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THE CLOSED STRING GERVAIS-NEVEU LIOUVILLE THEORY

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ABSTRACT

We extend the Gervais-Neveu quantization scheme of the Liouville theory to the closed string case, for both weak and strong coupling regions. The consequences of locality and modular invariance are analyzed, which lead to quantum modifications of the boundary conditions. The critical dimensions and spectra found by Gervais and Neveu for open strings are then seen to apply also in the closed string case.

1. Introduction

The formulation of string theories outside their critical dimensions [1] necessitates the proper understanding of two-dimensional gravity coupled to matter. In a sequence of papers [2-8], Gervais and Neveu (GN) proposed an elegant quantization scheme for two-dimensional gravity in the conformal gauge, the bosonic Liouville theory (later generalized to the supersymmetric case by Arvis and Babelon [9], [10]). Using an inverse scattering approach, they were able to find a free field representation of the theory such that the Hilbert space becomes just that of a free boson in the presence of a background charge. The inverse scattering method shows [11] that the natural set of operators obeying a closed algebra is given by the truncation of the theory to a minimal conformal field theory (MCFT), involving only fusions of four basic vertex operators. In this way, GN constructed the inverse square root of the metric $e^{-\varphi/2}$ as a quantum field for the open string. However, the construction applies only to the weak coupling regime with space time dimension $D \leq 1$, $D \geq 25$.*

More recently, the work of Polyakov [12] and KPZ [13] showed that one can also solve the weak coupling gravity theory by quantizing it in a light-cone gauge. The remarkable property of the light-cone action is the appearance of an $SL(2, \mathbb{C})$ current algebra. The scaling dimensions of the conformal fields obtained by these authors were reproduced in the conformal gauge by David [14] and Distler, Kawai [15], assuming that the renormalized coefficient of the cosmological term vanishes. Then the Liouville action itself becomes that of a free boson with a background charge; in particular, the energy-momentum tensor has the same form as the one of GN in terms of the free field. Nevertheless, correlation functions of the metric do by no means agree, the theories are not equivalent.

To obtain non-critical strings in the more interesting strong coupling region $1 < D < 25$, GN [6] proposed to consider a different local operator composed from the basic vertex operators in such a way that the conformal dimension is real. The locality requirement leads them to a set of new critical dimensions (7,

* $D \geq 25$ is the region of negative Planck's constant which has little meaning a priori. However, the GN quantization scheme can formally be applied for $\hbar < 0$ as well, if one is willing to accept negative norm states. Reasons for considering this case are indicated in the last section.

13, 19 in the bosonic case) together with a discrete tachyon-free spectrum on which this operator acts consistently.

In the present paper, we will provide a more systematic discussion of locality, addressing in particular the question of uniqueness of the critical dimensions and spectra found by GN. In addition, we generalize the GN approach to include closed strings. In this context the treatment of the boundary conditions in the quantum theory will turn out to be crucial. We discuss its relevance both for locality and modular invariance and show in particular that a quantum modification of the boundary conditions is needed to avoid a continuous spectrum for the closed string.

The paper is organized as follows. We start out with a brief review of the GN approach in section two. The conformal structure of the theory is discussed in section three. We point out that in contrast to conventional MCFTs, highest weight states corresponding to Kac's zeros are not allowed, a fact that was not recognized properly before. Section four starts with a construction of $e^{-\varphi/2}$ for the closed string. We show that in a discrete set of dimensions there exists another local operator in the weak coupling regime for both open and closed string cases. In the sequel we study locality in the strong coupling region, again for both open and closed boundary conditions. Finally, we analyze the consequences of modular invariance. The last section contains a brief discussion of the results and an outlook on open questions.

2. Liouville action and mode separation

Our starting point will be the well-known Liouville action in the Euclidean formulation [16],

$$S_L = \frac{1}{16\pi\hbar} \int d^2\xi \sqrt{g} \left\{ \frac{1}{2} g^{ab} \partial_a \partial_b \varphi + Q \hat{R} \varphi + \mu^2 e^\varphi \right\} \quad (2.1)$$

where $\hat{g}_{ab}$ is a reference metric such that $g_{ab} = e^\varphi \hat{g}_{ab}$. Following Distler and Kawai we have introduced the background charge Q and the effective Planck's constant $\hbar$ as free parameters in order to account for the transition from a measure $\mathcal{D}_g \varphi$ regularized in the full metric to the (flat) measure $\mathcal{D}_g \varphi$. In contrast to their

analysis, however, we will be interested in the case of nonvanishing cosmological constant. Classically, the background charge equals one, and — as will be pointed out briefly below — we can keep this assignment also in the quantum theory. (We remark that the value for Q found by DK, which differs from one, really amounts only to the convention that the quantum operator corresponding to the metric should be precisely equal to e^φ (in the notation of (2.1)) and not $e^{\beta\varphi}$: with some $\beta(\hbar)$ obeying $\lim_{\hbar \rightarrow 0} \beta(\hbar) = 1$).[†] The Planck's constant, on the other hand, is fixed by the requirement that the total conformal anomaly vanishes. In order to quantize the interacting system (2.1), it is necessary to first transform to a complete set of action-angle-variables. In the GN theory, this is achieved by use of the inverse scattering method. It exhibits very clearly the conformal structure of the theory and reveals a deep relation between its integrability and conformal properties [11]. The result of the inverse scattering analysis can essentially be summarized in the simple relation [2]

$$\begin{aligned} T_{++}(z) &:= \frac{Q}{16\hbar z^2} + \frac{1}{8\hbar} \left\{ -\frac{1}{2}(\partial_z \varphi)^2 + Q\partial_z^2 \varphi \right\} = \frac{1}{2\hbar}(P^2 + iQ\partial_z P) \\ &= \frac{1}{2\hbar}(\tilde{P}^2 + iQ\partial_z \tilde{P}) \end{aligned} \quad (2.2)$$

with $z := e^{r+i\sigma}$

and similarly for $T_{--}(z)$ in the closed string case. Here, P and $\tilde{P}$ denote an equivalent set of periodic free fields related by a complicated canonical transformation [3]. The connection between the zero modes, however, is simply $\tilde{p}_0 = -p_0$. As for the conventional string — though in a more intricate way — the open string boundary conditions [2] couple left and right movers, and one ends up in this case with an analytic energy-momentum tensor T_{++} only. To clarify its role for the conformal structure, we have kept a general background charge Q here, even though we shall pick the GN convention $Q = 1$ later.

The Poisson brackets of P and $\tilde{P}$ follow from the classical Virasoro algebra, in terms of the Fourier modes $p_n, \tilde{p}_n$ they read [2]

$$\{p_n, p_m\} = -i\hbar\delta_{n,-m} = \{\tilde{p}_n, \tilde{p}_m\}. \quad (2.3)$$

The quantization of this system of harmonic oscillators is now performed in the

[†] In the GN convention $Q = 1$, the metric is given (for $\mu^2 = 0$) by $e^{\beta-\varphi}$: (see (3.9) for η_-).

obvious way:

$$i\{p_n, p_m\} \longrightarrow [p_n, p_m] = n\hbar\delta_{n,-m}. \quad (2.4)$$

3. Conformal structure

Introducing canonical normal orderings for the p_n resp. $\tilde{p}_n$ modes, we can write the Virasoro generators as^{*}

$$\begin{aligned} L_n &= \frac{1}{2\hbar} \left(\sum_r : p_r p_{n-r} : - i n Q p_n \right) \\ &= \frac{1}{2\hbar} \left(\sum_r : \tilde{p}_r \tilde{p}_{n-r} : - i n Q \tilde{p}_n \right), \quad (n \neq 0) \\ L_0 &= \frac{1}{2\hbar} \left(\sum_r : p_r p_{-r} : + \frac{Q^2}{4} \right) \\ &= \frac{1}{2\hbar} \left(\sum_r : \tilde{p}_r \tilde{p}_{-r} : + \frac{Q^2}{4} \right). \end{aligned} \quad (3.1)$$

(The normal ordering constant in L_0 is chosen such that the SL_2 -subalgebra is anomaly-free.) The corresponding algebra has central charge

$$c_L = 1 + 3 \frac{Q^2}{\hbar}, \quad (3.2)$$

so that the requirement $c_L = 26 - D$ leads (for $Q = 1$) to

$$\hbar = \frac{3}{25 - D}. \quad (3.3)$$

We also record for later use that $L_n^\dagger = L_{-n}$ implies

$$p_0^\dagger = \pm p_0. \quad (3.4)$$

^{*} We write down only the formulae for the open string resp. the left-moving part of the closed string.

The free fields P and $\tilde{P}$ give rise to the basic vertex operators

$$\begin{aligned}\Psi_1^\eta(z) &= z^{-\delta} e^{\delta_0 \eta} : e^{i\eta \sum_{n \neq 0} \frac{\tilde{p}_n}{n} z^{-n}} : , \\ \Psi_2^\eta(z) &= z^{-\delta} e^{\delta_0 \eta} : e^{i\eta \sum_{n \neq 0} \frac{\tilde{p}_n}{n} z^{-n}} : ,\end{aligned}\quad (3.5)$$

where $q_0, \tilde{q}_0$ are the zero modes conjugate to $p_0, \tilde{p}_0$:

$$[p_0, q_0] = -i\hbar, \quad [p_n, q_0] = 0 \quad (n \neq 0). \quad (3.6)$$

(Note that $\tilde{p}_0 = -p_0$ but $\tilde{q}_0 \neq -q_0$). Their weights are given by

$$\delta_1 = \delta_2 = -\frac{1}{2}(Q\eta + \hbar\eta^2) \quad (3.7)$$

and the zero-mode shifts by

$$(\Delta p_0)_1 = -(\Delta p_0)_2 = -i\hbar\eta. \quad (3.8)$$

Notice that arbitrary correlators $\langle p_0 | \Psi_1^\eta(z_1) \dots \Psi_2^\eta(z_k) | p_0 \rangle$ only depend on the combinations $p_0/Q, \eta_j \cdot Q, \hbar/Q^2$ so it is really without loss of generality to put $Q = 1$.

The "tachyon vertex operators" Ψ_1^η, Ψ_2^η appear naturally in the inverse scattering approach as components of the transport matrix, with $\eta = \eta_\pm$ a solution of $(Q = 1)$

$$2\hbar\eta_\pm^2 - \eta_\pm + 1 = 0. \quad (3.9)$$

(3.9) characterizes Ψ_1^η, Ψ_2^η as degenerate primary fields of dimension $\delta_{(1,2)}$ ($\eta = \eta_-$) resp. $\delta_{(2,1)}$ ($\eta = \eta_+$) in the usual notation for the Kac's zeros:

$$\delta_{(m_+, m_-)} = -\frac{1}{2}[(m_+ - 1)\eta_+ + (m_- - 1)\eta_- + (m_+ - 1)\eta_-]^2 \quad (3.10)$$

with $m_+ \cdot m_- > 0$. In fact, (2.2) can be rewritten in terms of $\Psi_i^{\eta_\pm}$ as [4]

$$\begin{aligned}z^2 \frac{d^2 \Psi_i^\eta}{dz^2} + 3\delta z \frac{d\Psi_i^\eta}{dz} + \left(\delta^2 + \frac{1}{36}(2\delta + 1)^2 \right) \Psi_i^\eta = \\ \frac{2}{3}(2\delta + 1) \left[\sum_{n>0} L_{-n} z^n \Psi_i^\eta + \frac{1}{2} \tilde{L}_0 \Psi_i^\eta + \frac{1}{2} \Psi_i^\eta \tilde{L}_0 + \Psi_i^\eta \sum_{n>0} L_n z^{-n} \right]\end{aligned}\quad (3.11)$$

where

$$\begin{aligned}\delta &= \delta_{(1,2)} \text{ resp. } \delta_{(2,1)}, \quad i = 1, 2 \\ \tilde{L}_0 &:= L_0 - \frac{Q^2}{8\hbar}.\end{aligned}$$

(3.11) is the characteristic differential equation for the two simplest nontrivial degenerate fields. On the other hand, it can also be viewed as a second order formulation for the auxiliary linear system in the inverse scattering approach [11]. The algebra of $\Psi_1^{\eta_\pm}$ and $\Psi_2^{\eta_\pm}$,

$$\begin{aligned}\Psi_i^{\eta_\pm}(z) \Psi_j^{\eta_\pm}(z') &= S^{\pm jk}_{ii} \left(p_0, e^{\left(\frac{z}{z'} \right)} \right) \Psi_k^{\eta_\pm}, \\ e^{\left(\frac{z}{z'} \right)} &:= \text{sgn} \arg \left(\frac{z}{z'} \right) \quad (\text{for } \arg \left(\frac{z}{z'} \right) \in (-\pi, \pi))\end{aligned}\quad (3.12)$$

is then seen to be given by the quantum R matrix. It was determined by GN in [5] together with the corresponding Yang-Baxter-equation by applying (3.11) to the ground state expectation values $\langle p_0 | \Psi_1^{\eta_\pm}(z) \Psi_2^{\eta_\pm}(z') | p_0 \rangle$ resp $1 \leftrightarrow 2$ and performing a monodromy analysis for the solutions. Arbitrary degenerate fields are built up by fusions of the $\Psi_1^{\eta_\pm}, \Psi_2^{\eta_\pm}$ [4]. Note that $\Psi_1^{\eta_\pm}, \Psi_2^{\eta_\pm}$ are related by the formula

$$B(\eta_\pm, 2i\eta_\pm p_0) \Psi_2^{\eta_\pm}(\tau, \sigma) = i \Psi_1^{\eta_\pm}(\tau, \sigma) \int_{-\infty}^{\sigma} \Psi_1^{-2\eta_\pm}(\tau, \sigma') d\sigma' \quad (3.13)$$

where $\Psi_{1,2}^\eta(\tau, \sigma) := z^\delta \Psi_{1,2}^\eta(z)$ and $B(x, y)$ is the Beta function. (2.17) has immediate meaning in a certain range of p_0 -values ($\text{Re}(-2i\hbar\eta_\pm^2 - 2\eta_\pm p_0) > 0$ for matrix elements between states of finite level) and is defined outside this region by analytic continuation.*

The integral in (3.13) is just a screening operator, and in fact the GN Liouville theory provides a convenient framework to rederive the results of Fateev and Dotsenko [17] on the correlators of MCFTs. However, it is important to realize that the GN theory, based on the existence of the free fields P and $\tilde{P}$, must not

* Alternatively, one can use the formula

$$B(\eta_\pm, 2i\eta_\pm p_0) \Psi_2^{\eta_\pm}(\tau, \sigma) = i \Psi_1^{\eta_\pm}(\tau, \sigma) \int_0^{\sigma} \Psi_1^{-2\eta_\pm}(\tau, \sigma') d\sigma' \quad (3.14)$$

which is defined for all p_0 .

be considered as an SL_2 -invariant MCFT itself. The absence of an SL_2 -invariant vacuum, which was noted already by D'Hoker and Jackiw [18] for the Liouville field theory, is seen immediately by considering the condition

$$L_{-1}|0\rangle = 0. \quad (3.14)$$

Use of the P -representation for L_{-1} leads to

$$p_0 = \frac{iQ}{2} \quad (3.15)$$

while the $\tilde{P}$ -representation would predict

$$\tilde{p}_0 = \frac{iQ}{2} \quad (3.16)$$

in obvious disagreement with (3.15) since $\tilde{p}_0 = -p_0$. The contradiction results from the breakdown of the canonical transformation between P and $\tilde{P}$ for certain values of the zero mode. In fact, the (quantum) series expansion e.g. for $\tilde{P}$ in terms of P [3] has manifest poles at

$$2p_0 = imQ, \quad m \in \mathbb{Z} \setminus \{0\}. \quad (3.17)$$

One easily shows that classically, for these values of p_0 , the Liouville field φ cannot be real. Moreover, the derivation of the classical Poisson brackets for P and $\tilde{P}$ from those of T_{++} breaks down because δP is not of first order in δT_{++} for $\delta T_{++} \rightarrow 0$. Now (3.17) is just a special case of Kac's formula, expressed in terms of the zero mode ($\delta = \frac{1}{2\hbar}(p_0^2 + \frac{Q}{2})$),

$$p_0 = i\hbar(m_+ \eta_+ + m_- \eta_-), \quad m_+ m_- \geq 1. \quad (3.18)$$

Note also that (3.18) reduces to (3.17) in the classical limit, and that $m_+ \neq m_-$ is only allowed for $D < 1$ due to (3.4). One easily convinces oneself by studying null states on these zero modes that the same kind of singularity arises for all Kac's zeros. Working in the P -representation, one finds that the null states with $\text{Im } p_0 < 0$ vanish identically, whereas those with $\text{Im } p_0 > 0$ have nonvanishing

matrix elements in oscillator space, and conversely for the $\tilde{P}$ -representation. This implies in particular that the hermiticity of the Virasoro generators cannot be maintained upon the Kac's zeros. In addition to being physically unacceptable, this is fatal in the GN approach because the relation $L_n^+ = L_{-n}$ is used to define the scalar product in oscillator space ($p_n^+ = p_{-n}$ for p_0 real, $p_n^+ = \tilde{p}_{-n}$ for p_0 purely imaginary, as in (3.18)). The above arguments should convince the reader that fundamental inconsistencies occur when one tries to include the points (3.18) in the Hilbert space. On the other hand, this should seem rather strange because in a standard BPZ [19]-type CFT, the highest weight states are determined by the conformal dimensions of the primary fields, which are given precisely by the Kac's zeros in the minimal GN theory. The point is, of course, that the BPZ framework assumes SL_2 -invariance. Further insight is gained by comparing the GN theory with the bc-ghost system, for which the $U(1)$ -stress-energy-tensor has the same form. One realizes that the problem can be avoided if one gives up the simultaneous existence of P and $\tilde{P}$ (or $\Psi_1^{\eta_+}$ and $\Psi_2^{\eta_-}$) and works with the modified zero mode scalar product

$$(|p_0\rangle, |p'_0\rangle) := \delta_{p_0 p'_0} \quad (3.19a)$$

instead of

$$(|p_0\rangle, |p'_0\rangle) := \delta_{p_0 p'_0}. \quad (3.19b)$$

However, this is problematic in the Liouville theory because both $\Psi_1^{\eta_+}$ and $\Psi_2^{\eta_-}$ enter into the fundamental local field operators like $e^{-\varphi/2}$ (see below). Actually, it turns out that for $\eta = \eta_{\pm}$, the limits of correlators $\langle p'_0 | \Psi_1^{\eta_{\pm}}(z_1) \dots \Psi_n^{\eta_{\pm}}(z_n) | p_0 \rangle$ to the points (3.18) can be defined in general, and therefore no contradiction to the results of Fateev and Dotsenko occurs. Therefore, one could formally consider the minimal GN theory (with no extension possible!) on the highest weights (3.18). However, this is meaningless for the description of the Liouville system, because, as we have seen by considering the classical limit, the points (3.18) do not correspond to allowed configurations of the Liouville field. Fortunately, we will see below that in the construction of local field operators, the highest weight states of the theory become determined in such a way that no Kac's zeros occur among them.

4. Local operators

4.1 CONSTRUCTION OF $e^{-\varphi/2}$ FOR WEAK COUPLING

Classically, the inverse scattering approach leads to a very simple relation between $e^{-\varphi/2}$ and the classical limits^{*} $\Psi_{1,2}$, $\bar{\Psi}_{1,2}$ of $\Psi_{1,2}^{\eta-}$ resp. their right-moving counterparts $\bar{\Psi}_{1,2}^{\eta-}$:

$$e^{-\varphi/2} = \frac{i}{2\sqrt{2}} \left(\left(-\frac{1}{2p_0} \right) \Psi_1 \bar{\Psi}_2 - \left(-\frac{1}{2\bar{p}_0} \right) \bar{\Psi}_1 \Psi_2 \right). \quad (4.1)$$

(In the zero mode factors on the RHS we recognize the classical limits of the normalization factors of (3.13) resp. its right-moving counterpart). It turns out that on the quantum level, an equation of this form can be upheld (whereas a quantum generalization of the reconstruction formula for the field φ itself is not known). However, the p_0 - resp. $\bar{p}_0$ -dependence of the R -matrix leads to strong constraints on the spectrum of zero modes for $e^{-\varphi/2}$. For the open string case, where the boundary conditions couple left and right movers, one can express $\Psi_{1,2}$ in terms of $\bar{\Psi}_{1,2}$. This led GN to the ansatz [5]

$$i2\sqrt{2}e^{-\varphi/2} = a(p_0)\bar{\Psi}_1^-(z)\bar{\Psi}_2^-(\bar{z}) + b(p_0)\bar{\Psi}_2^-(z)\bar{\Psi}_1^-(\bar{z}) \\ + c(p_0)\bar{\Psi}_1^-(z)\bar{\Psi}_1^-(\bar{z}) + d(p_0)\bar{\Psi}_2^-(z)\bar{\Psi}_2^-(\bar{z}) \quad (4.2)$$

with $\bar{\Psi}_{1,2}^- := \bar{\Psi}_{1,2}^{\eta-}$. They showed that there exists a discrete spectrum such that for a suitable choice of the coefficients and the free parameter of the open string boundary conditions, $e^{-\varphi/2}$ acts consistently on the corresponding Hilbert space and does not couple to negative norm states with $p_0^2 + \frac{1}{4} < 0$ (for $D < 1$).

The corresponding ansatz for the closed string periodic boundary conditions is

$$i2\sqrt{2}e^{-\varphi/2} = a(p_0, \bar{p}_0)\bar{\Psi}_2^-(z)\bar{\Psi}_1^-(\bar{z}) - b(p_0, \bar{p}_0)\bar{\Psi}_1^-(z)\bar{\Psi}_2^-(\bar{z}) \quad (4.3)$$

We are using here (and in the sequel) a Fubini-Veneziano-representation [20] for the closed string operators, so that one formally assigns to the left and right

^{*} Note that $\lim_{\eta \rightarrow 0} \eta_- = 1$ whereas η_+ has no classical limit.

zero modes the same commutation relations as for the open string, and $[p_0, \bar{q}_0] = [\bar{p}_0, q_0] = 0$. (This simplifies the notation and is known to reproduce the correct amplitudes.) The classical analysis suggests that the boundary conditions should imply [21]

$$p_0 - \bar{p}_0 = \frac{im}{2}, \quad m \in \mathbb{Z} \setminus \{0\} \quad (4.4a)$$

for $p_0, \bar{p}_0$ purely imaginary,

respectively

$$p_0 = \bar{p}_0 \quad (4.4b)$$

for $p_0, \bar{p}_0$ real

However, upon imposing locality and (anti)periodicity of $e^{-\varphi/2}$ for imaginary zero modes one finds after straightforward calculations that it is not sensible to insist on (4.4a) for the quantum theory. Rather, one is led to replace (4.4a) by

$$p_0 - \bar{p}_0 = \frac{im}{2\eta_-}, \quad m = \pm 1 \quad (4.5)$$

The values $m = \pm 1$ are the only ones consistent with unitarity (see (4.7)) for arbitrary $\hbar < 1/8$.^{*} Then both locality and periodicity are guaranteed without any further restrictions to the spectrum, i.e. $p_0, \bar{p}_0$ are a priori continuous. The coefficients a and b are given by

$$a = N^{-1} \frac{\sin(-\eta_- \omega)}{\sin(\eta_- \omega + \eta_- \pi)} \\ b = \bar{N}^{-1} \frac{\sin(\eta_- \bar{\omega})}{\sin(-\eta_- \bar{\omega} + \eta_- \pi)} \quad (4.6)$$

with

$$\omega := 2\pi i p_0 \in \mathbb{R},$$

$$\bar{\omega} := 2\pi i \bar{p}_0 \in \mathbb{R},$$

The appearance of the normalization factors $N, \bar{N}$ of (3.13) resp. its right-moving counterpart comes as no surprise, and the trigonometric factors represent

^{*} $m = 0$ is excluded in (4.4a) and (4.5) because it leads to a complex φ at least in the classical limit.

the quantum corrections to (4.1) which are not accounted for by the quantization of the elementary building blocks themselves. Note that the solution (4.6) is unique only up to functions f_σ, f_δ with $f_\sigma(p_0, \bar{p}_0)f_\delta(p_0 - i\hbar\eta_-, \bar{p}_0 - i\hbar\eta_-) = 1$, which tend to 1 in the classical limit.

On the other hand, local behaviour cannot be achieved for real zero modes. Thus, p_0 and $\bar{p}_0$ are purely imaginary, as for the open string. However, for purely imaginary zero modes there is always a potential problem with negative norm states. These will occur outside the region

$$p_0^2 + \frac{1}{4} > 0, \quad \bar{p}_0^2 + \frac{1}{4} > 0. \quad (4.7)$$

Taking into account the physical state condition $(L_0 - \bar{L}_0)|\text{phys}\rangle = 0$ (which discretizes the spectrum!), one finds that there exists no restriction of $e^{-\varphi/2}$ † to the positive norm states. Our point of view on this problem is, however, that one should consider the construction of $e^{-\varphi/2}$ only as a mathematically well-defined starting-point, yielding a quantum theory with a smooth classical limit. The full set of physical consistency conditions should only be imposed in the range $D > 1$ relevant for the string application, that is, we do not insist on a probability interpretation in the unphysical region.

Before we turn to the strong coupling theory we wish to discuss another field operator for $D < 1$ which, however, is local only in a discrete set of dimensions. The open string operator is simply

$$\Phi = \Psi_1^\eta \Psi_2^\eta, \quad \eta = \eta_\pm, \quad (4.8)$$

the product being defined by point-splitting (fusion). It acts locally on the spectrum

$$z = -m - \frac{1}{2} + l, \quad 0 \leq l \leq 2m + 1, \quad (4.9)$$

provided the dimension D takes the values

$$D = 25 - \frac{6(m + \frac{3}{2})^2}{m + \frac{1}{2}}, \quad m \in \mathbb{N}_0 \quad (\eta = m + \frac{3}{2}). \quad (4.10)$$

Φ is periodic and hermitian and does not change the zero mode. Therefore, it

† More precisely, one needs to consider $(e^{-\varphi/2})^2$ on this spectrum.

can trivially be restricted to the spectrum (4.9), which respects unitarity, and is tachyon-free for $1 \leq l \leq 2m$. Φ behaves like a bosonized fermion,

$$\Phi(\tau, \sigma)\Phi(\tau, \sigma') = -\Phi(\tau, \sigma')\Phi(\tau, \sigma). \quad (4.11)$$

The mechanism through which locality occurs here is rather different from that for $e^{-\varphi/2}$ as well as that of section 4.2 and therefore deserves some comment.

The correlators

$$\begin{aligned} \varphi_1(\sigma, \sigma') &:= \langle p_0 | \Phi(\sigma) \Phi(\sigma') | p_0 \rangle, \\ \varphi_2(\sigma, \sigma') &:= \langle p_0 | \Psi_1^\eta \Psi_2^\eta(\sigma) \Psi_1^\eta \Psi_2^\eta(\sigma') | p_0 \rangle, \\ \varphi_3(\sigma, \sigma') &:= \langle p_0 | \Psi_2^\eta \Psi_2^\eta(\sigma) \Psi_1^\eta \Psi_1^\eta(\sigma') | p_0 \rangle \end{aligned} \quad (4.12)$$

(we set $\tau = 0 = \tau'$ here) form a system of “isobar” ($\delta = \delta_{(1,3)}$ resp $\delta_{(3,1)}$) for all operators involved) expectation values fulfilling the same 3rd order differential equation [8] with regular singular points $\frac{z}{2} = 0, 1, \infty$. The system transforms linearly into itself under the exchange $\sigma \longleftrightarrow \sigma'$ and so do the corresponding operator products [5] (i.e. (4.12) without expectation values). For certain “degenerate” values of the parameter p_0 , the leading orders of power series solutions around a singular point will differ by integers, in particular those of the φ_i , which can be understood as series solutions around $\frac{z}{2} = 0$. It is well known from the general theory of linear differential equations [22] that for the degenerate values of p_0 , two of the φ_i may develop singularities, with leading term proportional to the φ_k with highest exponent at $\frac{z}{2} = 0$. Clearly, this is a situation where we can hope that the exchange algebra

$$\varphi_1(\sigma', \sigma) = c_1 \varphi_1(\sigma, \sigma') + c_2 \varphi_2(\sigma, \sigma') + c_3 \varphi_3(\sigma, \sigma') \quad (4.13)$$

degenerates into

$$\varphi_1(\sigma', \sigma) = \tilde{c}_1 \varphi_1(\sigma, \sigma'). \quad (4.14)$$

$\tilde{c}_1, c_i$ are functions of p_0 and $\text{sgn}(\sigma - \sigma')$. A more detailed analysis shows that the integer exponent difference is a necessary condition for quadratic fusions of the $\Psi_{1,2}^\pm$ to commute with themselves at equal times (up to a factor). On the other hand one finds that (4.14) can be established only in the dimensions (4.10).

A trivial exception to this picture occurs at the dimensions

$$D = 25 - \frac{6m^2}{m-1} \quad (\eta = m), \quad m = 2, 3, \dots \quad (4.15)$$

where the elementary algebra of the $\Psi_{1,2}^\pm$ themselves degenerates, i.e. $S_{12}^{\pm 21} = 0 = S_{21}^{\pm 12}$. Then all fusions are local, with no restrictions on the spectrum.

The closed string extension of these results is straightforward. Again, one finds that for

$$\Phi_{\text{closed}}(z, \bar{z}) = \Phi(z) \otimes \Phi(\bar{z}) \quad (4.16)$$

the modified boundary conditions (4.5) should be used instead of (4.4).

4.2 LOCALITY IN THE STRONG COUPLING REGION

Even though $(e^{-\varphi/2})_{\text{open}}$ and $(e^{-\varphi/2})_{\text{closed}}$ cannot be maintained as consistent local operators for $1 < D < 25$, the vertex operators $\Psi_1^\pm, \Psi_2^\pm$ continue to exist (with complex dimensions). The natural idea is therefore to combine $\Psi_1^\pm, \Psi_2^\pm, \bar{\Psi}_1^\pm, \bar{\Psi}_2^\pm$ in a different way into a new local field operator Ψ with real dimensions. To retain the simple correspondence between open and closed theories familiar from conventional string theory, we will concentrate here on closed string operators which factorize in chiral parts:

$$\Psi_{\text{closed}}(z, \bar{z}) = \Psi(z) \otimes \Psi(\bar{z}). \quad (4.17)$$

For $\Psi(z)$ we admit an arbitrary quadratic form in $\Psi_1^\pm$. Since the mechanism discussed in the last section does not work for $D > 1$ where $\eta_\pm$ become complex, $\Psi(z)$ must be a linear combination of (at least) two quadratic products, allowing for mutual cancellation of unwanted terms arising in the commutation of $\Psi(z)$ with itself.* We have performed the locality analysis for the most general $\Psi(z)$ in this setting, and ended up with only two possibilities:

$$\Psi_1(z) = a(p_0)\Psi_1^\dagger\Psi_1^\dagger(z) + b(p_0)\Psi_2^\dagger\Psi_2^\dagger(z), \quad (4.18a)$$

$$\Psi_2(z) = a(p_0)\Psi_1^\dagger\Psi_2^\dagger(z) + b(p_0)\Psi_2^\dagger\Psi_1^\dagger(z) \quad (4.18b)$$

with $\eta = \eta_+, \bar{\eta} = \eta_-$ or vice versa.

* One can try to make use of such a cancellation also in the weak coupling region, but we find that no new possibilities arise in this way.

Both operators are local only for

$$D = 25 - 6m, \quad m = 1, 2, 3, 4. \quad (4.19)$$

$\Psi(z)$ is just the operator suggested by GN [6] and lives on the real spectrum

$$p_0 = \frac{n}{2}\sqrt{8h-1}, \quad n \in \mathbb{Z} \setminus \{0\} \quad (4.20)$$

which is tachyon-free and manifestly unitary.

Ψ_1 , on the other hand, requires purely imaginary zero modes:

$$p_0 = \frac{ik_0}{2m} + n\frac{i}{2}, \quad n \in \mathbb{Z}, \quad k_0 \in \{1, \dots, m-1\} \text{ fixed} \quad (4.21)$$

and we find that Ψ_1 must be discarded since it couples to negative norm states.

Thus we are led to consider

$$\Psi_{2,\text{closed}}(z, \bar{z}) = \Psi_2(z) \otimes \Psi_2(\bar{z}). \quad (4.22)$$

However, it is immediately seen that this operator does not exist if we work with the "classical" boundary conditions (4.4)[†], and the modification (4.5) is of no use since $p_0, \bar{p}_0$ are real here. But then we realize that (4.5) is just the special case $m_- = 0$ of the more general ansatz

$$p_0 - \bar{p}_0 = i\hbar(m_+\eta_+ + m_-\eta_-), \quad m_+, m_- \in \mathbb{Z} \quad (4.23)$$

where in the RHS we recognize Kac's formula, generalized to arbitrary m_+, m_- . In fact, (4.23) is the most general boundary condition for the operators (4.17) which is compatible with the zero mode shifts of the degenerate fields, and it reduces to (4.4) in the classical limit.* (3.4) requires for $1 < D < 25$ that

$$m_+ = \pm m_- \quad (4.24)$$

The choice $m_+ = -m_-$ applies to the spectrum of Ψ_2 and leads to a complete decoupling of right and left zero modes for the values (4.20). The operator

[†] (4.4) would allow for the ansatz $\Psi_{2,\text{closed}} = (p_0)\Psi_2^\dagger\Psi_2^\dagger(z) + (p_0)\Psi_2^\dagger\Psi_1^\dagger(z) + (p_0)\Psi_1^\dagger\Psi_2^\dagger(z)$ with $p_0 = \bar{p}_0$, but one realizes quickly that this operator cannot be local.

* Notice that (4.4) is also contained in (4.23) for $h \neq 0$ as the subset $m_+ = m_-$.

$\Psi_{2,\text{closed}}$ now exists and is periodic. Similarly, the choice $m_+ = +m_-$ applies to the spectrum of Ψ_1 . Again, $\Psi_{1,\text{closed}}$ exists as a periodic field, but is physically irrelevant because of the unitarity problem.

Formulae (4.23), (4.24) in connection with (4.20) confirm the observation of [23] that the GN Liouville theory is governed by the generalized Kac's formula (i.e. the RHS of (4.23)).

4.3 MODULAR INVARIANCE OF THE ONE-LOOP PARTITION FUNCTION

A non-trivial check for the spectra obtained by the locality analysis is provided by modular invariance. However, it will be illuminating to first analyze the modular behaviour of the partition function in general, without taking recourse to the results of the last section. The Liouville string partition function, after integrating over the X^μ zero modes and summing over X^μ and φ oscillators, can be written in the form

$$Z = Z_{D+1} \cdot Z_{\mathcal{L}}$$

with

$$\begin{aligned} Z_{D+1} &= (\text{Im } \tau)^{-(D-1)/2} |\theta_1'(0|\tau)|^{-(D-1)/3} \cdot \\ Z_{\mathcal{L}} &= \sqrt{\text{Im } \tau} \cdot \sum_{p_0, \tilde{p}_0} \int e^{2\pi i \tau \cdot \frac{p_0^2}{2} - 2\pi i \tau \cdot \frac{\tilde{p}_0^2}{2}} \end{aligned} \quad (4.25)$$

where $\sum \int$ denotes a Lebesgue-Stieltjes integral, accounting for the most general Liouville spectrum with both discrete and continuous parts. $Z_{\mathcal{L}}$ contains the Liouville ground state contribution, as is obvious from the formula (2.5) for $L_0, \bar{L}_0$. The Liouville oscillators and the total intercept $1 - \frac{Q^2}{8k}$ have been included in Z_{D+1} , which is just the partition function for a conventional string in $D+1$ dimensions - except for the critical factor $|e^{i\pi\tau \frac{D+1-26}{12}}|^2$, which was cancelled by the Liouville intercept contribution. Thus, Z_{D+1} is modular invariant in arbitrary dimensions, not just $D+1 = 26$. Hence, Z will be invariant iff $Z_{\mathcal{L}}$ is.

Assuming the boundary conditions (4.4) together with real zero modes, it is not hard to show by use of well-known results [24] on the Fourier - Laplace transformation of tempered distributions that the only consistent spectrum is the

continuous one ($p_0 = \tilde{p}_0 \in (-\infty, \infty)$). For $p_0, \tilde{p}_0$ purely imaginary ($p_0 - \tilde{p}_0 = \frac{i\pi}{2}$), the situation is even worse because $Z_{\mathcal{L}}$ diverges when $p_0, \tilde{p}_0$ are unrestricted. Imposing the unitarity conditions

$$|p_0| < \frac{1}{2}, \quad |\tilde{p}_0| < \frac{1}{2} \quad (4.26)$$

one finds that the truncated continuous spectrum is not modular invariant. Any discrete spectrum would have to contain infinitely many points, causing $Z_{\mathcal{L}}$ to diverge again. We have nevertheless analyzed this situation as well by performing a formal Wick rotation of p_0 and $\tilde{p}_0$ (while keeping (4.26)). The result is that modular invariance cannot be achieved.

Note that the above discussion applies to both weak and strong coupling regions. On the other hand, the situation drastically changes in the strong coupling region $1 < D < 25$ if we implement the modified boundary conditions (4.23), (4.24). In this case, we can construct the operator $\Psi_{2,\text{closed}}$ with decoupled discrete zero mode spectrum

$$p_{0L} = \frac{n}{2} \sqrt{8\hbar - 1}, \quad p_{0R} = \frac{m}{2} \sqrt{8\hbar - 1}, \quad n, m \in \mathbb{Z}. \quad (4.27)$$

As shown already by the authors of [25], who considered the spectrum (4.20) $\otimes$ (4.20) without derivation as a plausible ansatz for the closed Liouville string, the zero modes (4.27) are in perfect accord with modular invariance in the critical dimensions, whereas those of $\Psi_{1,\text{closed}}$ are not.

We remark finally that in the weak coupling regime $D < 1, D > 25$, making use of the modified boundary conditions, (4.5) does not improve the modular invariance properties of the partition function.

5. Conclusions and outlook

We have shown that the GN Liouville string has a direct closed string extension with the same critical dimensions and analogous spectra as in the open string case. In the strong coupling region, one can construct “composite” closed string operators whose existence enforces the introduction of certain quantum corrections to the classical boundary conditions for the zero modes. These are not produced automatically by the inverse scattering approach, but were seen to be strongly restricted by general consistency requirements. It is very satisfying that the a priori unrelated conditions of locality and modular invariance are fulfilled by the same discrete set of decoupled left and right moving zero modes. Thus, the modification of the closed string boundary conditions plays a similar role as the mechanism of compactification in conventional string theory. On the other hand, modular invariance seems to require a continuous real spectrum in the weak coupling region. Though inadmissible in the GN construction, this spectrum is indeed realized in other approaches [26] to the closed Liouville theory. For $D \geq 25$ a continuous spectrum is physically sensible if we interpret the Liouville field as an uncompactified dynamical time variable [27], [28]. Indeed, the $D = 25$ model (with zero cosmological constant) has been shown to correspond exactly to the bosonic string in $25 + 1$ dimensional Minkowski space [29], [30].

Of course, the Liouville theory still contains many unsolved problems. For instance, the precise relation between the light-cone gravity theory and the free resp. interacting Liouville theories is still not clear. It will be interesting to see if light can be shed on this question by the most recent random matrix approach pioneered by Brezin, Douglas, Gross, Kazakov, Migdal, Shenker [31], [32], [33] and others. A more specific question which arises from the discussion of section three is how the failure of SL_2 -invariance can be reconciled with the usual world sheet interpretation of string scattering. Also, we should mention the problem that the truncation of the GN theory to a MCFT, while both natural and essential for the manageability of the theory, eliminates the possibility of obtaining operators of positive dimension, like the metric field e^φ . It would be desirable to find non-degenerate operators compatible with the spectra determined by the

minimal theory. An important step forward in this direction has been taken in a recent paper by Gervais [34], where it is shown that the operator algebra of the degenerate fields can be extended to negative powers of $\Psi_1^{\frac{1}{2}}, \Psi_2^{\frac{1}{2}}$.

Finally, perhaps the most interesting question regards the nature of the “magical barrier” at $D = 1$ which appears in all known quantization schemes, although in rather different ways. It is widely believed that some kind of phase transition must take place here, but the nature of this transition remains to be investigated. One expects that other approaches will then be found to admit $D > 1$ extensions, too. The discrete, unitary spectrum of the GN theory suggests for the light-cone quantization scheme that the strong coupling region should be governed by an integrable compact Kac-Moody algebra.

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REFERENCES

- [1] A.M. Polyakov, Phys. Lett. 103B (1981) 207.
- [2] J.L. Gervais and A. Neveu, Nucl. Phys. B199 (1982) 59; Nucl. Phys. B209 (1982) 125.
- [3] J.L. Gervais and A. Neveu, Nucl. Phys. B224 (1983) 329.
- [4] J.L. Gervais and A. Neveu, Nucl. Phys. B257 (1985) 59.
- [5] J.L. Gervais and A. Neveu, Nucl. Phys. B238 (1984) 125.
- [6] J.L. Gervais and A. Neveu, Phys. Lett. 151B (1985) 271.
- [7] J.L. Gervais and A. Neveu, Nucl. Phys. B238 (1984) 396; Nucl. Phys. B264 (1986) 557; Nucl. Phys. B Proc. Suppl. (Netherlands) Vol. 5B (1988) 119.
- [8] J.L. Gervais and A. Neveu, Nucl. Phys. B264 (1986) 557.
- [9] J.F. Arvis, Nucl. Phys. B212 (1983) 151; Nucl. Phys. B218 (1983) 309.
- [10] O. Babelon, Phys. Lett. 141B (1984) 353; Nucl. Phys. B258 (1985) 680.
- [11] O. Babelon, Phys. Lett. B215 (1988) 523; "From Integrable to Conformal Field Theory", Preprint PAR LPTHE 89/29; J.L. Gervais, "The Quantum Group Structure of 2D Gravity and Minimal Models", preprint LPTENS 89/14; E. Cremmer and J.L. Gervais, "The Quantum Group Structure associated with Nonlinear Extended Virasoro Algebras", preprint LPTENS 89/19.
- [12] A.M. Polyakov, Mod. Phys. Lett. A2 (1987) 893.
- [13] V.G. Knizhnik, A.M. Polyakov and A.B. Zamolodchikov, Mod. Phys. Lett. A3 (1988) 819.
- [14] F. David, Mod. Phys. Lett. A3 (1988) 1651.
- [15] J. Distler and H. Kawai, Nucl. Phys. B321 (1989) 509; "Hausdorff Dimensions of Continuous Polyakov's Random Surfaces", Cornell Preprint CLNS88/854.
- [16] D. Friedan, Lectures presented at Les Houches 1982, in "Advances in Field Theory and Statistical Mechanics", North Holland 1984.

- [17] V.S. Dotsenko and V.A. Fateev, Nucl. Phys. B240 (1984) 312; Nucl. Phys. B251 (1985) 691.
- [18] E. D'Hoker and R. Jackiw, Phys. Rev. D26 (1982) 3517; Phys. Rev. Lett. 50 (1983) 1719; E. D'Hoker, D.Z. Freedman and R. Jackiw, Phys. Rev. D28 (1983) 2583.
- [19] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, Nucl. Phys. B241 (1984) 332.
- [20] S. Fubini and G. Veneziano, Nuovo Cimento 67 A (1970) 29.
- [21] L. Johansson and R. Marnelius, Nucl. Phys. B254 (1985) 201.
- [22] E.L. Ince, "Ordinary Differential Equations", Dover, New York 1956.
- [23] A. Bilal and J.L. Gervais, Nucl. Phys. B284 (1987) 397.
- [24] A. Guichardet, "Calcul intégral", Librairie Armand Colin, Paris 1969.
- [25] A. Bilal and J.L. Gervais, Phys. Lett. B187 (1987) 39.
- [26] T.L. Curtright and C.B. Thorn, Phys. Rev. Lett. 48 (1982) 1309; E. Braaten, T.L. Curtright and C.B. Thorn, Phys. Lett. B118 (1982) 115; Ann. Phys. (N.Y.) 147 (1983) 365; E. Braaten, T.L. Curtright, G. Ghan-dour and C.B. Thorn, Phys. Rev. Lett. 51 (1983) 19; Ann. Phys. (N.Y.) 153 (1984) 147; H.J. Otto and G. Weigt, Phys. Lett. 159B (1985) 341; Z. Phys. C 31 (1986) 219.
- [27] J. Polchinsky, Nucl. Phys. B324 (1989) 123.
- [28] I. Antoniadis, C. Bachas, J. Ellis and D. Nanopoulos, Phys. Lett. B211 (1988) 393; Nucl. Phys. B328 (1989) 117.
- [29] S.R. Das, S. Naik and S.R. Wadia, Mod. Phys. Lett. A4 (1989) 1033.
- [30] A. Dhar, T. Yagaraman, K.S. Narain and S.R. Wadia, "The Role of Quantized 2-dim. Gravity in String Theory", preprint TIFR/TH/89-55.
- [31] E. Brezin and V.A. Kazakov, Phys. Lett. B236 (1990) 144.
- [32] M.R. Douglas and S.H. Shenker, Nucl. Phys. B335 (1990) 635.
- [33] D. Gross and A. Migdal, Phys. Rev. Lett. 64 (1990) 127.

- [34] J.L. Gervais, “*Critical Dimensions for Noncritical Strings*”,
preprint LPTENS 90/4.