theta))d\theta \quad (46)$$

where $\xi, \eta \in Lg$ and Tr denotes an invariant inner product on g . This defines an invariant closed two-form ω on $\Omega G = LG/G$ (see [21], for example) and thus defines a natural symplectic structure on ΩG . The Poisson bracket determined by the symplectic form ω on the phase space $\mathcal{X} = \Omega G$ is the natural Poisson bracket of the WZW model. To quantize the WZW model geometrically it is necessary, as we have seen in the context of Chern-Simons theory,

for the phase space $\mathcal{X}$ to have a Kähler structure. Let $LG_{\mathbb{C}}$ denote the complexification of LG and let $L^+G_{\mathbb{C}}$ denote the subgroup of $LG_{\mathbb{C}}$ consisting of loops ϕ which are the boundary values of holomorphic maps $\phi : \{z \in \mathbb{C} : |z| < 1\} \rightarrow G_{\mathbb{C}}$. It can be shown that ΩG may be identified with $LG_{\mathbb{C}}/L^+G_{\mathbb{C}}$ and that this latter space is a complex manifold [21]. The complex structure on $\Omega G = LG_{\mathbb{C}}/L^+G_{\mathbb{C}}$ is compatible with the symplectic structure defined by ω , and ΩG acquires therefore the structure of a Kähler manifold [21]. The standard methods of geometric quantization may now be used to quantize the phase space $\mathcal{X} = \Omega G$. There exists a holomorphic line bundle $\mathcal{L} \rightarrow \mathcal{X}$ with a connection whose curvature is equal to ω and the quantum Hilbert space $\mathcal{H}$ is the space $\Gamma(\mathcal{L})$ of holomorphic sections of $\mathcal{L}$. It is known from the representation theory of loop groups [21] that the space $\mathcal{H} = \Gamma(\mathcal{L})$ defines the fundamental representation of LG . The phase space $\mathcal{X} = \Omega G$ may or may not be connected, depending on the group G . The set of components of $\mathcal{X}$ is $\pi_0(\mathcal{X}) = \pi_0(\Omega G) = \pi_1(G)$. If G is simply-connected then $\mathcal{X}$ is connected. For the case $G = U(n)$, $\pi_0(\mathcal{X}) = \mathbb{Z}$ and the Hilbert space $\mathcal{H}$ decomposes into distinct sectors $\mathcal{H}_r$ ($r \in \mathbb{Z}$), such that $\mathcal{H}_r$ furnishes an irreducible representation of the component $(LU(n))_r$ of $LU(n)$.

For $G = U(n)$ there is a slightly different way of looking at the quantization of the WZW model which also relates it to the representation theory of $LU(n)$. The phase space $\mathcal{X}$ that we wish to quantize is $\Omega U(n)$. If $H^{(n)} = L^2(S^1; \mathbb{C}^n)$ denotes the Hilbert space of maps from S^1 to $\mathbb{C}^n$ and $\text{Gr}(H^{(n)})$ denotes the corresponding infinite Grassmannian [21], then it is known that $\Omega U(n)$ may be identified with the closed subset $\text{Gr}^{(n)}$ of $\text{Gr}(H^{(n)})$ which consists of subspaces W such that $zW \subset W$, where z is a scalar-valued function (see [21] for further details). The Grassmannian $\text{Gr}^{(n)}$ is a Kähler manifold [21]. There exists over $\text{Gr}^{(n)}$ a holomorphic line bundle Det : the fibre of Det at $W \in \text{Gr}^{(n)}$ is the renormalized top exterior power of W [21]. Thus Det is the determinant line bundle over $\text{Gr}^{(n)}$. The dual bundle $\text{Det}^* \rightarrow \text{Gr}^{(n)}$ of Det is isomorphic to the holomorphic line bundle $\mathcal{L} \rightarrow \mathcal{X}$ which featured in the geometric quantization of the WZW model. Hence the quantum Hilbert space $\mathcal{H}$ of the $U(n)$ WZW model is

$$\mathcal{H} = \Gamma(\text{Det}^*) \quad (47)$$

It is well known that $\Gamma(\text{Det}^*)$ yields the basic representation of $LU(n)$ [21]. The Hilbert space $\mathcal{H}$ decomposes into distinct sectors labelled by the integer corresponding to $\pi_0(\Omega U(n)) \simeq \mathbb{Z}$.

The r^{th} sector corresponds to the basic representation of $(LU(n))_r$, which was obtained in Section 4 from the quantization of Chern-Simons theory with a $U(n)$ gauge bundle of degree r .

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