CERN-TH.5718/90

FINITE NUMBER OF STATES IN UNITARY $d < 1$ STRINGS

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ABSTRACT

By proving a little theorem on Virasoro modules, and assuming that Liouville primaries with *given* conformal weight have finite multiplicity, we show that unitary minimal models coupled to $2d$ gravity have a finite number of physical states corresponding to their non-null primary fields.

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Minimal unitary models coupled to $2d$ gravity have attracted some attention recently as simple examples of string theories. It is believed that these theories have a finite number of physical states, corresponding to the appropriately dressed non-null primary fields of the unitary model. This has been shown by Horvath et al.^[1] and by Itoh^[2] in the light-cone gauge, where the $2d$ metric is described by an $SL(2, R)$ chiral current algebra. Here, by proving a little theorem on Virasoro modules, we will demonstrate this in the conformal gauge. In this gauge the statement has been discussed by Douglas and Shenker.^[3] All we will do is to readapt an argument used by Goddard and Thorn^[4] in their old no-ghost theorem for strings in $d \leq 26$ integer space-time dimensions.

Our basic assumption is that the Liouville theory can be viewed as a conformal model whose states are classified by a Virasoro algebra, and furthermore that the corresponding highest weight vectors of any given conformal weight $h(g)$ have finite multiplicity*. Since non-null physical states are, as we will show, both Liouville- and "matter"-primaries, and since the minimal (matter) model has a finite number of primaries, this demonstrates the desired result.

Let $L^{(m)}$ and $L^{(g)}$ stand for the matter- and gravity- (or Liouville-) Virasoro generators, and let $c^{(m)}, c^{(g)}$ be the corresponding central charges, which satisfy

$$c \equiv c^{(m)} + c^{(g)} = 26. \quad (1)$$

Physical states obey the conditions

$$L_n |\psi\rangle = 0 \quad n > 0 \quad (2)$$

$$(L_0 - 1) |\psi\rangle = 0 \quad (3)$$

with $L \equiv L^{(m)} + L^{(g)}$, i.e. they are total Virasoro primaries, with total conformal weight $h = 1$. We now state the theorem.

* If one represents the Liouville mode as a free field with a background charge, this multiplicity is generically two.

Theorem All solutions of eqs.(2) and (3) are of the form :

$$|\psi\rangle = |\text{matter primary}, h^{(m)}\rangle \otimes |\text{Liouville primary}, h^{(g)}\rangle + |\text{null}\rangle,$$

where $h^{(m)} + h^{(g)} = 1$, and $|\text{null}\rangle$ is orthogonal to all states obeying eqs.(2) and (3), including itself.

Note : We have here stated the theorem in a particular context, but as will become obvious from the proof it holds, in fact, more generally for any decomposition of L into two commuting Virasoro algebras whose central charges add to 26.

Proof A general state is a linear combination of terms, obtained by acting with the $L^{(m)}$ and $L^{(g)}$ raising operators on some highest-weight vector of both algebras, i.e.

$$|\psi\rangle = \prod L_{-n_i}^{(g)} \prod L_{-m_j}^{(m)} |h^{(g)}\rangle \otimes |h^{(m)}\rangle. \quad (4)$$

In order to satisfy condition (3), we must have $h^{(g)} + h^{(m)} = 1 - N$ for some non-negative integer N , and consider states at level $\sum n_i + \sum m_j = N$. Replacing $L^{(g)}$ by $L - L^{(m)}$, and commuting all L 's to the left in expression (4), we may uniquely decompose $|\psi\rangle$ as

$$|\psi\rangle = |\phi\rangle + |s\rangle, \quad (5)$$

where $|\phi\rangle \in \mathcal{P}^1$ is constructed only out of "matter" operators: $|\phi\rangle = \prod L_{-i_k}^{(m)} |h^{(g)}\rangle \otimes |h^{(m)}\rangle$, while the "spurious" state $|s\rangle \in \mathcal{S}^1$ is a linear combination of terms of the form $L_{-n} |h^{(g)}\rangle \otimes |h^{(m)}\rangle$ for some positive n . Note that the superscript in $\mathcal{P}, \mathcal{S}$ and χ denotes the total conformal weight of the corresponding states.

Now the trick of Goddard and Thorn is to construct the operators: $\tilde{L}_1 \equiv L_1$ and $\tilde{L}_2 \equiv L_2 + \frac{3}{2}L_1 L_1$, which together with their Hermitian conjugates generate the entire L -algebra. These operators have the property of preserving the decomposition (5), when acting on states

of conformal weight one, i.e. in an obvious notation

$$\tilde{L}_1|\phi\rangle \in \mathcal{P}^0, \quad \tilde{L}_2|\phi\rangle \in \mathcal{P}^{-1} \quad (6)$$

and

$$\tilde{L}_1|s\rangle \in S^0, \quad \tilde{L}_2|s\rangle \in S^{-1}. \quad (7)$$

The first statement eq.(6) is a trivial consequence of the fact that L commuted with $L^{(m)}$ gives $L^{(m)}$. To prove the other statement one writes $|s\rangle = \tilde{L}_{-1}|\chi^0\rangle + \tilde{L}_{-2}|\chi^{-1}\rangle$, and uses the fact that $[\tilde{L}_1, \tilde{L}_{-1}] = 2L_0$ and $[\tilde{L}_2, \tilde{L}_{-1}] = 6L_1L_0$ vanish when acting on a state of conformal weight zero, as does $[\tilde{L}_2, \tilde{L}_{-2}] = (\frac{5}{2} + 31L_0 + 18L_0^2) + 18L_{-1}L_1(1 + L_0)$ when acting on a state of conformal weight minus one, provided $c = 26$ which is here the case.

We are now almost done. Indeed, since $\tilde{L}_1$ and $\tilde{L}_2$ generate all lowering L -operators, we may write conditions (2) as $\tilde{L}_1|\psi\rangle = \tilde{L}_2|\psi\rangle = 0$. But from eqs.(6) and (7) and the fact that states in $\mathcal{P}^h$ and S^h are linearly independent, we conclude that these conditions must separately hold for $|\phi\rangle$ and $|s\rangle$. Thus $|\phi\rangle$, which by construction is a Liouville-primary, is also a primary of the matter-conformal model, while $|s\rangle$ is both a primary and a descendant of total Virasoro, and hence null. Q.E.D.

The proof is easily extended to the Neveu-Schwarz sector of the $N = 1$ superstring, where the rôle of $\tilde{L}_1$ and $\tilde{L}_2$ is played by the operators $G_{1/2}$ and $G_{3/2} + 2L_1G_{1/2}$.^[4]

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