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KNOT OPERATORS IN CHERN-SIMONS GAUGE THEORY

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ABSTRACT

The operator formalism for Chern-Simons gauge theory with gauge group $SU(N)$ is presented. The connection with rational conformal field theory is shown explicitly by identifying a basis for the Hilbert space of the theory with the set of characters corresponding to a Wess-Zumino-Witten model for $SU(N)$. Knot operators are constructed performing the calculation of matrix elements of Wilson line operators on this Hilbert space. Using these operators a representation of the Verlinde operators in the context of Chern-Simons gauge theory is obtained. As an application of the use of these operators to knot theory, the Jones polynomial for toral knots is explicitly computed.

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1. INTRODUCTION

Conformal Field Theories (CFT) have been a subject of great interest during the last few years because of their relation with the study of critical phenomena and string theory. One of the main problems in CFT is its classification. Among these theories, a subclass of them, Rational Conformal Field Theories (RCFT) have been widely studied. RCFT are simple cases of the general set of theories since they contain only a finite number of primary fields. The classification of RCFT [1] is a very important open question and constitutes a first step towards the classification of general CFT. At the moment, there are several approaches towards this classification. There are algebraic approaches in which the connection between RCFT and quantum groups is exploited [2], or the modular properties of differential equations satisfied by correlation functions and characters are explored [3]. A geometrical approach to the classification of RCFT consists of exploiting the connection between RCFT and Chern-Simons (CS) gauge theories in three dimensions [4]. In this approach only preliminary work regarding the problem of classification has been done [5]. Most of the effort so far has concentrated on the quantization of CS theories to explicitly construct the connection with RCFT. In this paper we will address this issue for the case of a gauge group $SU(N)$ and we will prove, using insight based on CS theories, some of the properties of RCFT.

The connection between three-dimensional CS theories and RCFT has been analyzed from the point of view of canonical quantization [6,7,8,9,10,11], and of current algebra [5,9] as originally proposed in [4]. One of the most fruitful approaches in demonstrating the connection with RCFT has been shown to be the construction of an operator formalism [8] similar to the one in string theory [12]. This construction involves several steps. First, the three-dimensional manifold in which the CS theory is defined is cut via a Heegaard splitting [13] and then, for each resulting g -handlebody a state is defined via a suitable Feynman path integral. The corresponding functional depends on the data given at the boundary of the g -handlebody. The gauge symmetry present in the theory allows one to deter-

mine the set of possible functionals and therefore a Hilbert space is constructed. This Hilbert space is finite-dimensional in agreement with the topological nature of the CS action and is associated to a given boundary of a g -handlebody, *i.e.*, to a Riemann surface of genus g which is specified via modular parameters. The states of a basis of the Hilbert space have the same properties under modular transformations as the characters of RCFT. This leads to their identification.

The interesting observables in CS theory are gauge invariant operators which do not depend on the three-dimensional metric. By virtue of the topological nature of the CS action the vacuum expectation values of these operators are topological invariants. Operators which have these features are Wilson lines. In fact, as shown in [4], the vacuum expectation values of Wilson lines are related to Jones polynomials [14] for knots with variable $t = \exp(\frac{2\pi i}{k+c_v})$, where k is the CS parameter and c_v is the quadratic Casimir in the adjoint representation of the gauge group. The operator formalism described in [8] associates a quantum operator to each observable. The main goal of this work is to construct operators associated to knots by evaluating Wilson lines. Since we will restrict ourselves in this paper to operator formalisms for genus 0 and 1, we will be able to construct only total knot operators. These operators seem to be more intrinsic to knots than polynomials. Polynomials are vacuum expectation values of these operators and so they depend on the three-manifold where the knot is enclosed. Knot operators, however, are independent of the three-manifold. Furthermore, knot operators are easy to compute in the framework of the operator formalism and they have a rather compact form. Their vacuum expectation values, on the other hand, require a lengthy and tedious calculation (although straightforward). Of course, our formalism should be extended to higher genus to be able to obtain knot operators for any knot.

For some specific cases the quantum knot operators constructed in this work constitute a representation of the Verlinde operators [15]. Properties of these operators can be obtained using arguments based on the quantization of CS theory. This allows one to prove certain properties of RCFT using rather simple arguments. For example, the fusion rules and their diagonalization are simple consequences of

the properties of Wilson line operators. The representation of Verlinde operators associated to contractible cycles in the solid torus turns out to be the one anticipated in [15], *i.e.*, it is diagonal and the eigenvalues are Weyl characters for a specific value of the argument. The representation which is obtained for the other type of Verlinde operators, the ones associated to non-contractible cycles in the solid torus, is new. It turns out to correspond to shifts in the index of a character (or a state of the basis of the Hilbert space) by all the weights of the representation carried by the Wilson line. It is shown that the corresponding matrix elements are just the constants appearing in the fusion rules of a Wess-Zumino-Witten model with group $SU(N)$.

Knot operators allow one to compute expectation values rather simply since their matrix elements in the Hilbert space are explicitly obtained. As an example, we compute in sect. 3 the vacuum expectation value of any toral knot in the three-sphere. Taking the case of the fundamental representation of $SU(2)$ in the resulting expression we obtain the Jones polynomial for toral knots given in [14]. Other approaches which have proved to be very fruitful in calculating vacuum expectation values of Wilson lines and graphs can be found in [16,17,18,19,20].

In [8] the operator formalism was briefly formulated and applied to the cases in which the gauge groups are $U(1)$ and $SU(2)$. In this paper we give full account of the construction and we generalize it to the case in which the gauge group is $SU(N)$. It is organized as follows. In sect. 2 we construct the operator formalism for $SU(N)$. The section is divided into two subsections dealing with the cases of genus 0 and 1 respectively. Sect. 3 is devoted to the analysis of knot operators, their connection to Verlinde operators, and their vacuum expectation values. Finally, in sect. 4 we state our conclusions. An appendix deals with a summary of our conventions.

2. FORMULATION OF THE OPERATOR FORMALISM

The aim of the first part of this section is to give the general guidelines concerning the construction of the operator formalism for CS gauge theories. Two subsections follow describing how the construction is carried out in the two specific cases which are considered in this paper. Operator formalisms have shown to be a rather useful tool for quantizing string theories [12]. The fundamental advantage of this formalism is that it provides a natural framework to take into account global properties. This type of properties are of great importance when considering quantum field theories on manifolds with non-trivial topology or observables whose relevance is linked to their global form. This is the case for CS theories and, in general, for topological quantum field theories. CS theories reveal their richest flavor when considered on arbitrary three-manifolds. Furthermore, their observables, Wilson lines and graphs, are global objects. A treatment of these theories from the point of view of the operator formalism seems to be the most appropriate to exploit the behavior linked to global properties. In [8] a brief description of the formalism applied to the case of gauge groups $U(1)$ and $SU(2)$ was presented. In this paper we give the full details of the construction, with explicit proofs of the main facts observed in [8], and we extend the formalism to the case in which the gauge group is $SU(N)$.

Let us consider a compact oriented three-manifold without boundary M and a gauge group $SU(N)$, and let us take a trivial $SU(N)$ -bundle E with its corresponding one-form connection A . The action of the theory is given by the integral of the Chern-Simons form

$$S_{\text{CS}}(A_\mu) = \frac{1}{8\pi} \int_M \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A), \quad (2.1)$$

where “Tr” denotes the trace in the fundamental representation. We use a normalization in which $\text{Tr}(T^i T^j) = -\delta^{ij}$. This choice is different than the standard one [4] and therefore the normalization of S_{CS} differs from the one in [4] by a

factor two. The simplest way to check that (2.1) contains the right factor ($\frac{1}{8\pi}$) is the following. Since all three-manifolds of the type considered can be realized as the boundary of a four-manifold B , we can write the action of our theory as

$$S_{\text{CS}}(A_\mu) = \frac{1}{8\pi} \int_B \text{Tr}(F \wedge F), \quad (2.2)$$

where F is the field strength associated to the extension of the gauge connection A to the four-manifold B , which necessarily exists since E is trivial. The factor in (2.1) has been chosen in such a way that $\exp(iS_{\text{CS}})$ is independent of the extension. To verify this, notice that the difference between two distinct choices of four-dimensional extensions can be written as

$$\frac{1}{8\pi} \int_{\mathcal{M}} \text{Tr}(F \wedge F) = -2\pi \times \frac{1}{16\pi^2} \sum_{a=1}^{N^2-1} \int F^a \wedge F^a = 2\pi c_2(\mathcal{E}), \quad (2.3)$$

where $\mathcal{M}$ is a boundaryless four-dimensional manifold and $\mathcal{E}$ an $SU(N)$ bundle obtained after identification of the boundaries of the four-manifolds of the two extensions. $c_2(\mathcal{E})$ in (2.3) is the Chern class, which is an integer. Therefore, $\exp(iS_{\text{CS}})$ is independent of the extension.

We will consider CS gauge theories with parameter k in such a way that the partition function takes the form

$$Z_k(M) = \int_M [DA_\mu]_M e^{ikS_{\text{CS}}(A_\mu)}, \quad (2.4)$$

where $[DA_\mu]_M$ represents the Feynman path integral measure over gauge orbits of connections defined on M . It is well known that the action (2.1) has very special properties just for the value of the coefficient which has been chosen [21] when k is an integer. Under gauge transformations, the gauge connection A transforms as,

$$A \rightarrow h^{-1}Ah + h^{-1}dh, \quad (2.5)$$

where h is a smooth map $h : M \rightarrow SU(N)$. For maps h which are connected to the identity map, S_{CS} is gauge invariant. However, for maps h which are not

connected to the identity, S_{CS} changes by 2π times an integer [21], being therefore $\exp(ikS_{\text{CS}})$ invariant.

CS gauge theories are topological quantum field theories. This follows simply from the fact that the action (2.1) does not depend on the metric of the manifold in which the theory is defined. The natural observables in this class of theories correspond to operators which do not depend on the metric and are gauge invariant. These observables must lead to topological invariants. For CS gauge theories, operators which satisfy both properties are Wilson lines^{*}. In fact, it was shown in [4] that the observables corresponding to these operators are Jones polynomials [14] for knots and links. Wilson line operators are defined as

$$W_R(L) = \text{Tr}_R \left(P e^{\int_L A} \right), \quad (2.6)$$

where R labels an irreducible representation of $SU(N)$, P denotes a path-ordered product and L is a map, $L : S^1 \rightarrow M$, i.e., a loop in M . In general, one considers the vacuum expectation value of a product of these operators:

$$Z_k(M; R_1, L_1, \dots, R_n, L_n) = \int_M [DA_\mu]_M \left(\prod_{i=1}^n W_{R_i}(L_i) \right) e^{ikS_{\text{CS}}(A_\mu)}. \quad (2.7)$$

This quantity depends on the topological properties of the manifold M and the loops L_i , i.e., on whether or not the different loops L_i are knotted or linked. They are related to Jones polynomials [4].

Before starting with the construction of the operator formalism let us discuss briefly some aspects of three-manifolds of the type considered here. All compact oriented boundaryless three-dimensional manifolds admit a Heegaard splitting [13]. This means that we may cut M along a Riemann surface Σ of genus g in such a way that $M = M_1 \cup M_2$, M_1 and M_2 being homeomorphic to a g -handlebody.

* Graphs, as defined in [22], are also suitable operators for CS gauge theories. We will not consider those in this paper.

The joining of M_1 and M_2 to build M is performed by identifying their boundaries ∂M_1 and ∂M_2 via a homeomorphism. In general a manifold M admits different Heegaard splittings. In this work we will be able to carry out the full construction only for the case in which the manifold M can be built by joining two solid tori or two solid balls. The second case is rather trivial. Only the manifold S^3 admits a genus 0 Heegaard splitting, just constructing it by gluing two S^2 . The manifolds which admit genus 1 Heegaard splitting are called lens spaces [13] and are constructed by performing modular transformations on the surface of one of the solid tori before joining their surfaces. Let us consider some examples. If two solid tori are joined without performing any homeomorphism on the surface of one of them, one builds the three-manifold $S^2 \times S^1$. Notice that loops which were not contractible in each of the solid tori remain non-contractible after the gluing. This fact manifests itself in the presence of S^1 in $S^2 \times S^1$. On the other hand, if one performs a homeomorphism corresponding to a modular transformation of type S on the surface of one of the tori before gluing, no non-contractible loop before the joining remains non-contractible. Thus, one obtains in this way S^3 . We observe from these examples that S^3 admits at least two different Heegaard splittings.

Let us carry out the construction of the operator formalism. Assume that we have a manifold M with some Wilson lines L_i on it. Let us make a Heegaard splitting in such a way that no Wilson line is cut[†]. The aim of the formalism is to define the vacuum expectation value (2.7) as an inner product of states in a Hilbert space. These states are defined as functional integrals over configurations on each of the manifolds in the Heegaard splitting with some boundary data. Let us assume that the manifold M has split into M_1 and M_2 with as common boundary a Riemann surface Σ , as shown in Fig. 1. Since the action S_C is first order we need to specify only the value of the field at the boundary. On the other hand, out of the three components of A we have to specify only one at a given boundary. The reason for this is that while one of the components

[†] The case in which Wilson lines are cut in this framework has been studied in [23]. We will restrict ourselves in this paper to the case in which no Wilson line is cut in the splitting.

could be traded at least locally by a gauge symmetry, the other two must be shared by the two boundaries. It is convenient to use complex variables which lead naturally to a quantum holomorphic representation. The first use of this type of representation for CS gauge theories was carried out in [6]. Let us therefore define local coordinates on Σ as $z = \sigma_1 + i\sigma_2$ and $\bar{z} = \sigma_1 - i\sigma_2$, and components of the gauge connection parallel to the surface as $A_z^a = \frac{1}{2}(A_1^a - iA_2^a)$ and $A_{\bar{z}}^a = \frac{1}{2}(A_1^a + iA_2^a)$. We will define wave functionals which are functional integrals over field configurations in the g -handlebodies resulting after the splitting with the value of A_z fixed at the boundary. The inner product, after complex conjugation and reversal of orientation on one of the wave functionals will consist of an integration over both components A_z and $A_{\bar{z}}$ on the common boundary.

To gain insight into the kind of inner product to be introduced let us look for a moment at the canonical quantization of the theory in a local region of M which we approximate by $\Sigma \times R^1$. Choosing the gauge in which the component of A along R^1 vanishes, which is consistent with our choice of boundary data, one finds from (2.1) that A_z and $A_{\bar{z}}$ are canonically conjugate,

$$[A_z^a(\sigma), A_{\bar{z}}^b(\sigma')] = \frac{\pi}{k} \delta^{ab} \delta^{(2)}(\sigma - \sigma'), \quad (2.8)$$

and that the states of the theory must satisfy the Gauss law,

$$F_{z\bar{z}}^a = 0, \quad (2.9)$$

where $F_{z\bar{z}}^a$ is the field strength on Σ , $F_{z\bar{z}}^a = \partial_z A_{\bar{z}}^a - \partial_{\bar{z}} A_z^a + f_{abc} A_z^b A_{\bar{z}}^c$, f_{abc} being the structure constants of the gauge group. Equation (2.8) tell us that we should try to make the construction in such a way that A_z^a and $A_{\bar{z}}^a$ are canonically conjugate. We will see that this is possible except for the fact that the coefficient of the delta functions on the right-hand side of (2.8) acquires some quantum corrections. The Gauss law (2.9) must be satisfied by our wave functionals when a small gauge transformation is performed. However, for non-trivial manifolds the

theory possesses invariance under certain gauge transformations not connected to the identity. The operator formalism provides a very useful framework to obtain the global Gauss law associated to this type of gauge transformations.

We are now in a position to define the inner product. Remember that A_z and $A_{\bar{z}}$ are canonically conjugate and we would like to fulfil the condition $A_z^\dagger = A_{\bar{z}}^a$, which is the quantum analogue of the classical condition $\overline{A_z^a} = A_{\bar{z}}^a$. Before proceeding further let us consider a simple example in elementary quantum mechanics that will serve to illustrate our quantization method. Suppose that we have two operators $\mathcal{O}$ and $\mathcal{O}^\dagger$ which satisfy the commutation relation

$$[\mathcal{O}^\dagger, \mathcal{O}] = \alpha, \quad (2.10)$$

with α a constant c -number. Obviously, $\mathcal{O}$ and $\mathcal{O}^\dagger$ can be represented as operators acting on the space of antiholomorphic functions of a single complex variable z as follows:

$$\mathcal{O}^\dagger f(\bar{z}) = \bar{z} f(\bar{z}), \quad \mathcal{O} f(\bar{z}) = -\alpha \frac{d}{d\bar{z}} f(\bar{z}). \quad (2.11)$$

One may ask now what is the inner product which makes $\mathcal{O}^\dagger$ the adjoint of $\mathcal{O}$. It is an easy exercise to find that the answer is

$$(f_1, f_2) = \int dz d\bar{z} e^{\frac{z\bar{z}}{\alpha}} \overline{f_1(\bar{z})} f_2(z). \quad (2.12)$$

This representation for the operators is usually called antiholomorphic representation. Notice that the commutation relations can be read from the coefficient that appears in the exponential of the inner product measure. The extension of this example to our case is straightforward. The resulting inner product can be written as

$$(\Psi_2 | \Psi_1) = \int [DA_z DA_{\bar{z}}]_{\Sigma} e^{\frac{1}{\alpha} \int_{\Sigma} d^2\sigma \sqrt{\gamma} \gamma^{\bar{z}z} \text{Tr}(A_z A_{\bar{z}})} \overline{\Psi_2[A_{\bar{z}}]} \Psi_1[A_z], \quad (2.13)$$

which can be represented as in Fig. 2. Several comments regarding this expression are in order. The measure $[DA_z DA_{\bar{z}}]_{\Sigma}$ is some gauge invariant measure on field

configurations on Σ . We will discuss it in more detail below. Its form will turn out to be rather important in our construction. In writing the exponential factor we have been forced to make a choice of metric on Σ which we have denoted by γ . One would expect that since the theory is topological its observables are independent of this choice. This is indeed the case as it will be shown below. Notice that since the exponential factor is reparametrization and Weyl invariant the dependence on the metric appears only through its corresponding complex structure.

The wave functionals are defined as functional integrals over field configurations over g -handlebodies. The same type of orientation is assigned to all the g -handlebodies. When computing the inner product one complex-conjugates the wave functional and reverses the orientation of the g -handlebody. In this way the inner product can be written as a functional integral over field configurations on the entire three-manifold. The inner product (2.13) must lead to vacuum expectation values of Wilson line operators. Imposing this requirement one obtains the explicit form of the wave functional $\Psi[A_z]$. Since we are considering the case in which no Wilson lines are cut let us assume that out of the n Wilson lines n_1 of them lie on the g -handlebody M_1 , and the rest on the other, M_2 . We define the wave functional associated to the g -handlebody with n_1 Wilson lines, M_1 , as

$$\Psi_1[A_z] = \int [DA_{\mu}]_{M_1} \left(\prod_{i=1}^{n_1} W_{R_i}(L_i) \right) e^{i k S_{CS}(A_{\mu}) - \frac{1}{2\alpha} \int_{\Sigma} d^2\sigma \sqrt{\gamma} \gamma^{\bar{z}z} \text{Tr}(A_z A_{\bar{z}})}. \quad (2.14)$$

In this definition $[DA_{\mu}]_{M_1}$ represents the Feynman path integral measure over gauge orbits such that $A_{\bar{z}}$ is fixed at Σ . A similar expression defines the wave functional $\Psi_2[A_{\bar{z}}]$ for the g -handlebody M_2 . Notice that a boundary term accompanies the action S_{CS} which is minus one-half the one entering the inner product (2.13). This factor $-1/2$ must be there to cancel the exponential factor in (2.13) after adding the contributions from $\Psi_1[A_z]$ and $\Psi_2[A_{\bar{z}}]$ in such a way that the vacuum expectation value (2.7) can be written as,

$$Z_k(M; R_1, L_1, \dots, R_n, L_n) = (\Psi_2 | \Psi_1). \quad (2.15)$$

Boundary terms like the one in (2.14) are typical of operator formalisms [12]. Its presence is of great importance. As we will observe below, it is because of this term that the wave functional is well defined, *i.e.*, it is antiholomorphic (only depends on $A_{\bar{z}}$) in any gauge. Also one can verify that the functional integral in (2.14) has its extremal for gauge field configurations which satisfy $F = 0$ in the body of M_1 , *i.e.*, the associated classical equations are the same as for the CS theory when defined on the three-manifold M .

We would like to end this first part of the section with some remarks. The theory under consideration is topological, as follows from the fact that (2.1) is independent of the metric in M . However, in our construction we have been forced to choose a two-dimensional metric on Σ which enters in the boundary terms present in the inner product (2.13) and in the definition of the wave functionals (2.14). Actually, because of Weyl and reparametrization invariance the formula- tion depends only on the corresponding conformal structure. Since the theory is topological one would expect that the construction is independent of the choice of metric on Σ . Due to the conformal anomaly this turns out not to be entirely true. Nevertheless, the distinction between different choices shows up as projective factors in the construction of the Hilbert space [4]. Those depend only on the choice of scale and so they do not manifest themselves at this stage since the boundary terms in (2.13) and (2.14) are conformal invariant. They appear when carrying out the Feynman path integral in the inner product (2.13) and are linked to a conformal anomaly. In this work we will make a choice of scale in which this factor is such that states of the Hilbert space acquire their simplest form in respect to their relation to RCFT. The observables of the theory, however, are the same irrespectively of the choice made.

Another important remark concerns the measure in the functional integral (2.13) and in the wave functionals (2.14). One has to make a choice with the requirement that it must be gauge invariant. There is a rather natural way to construct this measure which is familiar from the study of Wess-Zumino-Witten models [24,25,26]. Its construction depends on the type of g -handlebody under

consideration. In this work we will consider the cases $g = 0$ and $g = 1$. We will discuss them in turn in the following subsections.

2.1 QUANTIZATION FOR A GENUS ZERO HANDLEBODY.

Let M_1 be a solid ball and $\Sigma = S^2$ its boundary. In this case the measure in (2.13) corresponds to two-dimensional gauge fields on the two-sphere. On S^2 the gauge fields can be parametrized as

$$A_{\bar{z}} = u^{-1} \partial_{\bar{z}} u, \quad A_z = \bar{u}^{-1} \partial_z \bar{u}, \quad (2.16)$$

where u is a single-valued map $u : S^2 \rightarrow SU(N)^{\mathbb{C}} (SU(N)^{\mathbb{C}}$ is the complexification of $SU(N))$. The gauge transformation (2.5), when particularized to the surface of the g -handlebody takes the form

$$A_{\bar{z}} \rightarrow g^{-1} A_{\bar{z}} g + g^{-1} \partial_{\bar{z}} g, \quad A_z \rightarrow g^{-1} A_z g + g^{-1} \partial_z g, \quad (2.17)$$

where g is a map $g : \Sigma \rightarrow SU(N)$. In the parametrization (2.16) they read as $u \rightarrow ug$. Notice that since $A_{\bar{z}}^{\dagger} = A_z$, we have $u^{\dagger} = \bar{u}^{-1}$.

We would like to express the measure $[DA_z DA_{\bar{z}}]_{\Sigma}$ in terms of the infinite product of de Haar measures of $SU(N)^{\mathbb{C}}$. This involves the computation of the corresponding Jacobian which can be written as a fermionic determinant [24,26]. The result takes the form

$$[DA_z DA_{\bar{z}}]_{\Sigma} = e^{c_v \Gamma(u \bar{u}^{-1})} |\det \partial_z \partial_{\bar{z}}| du d\bar{u} \quad (2.18)$$

where c_v is the quadratic Casimir in the adjoint representation (for $SU(N)$, we have $c_v = N$) and $\Gamma(\alpha)$ is the Wess-Zumino-Witten (WZW) action [25],

$$\Gamma(\alpha) = \frac{1}{2\pi} \int_{\Sigma \equiv \partial M_1} d^2 \sigma \sqrt{\gamma} \gamma^{z\bar{z}} \text{Tr}(\alpha^{-1} \partial_z \alpha \alpha^{-1} \partial_{\bar{z}} \alpha) + \frac{i}{12\pi} \int_{M_1} \epsilon^{\mu\nu\rho} \text{Tr}(\tilde{\alpha}^{-1} \partial_{\mu} \tilde{\alpha} \tilde{\alpha}^{-1} \partial_{\nu} \tilde{\alpha} \tilde{\alpha}^{-1} \partial_{\rho} \tilde{\alpha}), \quad (2.19)$$

α being a map $\alpha : \Sigma \rightarrow SU(N)$ and $\tilde{\alpha}$ one of the extensions of the map α to the

interior of the solid ball. Notice that the second term involves the extension of the map α to the whole ball. It is well known [25] that $\exp(\Gamma(\alpha))$ is independent of the extension. Therefore, as c_v is an integer, the measure $[DA_z DA_{\bar{z}}]_{\Sigma}$ is well defined independently of the extension of $u\bar{u}^{-1}$ into the interior of the solid ball. In (2.18) $dud\bar{u}$ is an infinite product of the Haar measures of $SU(N)^c$. Also notice that the measure (2.18) is gauge invariant.

We could have used other measures in (2.13) as long as they are gauge invariant. The one chosen is rather natural and its generalization for the case of the torus makes the connection with RCFT immediate. Let us now discuss the form of the measure in the functional integral entering into the definition of the wave functional (2.14). Since we want (2.15) to hold, the measure in the wave functional must be related at least for field configurations on Σ to the measure in the inner product (2.13) that we have chosen in (2.18). This creates a problem, since the product of this part of the measure in the wave functional (2.14) and in its conjugate orientation reversed one must be just (2.18) and there is no way to split (2.18) into two parts symmetrically. Indeed, the WZW action satisfies the Polyakov-Wiegmann (PW) condition [24]:

$$\Gamma(\alpha\beta) = \Gamma(\alpha) + \Gamma(\beta) + \langle \alpha, \beta \rangle, \quad (2.20)$$

where we have defined:

$$\langle \alpha, \beta \rangle = \frac{1}{\pi} \int_{\Sigma} d^2\sigma \sqrt{\gamma} \gamma^{z\bar{z}} \text{Tr}(\alpha^{-1} \partial_z \alpha \partial_{\bar{z}} \beta \beta^{-1}). \quad (2.21)$$

Equation (2.20) shows that it is not possible to factorize the measure (2.18) because of the presence of a term of the form $\exp(c_v \langle u, \bar{u}^{-1} \rangle)$. This term is linked to the two-dimensional non-Abelian chiral anomaly [24]. The most natural choice we can make for the measure on the boundaries of M_1 and M_2 is to use the one induced by the full measure on M , i.e., just the measure in (2.18) without those factors that only depend on the gauge variables which are not being integrated

over in the path integral representation of $\Psi_1[A_z]$ and $\Psi_2[A_{\bar{z}}]$. Assuming that the radial component of A on Σ has been gauged away, this amounts to choosing

$$\exp\{c_v(\Gamma(\bar{u}^{-1}) + \langle u, \bar{u}^{-1} \rangle)\} d\bar{u}. \quad (2.22)$$

for $\Psi_1[A_z]$, and $\exp\{c_v(\Gamma(u) + \langle u, \bar{u}^{-1} \rangle)\} du$ for $\overline{\Psi_2[A_{\bar{z}}]}$. Doing this we have introduced an extra factor $\exp(c_v \langle u, \bar{u}^{-1} \rangle)$ which we have to account for when comparing with (2.18) if we want (2.15) to hold. This forces us to redefine the factor present in the inner product (2.13) in the following way:

$$\begin{aligned} e^{\frac{k}{\pi} \int_{\Sigma} d^2\sigma \sqrt{\gamma} \gamma^{z\bar{z}} \text{Tr}(A_z A_{\bar{z}})} \rightarrow \\ e^{-c_v \langle u, \bar{u}^{-1} \rangle} e^{\frac{k}{\pi} \int_{\Sigma} d^2\sigma \sqrt{\gamma} \gamma^{z\bar{z}} \text{Tr}(A_z A_{\bar{z}})} = e^{\frac{k+c_v}{\pi} \int_{\Sigma} d^2\sigma \sqrt{\gamma} \gamma^{z\bar{z}} \text{Tr}(A_z A_{\bar{z}})}, \end{aligned} \quad (2.23)$$

and therefore the inner product (2.13) now reads

$$(\Psi_2|\Psi_1) = \int |\det \partial_z \partial_{\bar{z}}| dud\bar{u} e^{\frac{(k+c_v)}{\pi} \int_{\Sigma} d^2\sigma \sqrt{\gamma} \gamma^{z\bar{z}} \text{Tr}(A_z A_{\bar{z}})} \overline{\Psi_2[A_{\bar{z}}]} \Psi_1[A_z]. \quad (2.24)$$

As we announced, the choice of measure introduces a quantum correction of non-perturbative nature which manifests itself in the new form of the exponential factor of the inner product. Going backwards now in the logic of the holomorphic quantization we can read from (2.24) the form of the quantum-corrected commutation relations for the components of the gauge field parallel to the surface Σ :

$$[A_z^a(\sigma), A_{\bar{z}}^b(\sigma')] = \frac{\pi}{(k+c_v)} \delta^{ab} \delta^{(2)}(\sigma - \sigma'), \quad (2.25)$$

which provides the operator representation

$$A_z^a = -\frac{\pi}{(k+c_v)} \frac{\delta}{\delta A_{\bar{z}}^a}. \quad (2.26)$$

Notice that if we want A_z^a and $A_{\bar{z}}^a$ to be canonically conjugate also in the quantum theory (i.e., to have commutation relations of the type (2.8)) we are

only allowed to add a factor $\exp\left[\frac{\lambda}{\pi} \int_{\Sigma} d^2\sigma \sqrt{\gamma} \gamma^{zz} \text{Tr}(A_z A_z)\right]$ to the measure of the Hilbert space. This term would produce a shift $k \rightarrow k + \lambda$ in equation (2.8). However, in order to keep $(\Psi_2|\Psi_1)$ gauge invariant this fact has to be compensated by similar terms in the measure of the path integral that defines Ψ_1 and Ψ_2 .

We have not discussed the form of the measure in the wave functional (2.14) for field configurations in the interior of the ball. There, since one is integrating over all components of the gauge field one finds gauge invariant measures easily. The simplest one which converges smoothly to the one on the surface is the following. Take a gauge in which the radial component of A vanishes. Now regard the ball as a set of continuous spherical shells, as depicted in Fig. 3, and for each shell take the measure (2.18). Other choices, however, are in principle possible. The arguments which follow are independent of this choice as long as it is gauge invariant.

Our next task is to determine the form of the wave functional (2.14). To achieve this we will work out the behavior of the wave functional (2.14) under a gauge transformation of the form (2.17). Let us consider therefore a map $g : \Sigma \rightarrow SU(N)$ and one of its extension to the interior of the solid ball $\tilde{g}$. We will observe that the transformation of the wave functional is independent of the extension chosen. Taking (2.14) and (2.17) one obtains (from now on we will omit the subindex 1 in (2.14) when not necessary),

$$\Psi(A_z) \rightarrow \Psi(g^{-1}A_z g + g^{-1}\partial_z g) = e^{-(k+c_V)(\Gamma(g)+\frac{1}{\pi} \int_{\Sigma} d^2\sigma \sqrt{\gamma} \gamma^{zz} \text{Tr}(A_z \partial_z g g^{-1}))} \Psi(A_z) \quad (2.27)$$

where we have taken into account the variation of the factor $\exp\{c_V(\Gamma(\tilde{u}) + \langle u, \tilde{u}^{-1} \rangle)\}$ contained in the measure of $\Psi(A_z)$. The transformed wave functional is independent of the extension g chosen since all the dependence on $\tilde{g}$ is in $\Gamma(g)$ and its factor in the exponential is an integer. Notice also that the antiholomorphic character of the wave functional is preserved under gauge transformations. The presence of the boundary term in the definition (2.14) is essential to cancel all the dependence in the integrand on A_z after the transformation and so to be able to factor out the dependence on A_z . It is also crucial to have on the boundary of M_1

a path integral measure as in (2.22). This is also because the two terms in (2.22) must be combined precisely as in this equation in order to give a gauge variation that only depends on A_z . This requirement of gauge covariance in the holomorphic quantization of the operator formalism eliminates completely the ambiguity in the finite renormalization of k that was alluded to above.

Using the PW property (2.20) one finds that the solution to (2.27) has the form

$$\Psi_k(A_z) = \xi \exp\{-(k+c_V)\Gamma(u)\}, \quad (2.28)$$

where ξ is a constant (independent of u). The Hilbert space for the case of the solid ball is therefore undimensional. No matter which configuration of Wilson lines is inside the solid ball the corresponding state is proportional to the wave functional (2.28). All the content related to the configuration is in the constant ξ which we are not able to obtain with our methods. A more explicit computation of the integral functional in (2.14) would be necessary. From (2.28) we cannot compute observables in S^3 . However, there is another approach to computing observables which does not require us to go through the calculation of the functional integral [4]. This approach is based on surgery and the knowledge of the Hilbert space, up to constants, for general g -handlebodies. For example, the observables for toral knots and links of toral knots in lens spaces can be obtained from the Hilbert space that we will construct in the next subsection. In CS gauge theory it is possible to normalize to unity the wave functional only for one three-dimensional topology. Once one fixes the normalization for one topology the norm of the states for the other topologies is obtained by surgery. As will be discussed in the next subsection, the most natural choice is the one in which the partition function (no Wilson lines) for the three-manifold $S^2 \times S^1$ is 1.

The state (2.28) satisfies the Gauss law (2.9) as can be verified either directly or by considering the infinitesimal form of the gauge transformation (2.27). Taking into account (2.21) and g infinitesimal in (2.27) one finds,

$$F_{\bar{z}z}^a \Psi(A_z) = 0, \quad (2.29)$$

where $F_{\bar{z}z}^a = \partial_z A_z^a - \partial_z A_z^a + f_{abc} A_z^b A_z^c$ and the operator representation (2.26) has been used. Notice that the order between A_z^b and A_z^c chosen in the last term of $F_{\bar{z}z}^a$ is irrelevant since the structure constants f_{abc} are antisymmetric. Equation (2.27) constitutes therefore the finite form of the Gauss law (2.29). For S^2 this is the only constraint on the states since it has trivial topology and therefore there are only gauge transformations connected to the identity. As we will observe in the next subsection the non-trivial homology present in the torus provides a richer framework in which global forms of the Gauss law appear. The operator formalism which we are constructing provides very naturally these constraints. They will be obtained in a very similar way to the finite form of the Gauss law (2.27).

2.2 QUANTIZATION FOR A GENUS-ONE HANDLEBODY.

Let us consider now the case in which M_1 is a solid torus. The gauge fields on the surface $\Sigma = T^2$ can be parametrized in the following way [27,26]:

$$A_{\bar{z}} = (u_a u)^{-1} \partial_{\bar{z}}(u_a u), \quad A_z = (u_a \bar{u})^{-1} \partial_z(u_a \bar{u}), \quad (2.30)$$

where u is a single-valued map $u : \Sigma \rightarrow SU(N)^\circ$ and u_a a non-single-valued one which takes the form,

$$u_a = \exp\left(\frac{i\pi}{\text{Im}\tau} \int_{\bar{z}} \overline{w(z')} a \cdot H - \frac{i\pi}{\text{Im}\tau} \int_z w(z') \bar{a} \cdot H\right). \quad (2.31)$$

Several comments are in order to explain this expression. The holomorphic (anti-holomorphic) one-form $\omega(z)$ ($\bar{\omega}(z)$) is the holomorphic Abelian differential of the torus. They are the complex form of the one-forms dual to the homology cycles of the torus. They are chosen so that $\int_\alpha \omega = 1$, $\int_\beta \omega = \tau$ and $\int_{T^2} \omega \wedge \bar{\omega} = \text{Im}\tau$, β being the non-contractible homology cycle in the solid torus and α the contractible one. H is an r -dimensional complex vector made by the Cartan matrices of $SU(N)$, H_i , $i = 1, \dots, r$, such that, $H = H_i \lambda^{(i)}$, being $\lambda^{(i)}$, $i = 1, \dots, r$, the fundamental weights, and r the rank of the group $SU(N)$ ($r = N - 1$). The parameter

a is an r -dimensional vector of the form $a = \sum_{i=1}^r a^{(i)} \alpha_{(i)}$ where $\alpha_{(i)}$, $i = 1, \dots, r$, are the fundamental roots of $SU(N)$. A brief summary of our group theoretical conventions is presented in the appendix. Finally, τ is the modular parameter of the torus under consideration. The maps u_a are not single-valued. Phases are generated when going around homology cycles. However, the parametrization is such that A_z and $A_{\bar{z}}$ are well defined on the torus and when $u = 1$, A_z and $A_{\bar{z}}$ constitute a flat connection that contains all the information about the holonomy of the gauge field around non-trivial homology cycles. Notice that u_a is in the maximal torus of the group and $u_a^\dagger = u_a^{-1}$. We have defined (2.31) in such a way that $A_{\bar{z}}$ depends on a while A_z on $\bar{a}$. In this way the antiholomorphic nature of the wave functional (2.14) will lead to a holomorphic behavior with respect to the complex parameter a . In (2.30) one has $u^{-1} = \bar{u}^\dagger$ so that $A_z^\dagger = A_{\bar{z}}$.

Our choice of measure for the inner product has a natural counterpart which is the generalization of (2.18) for the case of the torus [26]:

$$[DA_z DA_{\bar{z}}]_\Sigma = e^{v\Gamma(u\bar{u}^{-1}, C)} |\Pi(\tau, a)|^4 e^{-2cv < u_a, u_a^{-1} >} (\text{Im}\tau)^r du d\bar{u} du_a du_a^\dagger \quad (2.32)$$

where $\Gamma(g, B)$ is the gauged WZW action [28],

$$\Gamma(g, B) = \Gamma(g) - \frac{1}{\pi} \int_\Sigma d^2\sigma \sqrt{\gamma} \gamma^{z\bar{z}} \text{Tr} [g^{-1} B_{\bar{z}} g B_z - B_{\bar{z}} \partial_z g g^{-1} + g^{-1} \partial_{\bar{z}} g B_{\bar{z}} - B_z B_{\bar{z}}], \quad (2.33)$$

and,

$$\Pi(\tau, a) = e^{\frac{N a^2}{2 \text{Im}\tau}} \Theta_{N, \rho}^A(\tau, a), \quad (2.34)$$

$\Theta_{N, \rho}^A(\tau, a)$ being the Weyl-antisymmetrized theta function of level N defined in (A17). The definition and a brief summary of the properties of these theta functions can be found in the appendix. In (2.34) ρ is

$$\rho = \sum_{i=1}^r \lambda^{(i)} \quad (2.35)$$

where $\lambda^{(i)}$, $i = 1, \dots, r$, are the fundamental weights of $SU(N)$. In (2.33) g is a

map $g : T^2 \rightarrow SU(N)$ and B is a vector field on the torus T^2 . In the expression for the measure (2.32), C is the vector field,

$$C_{\bar{z}} = u_a^{-1} \partial_{\bar{z}} u_a, \quad C_z = u_a^{-1} \partial_z u_a, \quad (2.36)$$

and $du d\bar{u}$ are infinite products of de Haar measures, while

$$du_a du_a^\dagger = \frac{d^r a d^r \bar{a}}{(\text{Im} \tau)^r}. \quad (2.37)$$

The measure (2.32) has three important properties. First, it is gauge invariant as one can verify explicitly. In terms of the new parameters the gauge transformation (2.17) takes the form $u \rightarrow ug$. We will denote this transformation as type i):

$$\text{i) } u \rightarrow ug. \quad (2.38)$$

Second, (2.32) is also invariant under transformations which leave $A_{\bar{z}}$ invariant. We will denote these transformations as type ii):

$$\text{ii) } u \rightarrow \hat{g}^{-1} u, \quad u_a \rightarrow u_a \hat{g}, \quad (2.39)$$

where $\hat{g}$ is a map from T^2 into the Cartan torus of $SU(N)$. Third, the measure is modular invariant. Modular invariance is a very important property which must have our construction. It tells us that observables are independent of how we have labelled the two-dimensional surface which appears in the Heegaard splitting. Modular transformations are described in the appendix. They are generated by the transformations S and T in (A20) and (A21). Notice that in order to maintain u_a in (2.31) independent of the labelling, a must transform under S transformations as $a \rightarrow a/\tau$ while it must remain invariant under a T type of transformation. Since

$$\langle u_a, u_a^{-1} \rangle = \frac{\pi}{\text{Im} \tau} a \cdot \bar{a}, \quad (2.40)$$

as follows from (2.31), one has that all except $|\Pi(\tau, a)|^4 (\text{Im} \tau)^r$ is modular invariant in the measure (2.32) (C defined in (2.36) is certainly modular invariant). To

verify that this remaining part is also modular invariant one must use the modular transformation properties of the theta functions given in (A22) and (A23).

The analogue of the PW property (2.20) for the gauged WZW action takes the form,

$$\begin{aligned} \Gamma(u\bar{u}^{-1}, C) = & \Gamma(u) + \langle u_a, u \rangle + \Gamma(\bar{u}^{-1}) + \langle \bar{u}^{-1}, u_a^{-1} \rangle \\ & - \langle u_a, u_a^{-1} \rangle - \frac{1}{\pi} \int_{\Sigma} d^2 \sigma \sqrt{\gamma} \gamma^{z\bar{z}} \text{Tr}(A_z A_{\bar{z}}), \end{aligned} \quad (2.41)$$

which again does not allow us to factorize the measure (2.32) symmetrically into two parts to define the measure for field configuration on Σ in the wave functional (2.14). We are now forced to take

$$e^{c_V(\Gamma(u\bar{u}^{-1}, C) - \Gamma(u) - \langle u_a, u \rangle) du d\bar{u} du_a^\dagger}, \quad (2.42)$$

for $\Psi(A_{\bar{z}})$, and $\exp\{c_V(\Gamma(u\bar{u}^{-1}, C) - \Gamma(\bar{u}^{-1}) - \langle \bar{u}^{-1}, u_a^{-1} \rangle)\} du du_a$ for its conjugated orientation reversed. Comparing the product of these two factors with the one in the measure, one concludes that the exponential factor in the inner product (2.13) has to be corrected by a factor $\exp\{\frac{c_V}{\pi} \int_{\Sigma} d^2 \sigma \sqrt{\gamma} \gamma^{z\bar{z}} \text{Tr}(A_z A_{\bar{z}}) + c_V \langle u_a, u_a^{-1} \rangle\}$. Therefore, the inner product becomes,

$$\begin{aligned} (\Psi_2 | \Psi_1) = & \int du d\bar{u} du_a du_a^\dagger |\Pi(\tau, a)|^4 e^{-c_V \langle u_a, u_a^{-1} \rangle} \langle \text{Im} \tau \rangle^r \\ & \times e^{\frac{k+5v}{\pi} \int_{\Sigma} d^2 \sigma \sqrt{\gamma} \gamma^{z\bar{z}} \text{Tr}(A_z A_{\bar{z}})} \Psi_2(A_{\bar{z}}) \overline{\Psi_1(A_{\bar{z}})}. \end{aligned} \quad (2.43)$$

For the interior of the solid torus one may choose any gauge invariant measure. Again there is a natural one which converges smoothly to (2.32). First a gauge is taken in which the radial component of A vanishes. For the other components one chooses (2.32) for each concentric layer around the central axis of the solid torus.

Let us now determine the form of the wave functional (2.14) using symmetry transformation properties. First let us consider a map $g : \Sigma \rightarrow SU(N)$ and one of

its extensions to the interior of the torus. Performing a gauge transformation of type i) (eq. (2.38)) one finds

$$\Psi(A_{\bar{x}}) \rightarrow e^{-(k+cv)(\Gamma(\hat{g})+\langle u_a, u, g \rangle)} \Psi(A_{\bar{x}}), \quad (2.44)$$

which is certainly independent of the choice of extension of g . Using the PW property (2.20) one obtains that the general solution to this has the form,

$$\Psi(A_{\bar{x}}) = \xi \psi_{k+cv}(u_a u) \Lambda(u_a), \quad (2.45)$$

where ξ is a constant, $\Lambda(u_a)$ is arbitrary and $\psi_{k+cv}(u_a u)$ is a functional which satisfies

$$\psi_l(u_a v) = \psi_l(u_a) e^{-l(\Gamma(v)+\langle u_a, v \rangle)} \quad (2.46)$$

for any single-valued map $v : \Sigma \rightarrow SU(N)^c$. Before solving this equation let us perform a gauge transformation of the type ii) (eq. (2.39)). The most general form of $\hat{g}$ on the Cartan torus that is single valued on T^2 is

$$\hat{g}_{n,m} = \exp\left\{\frac{i\pi}{\text{Im}\tau}(n+m\tau) \cdot H \int \frac{\overline{w(z')}}{w(z')} - \frac{i\pi}{\text{Im}\tau}(n+m\bar{\tau}) \cdot H \int w(z)\right\} \quad (2.47)$$

where n and m are r -dimensional vectors in the root lattice of $SU(N)$, $n, m \in \Lambda_R$. The maps $\hat{g}_{n,m}$ correspond to gauge transformations which are not connected to the identity. They wind in some of the directions of the Cartan torus when going around non-trivial homology cycles in $\Sigma = T^2$. These cycles are depicted in Fig. 4. In particular, n corresponds to cycles which are non-contractible in the solid torus (β -cycles) and m to the ones which are contractible (α -cycles). One could think that only maps with $m = 0$ should be taken into account since maps with $m \neq 0$ must have a singularity in the interior of the torus. This is however not right. In the interior of the torus may be Wilson lines which do not permit us to shrink the map with $m \neq 0$ to a single point and therefore there is no singularity.

All the maps $\hat{g}_{n,m}$ are generated by two sets of maps $\hat{g}_{n_{[i]},0}$, $\hat{g}_{0,m_{[i]}}$, $i = 1, \dots, r$, with $n_{[i]}$ and $m_{[i]}$ units vectors in the i direction, i.e., $n_{[i]}^j = \delta^{ij}$ and $m_{[i]}^j = \delta^{ij}$, where $n_{[i]}^j$ and $m_{[i]}^j$ are components in the root basis.

Let us now perform a type ii) transformation (eq. (2.39)) where $\hat{g}$ is any of the generators of (2.47). In the wave functional (2.14) only the measure transforms since the rest depends on $A_{\bar{x}}$ and A_x which are left invariant by this type of transformation. One finds

$$\Psi(A_x) \rightarrow e^{cv(\Gamma(\hat{g})+\langle u_a, \hat{g} \rangle)} \Psi(A_{\bar{x}}), \quad (2.48)$$

which implies the following property for $\Lambda(u_a)$ in (2.45),

$$\Lambda(u_a \hat{g}) = e^{cv(\Gamma(\hat{g})+\langle u_a, \hat{g} \rangle)} \Lambda(u_a). \quad (2.49)$$

Comparing this equation and (2.46) it turns out that $\Lambda(u_a)$ and $\psi_l(u_a)$ are related in the following way:

$$\Lambda(u_a) = [\psi_{cv}(u_a)]^{-1}. \quad (2.50)$$

What is still left is to find the solution to (2.46). Let us consider that v in (2.46) is one of the generating maps (2.47) of the form $\hat{g}_{n_{[i]},0}$. Certainly, this map is a particular case of the ones in (2.46). Using (2.47) one finds the following equation for $\psi_l(u_a)$,

$$\psi_l(u_a + n_{[i]}) = e^{l(\frac{1}{2\text{Im}\tau} n_{[i]} \cdot n_{[i]} + \frac{1}{\text{Im}\tau} a \cdot n_{[i]})} \psi_l(u_a). \quad (2.51)$$

Similarly, considering now a generator of the form $\hat{g}_{0,m_{[i]}}$ one finds the relation

$$\psi_l(u_a + m_{[i]}\tau) = e^{l(\frac{1}{2\text{Im}\tau} m_{[i]} \cdot m_{[i]} + \frac{1}{\text{Im}\tau} a \cdot m_{[i]}\tau)} \psi_l(u_a). \quad (2.52)$$

The most general solution of these two equations is any linear combination of the functions

$$\psi_{l,p}(\tau, a) = e^{\frac{\pi l}{2\text{Im}\tau} a^2} \Theta_{l,p}(\tau, a), \quad (2.53)$$

where $\Theta_{l,p}(\tau, a)$ are the theta functions of level l (see eq. (A14) in the appendix), and $p \in \Lambda_W / l\Lambda_R$, Λ_R and Λ_W being the root and weight lattices of $SU(N)$ re-

spectively. The solution is not unique. This was in fact expected as we have only used symmetry arguments to determine the states, and the wave functional (2.14) behaves in the same way under a gauge transformation no matter which Wilson lines are inserted in the functional integral. In fact, the wave functionals (2.14) should be labelled by the representations R_i that appear in their functional integral expressions. It is well known in group theory that to each irreducible representation is associated a Weyl invariant set of vectors in Λ_W containing a highest weight vector that labels the representation. This is precisely the kind of labelling that has the solutions which we have found in (2.53). At this stage we can not determine which ones are the linear combinations of $\psi_{l,p}$ that enter into $\psi_{k+cv}(u_a)$ and $\psi_{cv}(u_a)$ but according to what we already know they must be such that the wave functional $\Psi(A_{\bar{\tau}})$ is Weyl invariant. We will see that this condition together with the form of the measure (2.32) will be enough to determine $\Psi(A_{\bar{\tau}})$. Let us assume that these combinations are arbitrary for the time being. Taking (2.45), (2.46) and (2.50) one finds for the wave functional:

$$\Psi(A_{\bar{\tau}}) = \xi e^{-(k+cv)(\Gamma(u)+<u_a, u>)} \frac{\psi'_{k+cv}(u_a)}{\psi_{cv}(u_a)}. \quad (2.54)$$

The solution found satisfies the Gauss law (2.9). Taking the infinitesimal form of (2.44) one easily obtains the relation found in the previous case leading to (2.29). However, in this case one has in addition a Gauss law related to gauge transformations not connected to the identity. This Gauss law is however more simply expressed when reducing the theory at hand to an equivalent quantum system in which the part of the gauge field connected to the identity (u in (2.30)) has been integrated out. Let us carry this out first.

Consider the inner product (2.13) for wave functionals of the form (2.54). Let us integrate out u and $\bar{u}$ to end with an effective quantum system. Taking the

explicit form of the measure (2.32) and (2.54) one finds, after using (2.41),

$$\begin{aligned} \langle \Psi' | \Psi \rangle &= \int du_a du_a^\dagger |\Pi(\tau, a)|^4 e^{-(k+2cv)<u_a, u_a^{-1}>} (\text{Im}\tau)^r \\ &\times \overline{\xi' \xi} \left[\frac{\psi'_{k+cv}(u_a)}{\psi_{cv}(u_a)} \right] \frac{\psi_{k+cv}(u_a)}{\psi_{cv}(u_a)} \int dud\bar{u} e^{-(k+cv)(\Gamma(u\bar{u}^{-1}, C))}. \end{aligned} \quad (2.55)$$

The functional integral over u and $\bar{u}$ has been computed in [26]:

$$\int dud\bar{u} e^{-(k+cv)(\Gamma(u\bar{u}^{-1}, C))} = (\text{Im}\tau)^{-\frac{r}{2}} |\Pi(\tau, a)|^{-2} e^{cv<u_a, u_a^{-1}>}. \quad (2.56)$$

Inserting this result in (2.55) one finds the effective form of the inner product (2.13):

$$\langle \Psi' | \Psi \rangle = \int du_a du_a^\dagger |\Pi(\tau, a)|^2 e^{-(k+cv)<u_a, u_a^{-1}>} (\text{Im}\tau)^{\frac{r}{2}} \overline{\xi' \xi} \left[\frac{\psi'_{k+cv}(u_a)}{\psi_{cv}(u_a)} \right] \frac{\psi_{k+cv}(u_a)}{\psi_{cv}(u_a)}. \quad (2.57)$$

So far we have constructed an effective quantum system described by the inner product (2.57). We do not know however how to choose a basis for the corresponding Hilbert space. In other words we do not know which combinations of functions (2.53) enter into (2.54). To make a choice let us argue in the following way. The most natural normalization of states is the one in which the partition function for the manifold $S^2 \times S^1$ is 1. The reason for this is that since the theory is topological the Hamiltonian is zero and therefore, taking the time direction as S^1 , the partition function reduces to the trace of the operator 1 in S^2 (see Fig. 5), *i.e.*, the dimension of the Hilbert space corresponding to S^2 . We have seen in the previous subsection that this dimension is in fact 1. Once the normalization is fixed for this manifold it is fixed for all the others via surgery. Looking at the form of (2.57) and the fact that $\Pi(\tau, a)$ involves the Weyl-antisymmetric combination of theta functions entering $\psi_{cv}(u_a)$ (compare eq. (2.34) and (2.53)) one finds that the requirement of orthonormality for the basis of the Hilbert space that we are

about to construct implies that $\psi_{c_v}(u_a) = \Pi(\tau, a)$. Weyl invariance then forces us to have also the Weyl-antisymmetric combination for $\psi_{k+c_v}(u_a)$ in the numerator of (2.54). Let us then define

$$\lambda_{l,p}(\tau, a) = \sum_{w \in W} \epsilon(w) \psi_{l,w(p)}(\tau, a) \quad (2.58)$$

where W is the Weyl group and $\epsilon(w)$ is the signature of the element w of the Weyl group. As discussed in (A19), the only independent $\lambda_{l,p}(\tau, a)$ are the ones in which p is in the fundamental chamber $\mathcal{F}_{k+c_v}$. For the case in which $l = c_v$ there is only one independent state in the fundamental chamber which corresponds to $\lambda_{c_v,\rho}(\tau, a)$. Therefore, necessarily, $\psi_{c_v}(u_a)$ in (2.54) must be

$$\psi_{c_v}(u_a) = \lambda_{c_v,\rho}(\tau, a). \quad (2.59)$$

Notice that $\Pi(\tau, a) = \lambda_{c_v,\rho}(\tau, a)$ so in the effective inner product (2.57) all dependence on $\Pi(\tau, a)$ cancels out and one has

$$(\lambda_{k+c_v,q} | \lambda_{k+c_v,p}) = |\xi|^2 \int d^r a d^r \bar{a} (\text{Im} \tau)^{-\frac{r}{2}} e^{-(k+c_v) \frac{\pi}{\text{Im} \tau} a \cdot \bar{a}} \overline{\lambda_{k+c_v,q}(\tau, a)} \lambda_{k+c_v,p}(\tau, a), \quad (2.60)$$

where we have made use of (2.37) and (2.40). The constant ξ is obtained by imposing orthonormality. It turns out to be

$$\xi = \sqrt{\frac{(2(k+c_v))^{\frac{r}{2}}}{N!}}. \quad (2.61)$$

Equation (2.60) represents the inner product of an effective theory for which we can read the commutation relations of its basic operators a and $\bar{a}$ following backwards the logic of the holomorphic quantization,

$$[\bar{a}^i, a_j] = \frac{\text{Im} \tau}{\pi(k+c_v)} \delta_j^i, \quad (2.62)$$

where $a_j = \alpha_j \cdot a$ (i.e., $a = a_j \lambda^{(j)}$). Notice that (2.62) means that we can represent

$\bar{a}$ as

$$\bar{a}^j = \frac{\text{Im} \tau}{\pi(k+c_v)} \frac{\partial}{\partial a_j}. \quad (2.63)$$

This relation will be very useful in the next section to construct explicitly the Wilson line operators for the effective theory. The formulation involving (2.60) and (2.62) is very simple. However, it has some inconvenients. The inner product (2.60) is certainly modular invariant by construction. It would therefore be convenient to have a Hilbert space whose states transform according to a unitary representation of the modular group. This is certainly not the case for the functions $\lambda_{k+c_v,p}(\tau, a)$. Their modular transformations can be obtained easily using (A22) and (A23). One way to realize unitarily the modular group is to go back to the inner product (2.55), leave the factor $|\Pi(\tau, a)|^2$ in the measure and define the effective wave functions of the states as,

$$\chi_{k+c_v,p}(\tau, a) = \frac{\lambda_{k+c_v,p}(\tau, a)}{\lambda_{c_v,\rho}(\tau, a)}, \quad (2.64)$$

so that the effective inner product is

$$(\chi_{k+c_v,q} | \chi_{k+c_v,p}) = |\xi|^2 \int \frac{d^r a d^r \bar{a}}{(\text{Im} \tau)^r} |\Pi(\tau, a)|^2 (\text{Im} \tau)^{\frac{r}{2}} e^{-(k+c_v) \frac{\pi}{\text{Im} \tau} a \cdot \bar{a}} \times \overline{\chi_{k+c_v,q}(\tau, a)} \chi_{k+c_v,p}(\tau, a), \quad (2.65)$$

Since $|\Pi(\tau, a)|^2 (\text{Im} \tau)^{\frac{r}{2}}$ is modular invariant (see the comment after (2.40)), the functions $\chi_{k+c_v,p}(\tau, a)$ must transform according to a unitary representation of the modular group. This is in fact the case. Using (A22) and (A23) one finds

$$\begin{aligned} \chi_{k+c_v,p}(\tau, a) | \tau = \chi_{k+c_v,p}(\tau + 1, a) &= e^{2\pi i \left(\frac{p^2 - p^2}{2(k+c_v)} - \frac{1}{24} \frac{k(N^2-1)}{k+c_v} \right)} \chi_{k+c_v,p}, \\ \chi_{k+c_v,p}(\tau, a) | s = \chi_{k+c_v,p} \left(\frac{-1}{\tau}, \frac{a}{\tau} \right) &= \sum_{p' \in \mathcal{F}_{k+c_v}} S_{p,p'} \chi_{k+c_v,p'}, \end{aligned} \quad (2.66)$$

where the S matrix elements are given by:

$$S_{p,p'} = \left(\frac{N}{k+c_v} \right)^{\frac{r}{2}} \frac{1}{N^{\frac{N}{2}}(i)} \sum_{w \in W} \epsilon(w) e^{\frac{-2\pi i p \cdot w(p')}{k+c_v}}. \quad (2.67)$$

The states represented by $\chi_{k+c_v,p}(\tau, a)$ can be identified with the characters of a $SU(N)$ Kac-Moody algebra [29]. Central charge and conformal dimensions can be read from the form of the T transformation in (2.66):

$$h_p = \frac{p^2 - \rho^2}{2(k + c_v)}, \quad c = \frac{k(N^2 - 1)}{k + c_v}. \quad (2.68)$$

In terms of the $\chi_{k+c_v,p}(\tau, a)$ the connection between CS gauge theories and RCFT is manifest. In the next section we will construct the Verlinde operators [15] of RCFT in the context of CS gauge theory. To do that it is simpler to use the version of the effective theory in terms of the $\lambda_{k+c_v,p}(\tau, a)$. We have two equivalent effective formulations at hand. In the first one the representation of operators is simple, however, modular properties are cumbersome. In the second formulation the modular group is realized unitarily, but the representation of operators is more complicated.

Before ending this section let us discuss the interpretation of the symmetry properties (2.51) and (2.52) in terms of a global Gauss law. We will do the analysis in the representation in terms of $\lambda_{k+c_v,p}(\tau, a)$ so that operators have a simple form. Using (2.62) and (A16) we can write eq. (2.51) and (2.52) as

$$\begin{aligned} G_{n_{[i]}, 0} \psi_l(u_a) &= \psi_l(u_a), \\ G_{0, m_{[i]}} \psi_l(u_a) &= \psi_l(u_a), \end{aligned} \quad (2.69)$$

where

$$G_{n, m} = e^{\frac{\pi(k+c_v)}{\text{Im}\tau} [\bar{a} \cdot (n+m\tau) - a \cdot (n+m\bar{\tau})]}. \quad (2.70)$$

This operator can be regarded as the quantum form of the integral form of the Gauss law $F_{z\bar{z}}^a = 0$, for the gauge fields $G_{\bar{z}}$ and G_z in (2.36) with parameter gauge $\varepsilon_{n, m}$, the one in the exponent of (2.47), $\varepsilon_{n, m} = \frac{i\pi}{\text{Im}\tau} (n+m\tau) \cdot H \int^z \bar{w}(z') - \frac{i\pi}{\text{Im}\tau} (n+m\bar{\tau}) \cdot H \int^{\bar{z}} w(\bar{z}')$

$m\bar{\tau}) \cdot H \int^z w(z)$. In fact, taking into account that

$$\begin{aligned} \partial_z \varepsilon_{n, m} &= -\frac{i\pi}{\text{Im}\tau} \omega(z) (n + m\bar{\tau}) \cdot H, \\ \partial_{\bar{z}} \varepsilon_{n, m} &= \frac{i\pi}{\text{Im}\tau} \overline{\omega(z)} (n + m\tau) \cdot H, \end{aligned} \quad (2.71)$$

one finds that, indeed,

$$G_{n, m} = e^{\frac{k+c_v}{\pi} \int_{\Sigma} d^2\sigma \sqrt{\gamma} \gamma^{\bar{z}z} (G_{\bar{z}} \partial_z \varepsilon_{n, m} - G_z \partial_{\bar{z}} \varepsilon_{n, m})}. \quad (2.72)$$

The group of gauge transformations parametrized by (2.47) is Abelian and has the following composition law:

$$(n, m) \circ (n', m') = (n + n', m + m'). \quad (2.73)$$

The Gauss law (2.69) ((2.51) and (2.52)) which we have imposed so far only takes the generators of this Abelian group. However, the whole group should be implemented as a Gauss law and the states should fulfil a trivial representation of this Abelian group at least projectively. We will see that in general the states correspond to a projective trivial representation.

Let us analyze the properties of the operators (2.70). First, it is simple to show using (2.62) that since the root lattice is an integer lattice these operators commute:

$$G_{n, m} G_{n', m'} = e^{2\pi i (k+c_v)(n \cdot m' - n' \cdot m)} G_{n', m'} G_{n, m}. \quad (2.74)$$

On the other hand, the composition law is not faithfully represented,

$$G_{n, m} G_{n', m'} = e^{i\pi(k+c_v)(n \cdot m' - n' \cdot m)} G_{n+n', m+m'}. \quad (2.75)$$

The exponential factor in this equation is ± 1 since we are considering an integer lattice. For special cases like $k + c_v$ even or for $SU(2)$ the composition law (2.73)

is faithfully represented but in general this representation is projective. The exponential factor in (2.75) has the form of a cocycle and, indeed, associativity holds for the operators (2.70).

To end this section we finally give the expression of these operators when acting on the states $\lambda_{k+c_v,p}(\tau, a)$. Using (2.62), (2.53), (2.58) and the properties of the theta functions compiled in the appendix (eqs. (A15) and (A16)) one finds

$$\begin{aligned} G_{n,m} \lambda_{k+c_v,p}(\tau, a) &= e^{i\pi(k+c_v)n \cdot m + 2\pi i m \cdot p} \lambda_{k+c_v,p+n(k+c_v)}(\tau, a) \\ &= e^{i\pi(k+c_v)n \cdot m} \lambda_{k+c_v,p}(\tau, a). \end{aligned} \quad (2.76)$$

In the last step we have used the fact that the weights p are dual to the roots n and that p in $\lambda_{k+c_v,p}(\tau, a)$ lives in the fundamental chamber $\mathcal{F}_{k+c_v}$. The form of $G_{n,m}$ in this basis shows clearly the properties of these operators that we have discussed above. For the generators (2.69) the exponential factor in (2.76) is 1. However, in general, one finds signs which reveal the projective nature of the representation. Finally, notice that although the operators (2.70) are not realized as 1 on the states, the measure (2.32) is invariant under the gauge transformations (2.47) corresponding to these operators.

3. KNOT OPERATORS

In this section we will compute the operators corresponding to Wilson lines. We will show that certain types of these operators constitute an explicit representation of the Verlinde operators of RCFT [15]. These Verlinde operators are usually defined in a rather abstract form in the framework of RCFT. CS gauge theory seems to be the only approach to provide an explicit representation for these operators.

The wave functionals (2.14) are associated to handlebodies that do not have any Wilson lines on the boundary Σ . In the operator formalism that we have developed so far, Wilson lines that lie completely on Σ can be represented as operators acting on the finite-dimensional Hilbert space of the theory. If we restrict ourselves to genus-one handlebodies we have to consider only toral knots. Toral knots are characterized by two integers n and m which must be coprimes. We will use the convention in which n (m) indicates the number of times that the path of the knot winds around the β -cycle (α -cycle). It is easy to see that these Wilson line operators will not take us out of the Hilbert space constructed in the previous section. In fact, as the theory is topological, any Wilson line lying on the boundary Σ can be continuously moved to the interior of the handlebody with no effect on the state (see Fig. 6). This means that we can represent the result of acting with the Wilson line operator as in equation (2.14) (*i.e.*, as an element of our Hilbert space). This, of course, can be explicitly checked (see below).

Let us therefore consider a toral knot (n, m) and the corresponding Wilson line in the irreducible representation Λ . We will denote irreducible representations of $SU(N)$ by their highest weight Λ . According to (2.6) we need to evaluate

$$W_{\Lambda}^{(n,m)} = \text{Tr}_{\Lambda} \left(P e^{\int_{n,m} A} \right), \quad (3.1)$$

where all the dependence on the path has been encoded on the labels n, m . As we have seen in the previous section, the dependence on the single-valued map u

in (2.30) can be integrated out in general leading to the effective theory described by (2.60) and (2.62). Thus, since we are interested in the representation of the operator (3.1) we may well assume that $u = 1$, *i.e.*, that the operator $W_\Lambda^{(n,m)}$ in (3.1) depends only on a and $\bar{a}$. Using (2.30) and (2.31) one easily finds that,

$$\begin{aligned} W_\Lambda^{(n,m)} &= \text{Tr}_\Lambda \left(\exp \left(\frac{i\pi}{\text{Im}\tau} (n\bar{\tau} + m) \bar{a}^j H_j - \frac{i\pi}{\text{Im}\tau} (n\tau + m) \bar{a}^j H_j \right) \right) \\ &= \sum_{\mu \in M_\Lambda} \exp \left(-\frac{\pi}{\text{Im}\tau} (n\bar{\tau} + m) a \cdot \mu + \frac{n\tau + m}{k + c_v} \mu \cdot \frac{\partial}{\partial a} \right), \end{aligned} \quad (3.2)$$

where we have used the fact that H_j are diagonal and have as entries i times the components of the set of weights $\mu \in M_\Lambda$ corresponding to the $SU(N)$ irreducible representation of highest weight Λ . In (3.2) we have made use of (2.62). From the differential form of $W_\Lambda^{(n,m)}$ in (3.2) one easily finds its matrix form on the Hilbert space of the effective theory under consideration. This provides the explicit form of toral knot operators. Using standard properties of the theta functions, which are conveniently compiled in the appendix (eq. (A16)), one finds

$$W_\Lambda^{(n,m)} \lambda_{k+c_v,p} = \sum_{\mu \in M_\Lambda} e^{i\pi\mu^2 \frac{p\bar{p}}{k+c_v} + 2\pi i \frac{p\bar{p}}{k+c_v} p \cdot \mu} \lambda_{k+c_v,p+n\mu}. \quad (3.3)$$

Notice that in this equation the label of $\lambda_{k+c_v,p}$ is shifted by some amount. When getting out of the fundamental chamber $\mathcal{F}_{k+c_v}$ by the shift one must express the corresponding wave function in terms of the one in the fundamental chamber using the periodicity properties discussed in the previous section for $\lambda_{k+c_v,p}$.

Among all the operators for toral knots which we have constructed, two types of them are of particular importance: those consisting of a single wind around the β -cycle, which take the form

$$W_\Lambda^{(1,0)} \lambda_{k+c_v,p+\mu} = \sum_{\mu \in M_\Lambda} \lambda_{k+c_v,p+\mu}, \quad (3.4)$$

and those which winds only once around the α -cycle,

$$W_\Lambda^{(0,-1)} \lambda_{k+c_v,p} = \sum_{\mu \in M_\Lambda} e^{-2\pi i \frac{p\bar{p}}{k+c_v}} \lambda_{k+c_v,p}. \quad (3.5)$$

Notice that this last operator is diagonal and that the sum entering into the last expression is a well-known quantity. This quantity corresponds to the Weyl character of the representation of highest weight Λ ,

$$W_\Lambda^{(0,-1)} \lambda_{k+c_v,p} = \text{ch}_\Lambda \left(-2\pi i \frac{p}{k+c_v} \right) \lambda_{k+c_v,p}. \quad (3.6)$$

which is defined in (A12) in the appendix. Let us concentrate our attention on the behavior of the operators (3.4) and (3.5), which will be identified with the Verlinde operators. First of all, it follows trivially from (3.4) that the action of two of the operators of the first type corresponding to different representations, Λ and Λ' , commute:

$$W_\Lambda^{(1,0)} W_{\Lambda'}^{(1,0)} = W_{\Lambda'}^{(1,0)} W_\Lambda^{(1,0)}. \quad (3.7)$$

The same property also holds for the operators of the second type. The Verlinde operators related to the β -cycle (3.4) possess the property that when acting on the vacuum of the theory they generate a state corresponding to the representation carried out by the operator. This behavior is the most important property shared by the operators $W_\Lambda^{(1,0)}$. It turns out that, indeed,

$$W_\Lambda^{(1,0)} \lambda_{k+c_v,p} = \lambda_{k+c_v,p+\Lambda}. \quad (3.8)$$

The fact that this is true for the operators (3.4) is by no means trivial. The proof of (3.8) is based on the Weyl character formula. First, notice that taking into account (2.58) and (2.53), proving (3.8) is equivalent to proving the following relation among theta functions,

$$\sum_{w \in W} \sum_{\mu \in M_\Lambda} \epsilon(w) \Theta_{l,w(\mu+\rho)}(a, \tau) = \sum_{w \in W} \epsilon(w) \Theta_{l,w(\Lambda+\rho)}(a, \tau), \quad (3.9)$$

i.e., out of the sum over all $\mu \in M_\Lambda$ only the contribution from the highest weight does not cancel out. The strategy to obtain a proof of (3.9) is the following. First

we will show, using the Weyl character formula (A13), that a relation like (3.9) follows for the exponential function. Then we will show that the theta functions (A14) admit a Fourier expansion.

Using (A12) and (A13), one finds

$$\begin{aligned} \sum_{w \in W} \epsilon(w) e^{w(\Lambda + \rho) \cdot a} &= \text{ch}_\Lambda(a) \sum_{w \in W} \epsilon(w) e^{w(\rho) \cdot a} \\ &= \sum_{\mu \in M_\Lambda} \sum_{w \in W} \epsilon(w) e^{a \cdot \mu + w(\rho) \cdot a} \\ &= \sum_{\mu \in M_\Lambda} \sum_{w \in W} \epsilon(w) e^{w(\mu + \rho) \cdot a}, \end{aligned} \quad (3.10)$$

where in the last step we have used the fact that M_Λ is transformed into itself under the action of the Weyl group. This equation proves that the exponential function satisfies a property like (3.9). This implies that any function $f: \Lambda_W \rightarrow \mathbb{C}$ which admits a Fourier series expansion also verifies (3.9). Let us prove that this is the case for the level l theta functions. From (A15) it follows that $\Theta_{l,p}(\tau, a)$ is defined in $\Lambda_W/l\Lambda_R$ in respect to the index p . Let us prove that there exist functions $\theta_{l,y}(\tau, a)$, $y \in (\Lambda_W/l)$, ($l\Lambda_R$ and Λ_W/l are dual to each other), such that

$$\Theta_{l,p}(\tau, a) = \sum_{y \in (\Lambda_W/l)} \theta_{l,y}(\tau, a) e^{-2\pi i y \cdot p}, \quad (3.11)$$

i.e., that $\Theta_{l,p}(\tau, a)$ admits a Fourier series expansion. To show this we must demonstrate that the functions $\theta_{l,y}(\tau, a)$, $y \in (\Lambda_W/l)$, as defined by the Fourier relation,

$$\theta_{l,y}(\tau, a) = \frac{1}{\text{Vol}(l\Lambda_R)} \int_{\mathcal{U}} d^r p \Theta_{l,p}(\tau, a) e^{2\pi i y \cdot p}, \quad (3.12)$$

are well defined. In (3.12) $\mathcal{U}$ denotes the unit cell of the lattice $l\Lambda_R$. Using now

the definition (A14), it turns out that

$$\begin{aligned} \theta_{l,y}(\tau, a) &= \frac{1}{\text{Vol}(l\Lambda_R)} \int_{\mathcal{U}} d^r p \sum_{\nu \in \Lambda_R} \exp\left\{i\pi \tau l\left(\nu + \frac{p}{l}\right)^2 + 2\pi i l\left(\nu + \frac{p}{l}\right) \cdot a + 2\pi i y \cdot p\right\} \\ &= \frac{1}{\text{Vol}(l\Lambda_R)} \int_{\mathbb{R}^r} d^r p \exp\left\{i\pi \tau \frac{p^2}{l} + 2\pi i p \cdot (a + y)\right\}, \end{aligned} \quad (3.13)$$

where in the last step we have made the change of variable $p + l\nu \rightarrow p$ and used the fact that $l\nu \cdot y$ is an integer since $y \in (\Lambda_W/l)$ and $\nu \in \Lambda_R$. This last integral is convergent for any $y \in (\Lambda_W/l)$ in the range of values of τ where the theta functions (A14) are defined, $\text{Im}\tau > 0$. In fact, its computation is straightforward,

$$\theta_{l,y}(\tau, a) = \frac{1}{(-i l \tau)^{\frac{r}{2}}} e^{-\frac{i\pi l}{\tau}(a+y)^2}. \quad (3.14)$$

Eq. (3.13) proves that the theta functions admit a Fourier series expansion of the type (3.11) whose coefficients are (3.14) (notice that the resulting series (3.11) is convergent when $\text{Im}\tau > 0$). From this fact and the result (3.10) the proof of (3.9) follows. Therefore, (3.8) is proven.

So far we have considered Wilson lines in arbitrary irreducible representations of $SU(N)$. It is well known in RCFT that only some distinguished representations play a relevant role. In particular, the Verlinde operators should exist only for the so-called integrable representations. An irreducible representation is integrable if its highest weight Λ is such that $\Lambda + \rho$ is in the fundamental chamber $\mathcal{F}_{k+c_v}$. Notice that there is a one-to-one correspondence between integrable representations and the elements of a basis of the Hilbert space of the effective quantum system under consideration. Since $W_\Lambda^{(0,-1)}$ is well defined for any irreducible representation one would expect that it possesses some periodicity properties in such a way that the only relevant irreducible representations are the integrable ones. This is indeed the case. Consider two highest weights Λ and Λ' such that $\Lambda' + \rho = w(\Lambda + \rho + (k + c_v)\alpha)$ where α is a root, $\alpha \in \Lambda_R$, and w is an element of the Weyl group, $w \in W$. Using

the Weyl character formula (A13), one finds,

$$\text{ch}_{\Lambda'} \left(-2\pi i \frac{p}{k+c_v} \right) = \frac{\sum_{w \in W} \epsilon(w) e^{-2\pi i \frac{w(\Lambda'+p)p}{k+c_v}}}{\sum_{w \in W} \epsilon(w) e^{-2\pi i \frac{w(p)p}{k+c_v}}}, \quad (3.15)$$

and since,

$$\begin{aligned} \sum_{w \in W} \epsilon(w) e^{-2\pi i \frac{w(\Lambda'+p)p}{k+c_v}} &= \epsilon(w) \sum_{w' \in W} \epsilon(w') e^{-2\pi i \frac{w'(\Lambda+p+(k+c_v)\rho)p}{k+c_v}} \\ &= \epsilon(w) \sum_{w' \in W} \epsilon(w') e^{-2\pi i \frac{w'(\Lambda+p)p}{k+c_v}}, \end{aligned} \quad (3.16)$$

we have,

$$W_{\Lambda'}^{(0,-1)} \lambda_{k+c_v,p} = \epsilon(w) W_{\Lambda}^{(0,-1)} \lambda_{k+c_v,p}. \quad (3.17)$$

Therefore, the only relevant representations which the operators $W_{\Lambda}^{(0,-1)}$ carry are the ones labelled by a highest weight Λ such that $\Lambda + \rho$ is in the fundamental chamber $\mathcal{F}_{k+c_v}$, i.e., Λ corresponds to an integrable one. Actually, it is rather simple to prove that this fact is valid for any Wilson line of the type (3.1). Taking the explicit form of $W_{\Lambda}^{(n,m)}$ in (3.2) one finds that after performing modular transformations of the types S and T as in (A20) and (A21), Wilson line operators transform into themselves. Indeed, one obtains

$$\begin{aligned} S^{-1} W_{\Lambda}^{(n,m)} S &= W_{\Lambda}^{(m,-n)}, \\ T^{-1} W_{\Lambda}^{(n,m)} T &= W_{\Lambda}^{(n,m+n)}. \end{aligned} \quad (3.18)$$

This implies that any Wilson line operator of the type considered can be obtained from $W_{\Lambda}^{(0,-1)}$ by performing appropriate modular transformations. Thus, the property (3.17) extends to any $W_{\Lambda}^{(n,m)}$, i.e., in general, the only relevant representations are the integrable ones.

In RCFT Verlinde operators close among themselves and the constants appearing in their algebra correspond to the fusion rules. Let us analyze this aspect in our representation of these operators. The fact that the operators $W_{\Lambda}^{(1,0)}$ form a closed algebra follows from (3.7) and (3.8). There is no need to know their explicit form (3.4). The argument is rather standard. From now on we will denote the label of the states of the Hilbert space by $\lambda_{k+c_v,\rho+\Lambda}$ $\rho+\Lambda$ being in the fundamental chamber $\mathcal{F}_{k+c_v}$. Let us write (3.4) in matrix form as

$$W_{\Lambda}^{(1,0)} \lambda_{k+c_v,\rho+\Lambda'} = N_{\Lambda\Lambda'}^{\Lambda''} \lambda_{k+c_v,\rho+\Lambda''}, \quad (3.19)$$

where a summation over repeated indices is understood. From (3.7) and (3.8) it follows that

$$\begin{aligned} N_{\Lambda\Lambda'}^{\Lambda''} &= N_{\Lambda'\Lambda}^{\Lambda''}, \\ N_{\Lambda\Lambda'}^{\Upsilon} N_{\Lambda''\Upsilon}^{\Lambda'''} &= N_{\Lambda\Upsilon}^{\Lambda'''} N_{\Lambda''\Lambda'}^{\Upsilon}, \end{aligned} \quad (3.20)$$

which imply,

$$W_{\Lambda}^{(1,0)} W_{\Lambda'}^{(1,0)} = N_{\Lambda\Lambda'}^{\Lambda''} W_{\Lambda''}^{(1,0)}. \quad (3.21)$$

From this equation we observe that the matrix elements of $W_{\Lambda}^{(1,0)}$ are the constants entering the algebra of these operators. Certainly these constants are integers, as follows from (3.4). Furthermore, these constants are non-negative. We will not prove this from a pure Chern-Simons point of view. We will carry out instead the full connection with RCFT where these constants are identified with the ones entering the fusion rules, which are known to be positive.

In practice there are several ways to compute the constants $N_{\Lambda\Lambda'}^{\Lambda''}$. One may just take (3.4) and plug back into the fundamental chamber $\mathcal{F}_{k+c_v}$ all the representations that appear. This is a rather tedious way to proceed. Another

possibility is to take (3.21) on the vacuum and to realize that,

$$\begin{aligned}
 W_{\Lambda'}^{(1,0)} W_{\Lambda}^{(1,0)} \lambda_{k+c_v, \rho} &= W_{\Lambda}^{(1,0)} \sum_{\mu \in M_{\Lambda}} \lambda_{k+c_v, \rho+\mu} \\
 &= \sum_{\mu' \in M_{\Lambda'}, \mu \in M_{\Lambda}} \lambda_{k+c_v, \rho+\mu'+\mu} = \sum_{\mu'' \in M_{\Lambda \otimes \Lambda'}} \lambda_{k+c_v, \rho+\mu''} \quad (3.22) \\
 &= \sum_{\oplus \Lambda'} \sum_{\mu \in M_{\Lambda'}} \lambda_{k+c_v, \rho+\mu} = \sum_{\oplus \Lambda'} W_{\Lambda'}^{(1,0)} \lambda_{k+c_v, \rho},
 \end{aligned}$$

where we have denoted by Λ^i the highest weights of the irreducible representations which appear in the tensor product of the representations of highest weights Λ and Λ' . From (3.22) one may extract the constants $N_{\Lambda \Lambda', \Lambda''}$ after putting back the representations labelled by Λ^i into the fundamental chamber $\mathcal{F}_{k+c_v}$. This is in fact the so-called depth rule for computing the constants entering into the fusion rules of RCFT.

There is still another way to identify the constants $N_{\Lambda \Lambda', \Lambda''}$ with those in the fusion rules which is even more explicit. From (3.18) it follows that

$$W_{\Lambda}^{(1,0)} = S W_{\Lambda}^{(0,-1)} S^{-1}. \quad (3.23)$$

Using (3.6), (2.67), and the Weyl character formula (A13) one finds

$$\left[W_{\Lambda}^{(0,-1)} \right]_{\Lambda', \Lambda''} = \frac{S_{\rho+\Lambda, \rho+\Lambda'}}{S_{\rho+\Lambda'}} \delta_{\Lambda', \Lambda''}. \quad (3.24)$$

Combining this result with (3.23) one obtains the following expression for the constants $N_{\Lambda \Lambda', \Lambda''}$:

$$N_{\Lambda \Lambda', \Lambda''} = \sum_{p \in \mathcal{F}_{k+c_v}} \frac{S_{\rho+\Lambda', p} S_{\rho+\Lambda, p} (S^{-1})_{\rho+\Lambda'', p}}{S_{\rho, p}}. \quad (3.25)$$

This expression has the advantage that all the indices are in the right range. There is no need to go back to the fundamental chamber $\mathcal{F}_{k+c_v}$ as in the approach

leading to the depth rule. However, from (3.25) it is not manifest that $N_{\Lambda \Lambda', \Lambda''}$ are integers. The form (3.25) for these constants allows us to identify them with those appearing in the fusion rules after taking into account the proof in [1] of Verlinde's conjecture (which implies the validity of (3.25) when considering $N_{\Lambda \Lambda', \Lambda''}$ as the constants appearing in the fusion rules).

As we have discussed in this section, CS gauge theory permits us to associate an operator to each knot. The operators (3.3) constitute an example of this fact for toral knots. Clearly, the construction of our formalism for arbitrary genus would allow us to obtain operators for any knot. These operators seem more intrinsic to a knot than knot polynomials. As we will show below, the vacuum expectation value of these operators leads to knot polynomials. However, vacuum expectation values depend on the manifold where the knot is lying while expressions like (3.3) only depend on the knot structure. In this sense, the knot operators (3.3) seem more intrinsic. Furthermore, they constitute a very useful tool to compute the resulting topological invariants via vacuum expectation values.

Since our analysis has been done only up to genus 1, we are limited to the case of toral knots or links of toral knots living in lens spaces. Let us compute as an example the vacuum expectation value of a toral knot in S^3 carrying an irreducible representation of $SU(N)$ labelled by the highest weight Λ . Since the knot is a toral knot, it is possible to select a solid torus inside S^3 such that the knot lies in its surface. Let us then make a Heegaard splitting of S^3 separating this solid torus and its complementary in S^3 , which is topologically another solid torus. Now, since we are going to treat the two solid tori on an equal footing, to reobtain S^3 by joining them one must make first an S modular transformation on, say, the surface of the empty solid torus (no knot inside). To each of the solid tori there is a state associated as defined by the wave functional (2.14). For the empty solid torus this state is just the vacuum. For the other solid torus, however, since it contains a toral knot on the surface, the corresponding state is just the action of the operator (3.3) on the vacuum. Thus, we have that the vacuum expectation value of a Wilson line carrying an irreducible representation Λ and associated to

a toral knot n, m in S^3 is

$$V_{n,m}^\Lambda = \left[S' W_\Lambda^{(n,m)} \right]_{00} \quad (3.26)$$

where S' is just the S -transformation matrix (2.67) in terms of the labels Λ, Λ' with $\Lambda + \rho, \Lambda' + \rho \in \mathcal{F}_{k+c_v}$, i.e., $S'_{\Lambda\Lambda'} = S_{\rho+\Lambda, \rho+\Lambda'}$. For other lens spaces one must introduce the modular transformation leading to such space instead of S' in (3.26). There are in general different modular transformations leading to the same lens space, giving therefore a certain ambiguity in the computation of these vacuum expectation values [30]. However, the way to control this problem is well known [4,31]. CS gauge theory leads to invariants associated to framed manifolds. Different framings in topologically equivalent manifolds lead to vacuum expectation values which differ by some factor which depends simply on the central charge (2.68). A similar problem occurs when considering Wilson lines since a toral knot can be constructed in many ways from a knot on the surface of the torus. For example, the knots in Fig. 7a and Fig. 7b both lead to the trivial knot in S^3 . However, their vacuum expectation values computed by (3.26) are different. As for manifolds, CS gauge theory provides invariants for framed knots. This means that one should consider the knots in Fig. 7a and Fig. 7b as ribbons. This is shown in Fig. 8a and Fig. 8b. Clearly, when looking at the corresponding ribbons in S^3 the one resulting from Fig. 8b is going to have a twist with respect to the one resulting from Fig. 8a. This fact is displayed in Fig. 9a and Fig. 9b. CS gauge theory feels this twist and so it leads to invariants for framed knots. However, similarly to the case of the dependence on the frame of the manifold, the effect of a twist like the one in Fig. 9b is easy to control. As shown in [4], the ratio between the vacuum expectation value associated to the knots in Fig. 9a and 9b is of the form

$$e^{2\pi i h_{\rho+\Lambda}}, \quad (3.27)$$

where $h_{\rho+\Lambda}$ is the conformal weight (2.68) of the representation Λ carried by the knots. We will observe this effect below when computing (3.26) explicitly.

Let us calculate the vacuum expectation value (3.26) for the case of a Wilson line carrying the fundamental representation of $SU(2)$. This computation must lead to the expression of the Jones polynomial for the toral knots given in [14]. In computing (3.26) first notice that in the sum due to the product of S' and $W_\Lambda^{(n,m)}$ we do not need to worry about going back to the fundamental chamber, $\mathcal{F}_{k+c_v}$, when applying (3.8) since the matrix S' has the right periodicity properties $S_{\rho+(k+c_v)\Lambda_R, \rho'} = S_{\rho, \rho'}$ and $S_{w(\rho), \rho'} = \epsilon(w)S_{\rho, \rho'}$, as follows from its expression in (2.67). In general, this fact simplifies notably the calculation of (3.26).

Let us denote $SU(2)$ irreducible representations of highest weight Λ by the Dynkin label λ (or isospin $\lambda/2$) such that $\Lambda = \lambda\alpha^{(1)}$. With this notation the matrix S' obtained from (2.67) takes the form

$$S'_{\lambda\lambda'} = \sqrt{\frac{2}{k+2}} \sin \frac{\pi(\lambda+1)(\lambda'+1)}{k+2}, \quad (3.28)$$

where we have used the fact that $\alpha^{(1)} \cdot \alpha^{(1)} = 1/2$ and $c_v = 2$. The integrable representations of $SU(2)$ correspond to Dynkin labels, $\lambda = 0, 1, \dots, k+1$, (see (A19)). Given one of these representations, λ , the corresponding weights take the form $\mu_j = (\lambda - 2j)\alpha^{(1)}$, $j = 0, 1, \dots, \lambda$, and all have multiplicity one. Therefore the irreducible representation has dimension $\lambda+1$. For the fundamental representation, $\lambda = 1$, and, after taking (3.26) and (3.28),

$$\begin{aligned} V_{n,m}^1 &= \sqrt{\frac{2}{k+2}} e^{i\pi \frac{n}{k+2}} \left[e^{i\pi \frac{n}{k+2}} \sin \frac{\pi(1+n)}{k+2} + e^{-i\pi \frac{n}{k+2}} \sin \frac{\pi(1-n)}{k+2} \right] \\ &= \frac{1}{2i} \sqrt{\frac{2}{k+2}} t^{\frac{1}{4}(mn-2m-2n+2)} (1 - t^{n-1} - t^{m-1} + t^{m+n}), \end{aligned} \quad (3.29)$$

where in the last step we have introduced the variable,

$$t = e^{\frac{2\pi i}{k+2}}. \quad (3.30)$$

Notice that (3.29) has a polynomial form when expressed in terms of this variable. The resulting polynomial has the property that it is symmetric under the interchange of n and m . This should be expected since the result for S^3 must be blind to the way we labelled the α and β -cycles of the solid torus.

To obtain the polynomial for toral knots from the vacuum expectation value (3.29) we must take care of a few facts. First, notice that for the unknot of Fig. 7a, $n = 1$ and $m = 0$,

$$V_{1,0}^1 = \frac{1}{2i} \sqrt{\frac{2}{k+2}} (t - t^{-1}). \quad (3.31)$$

However, the unknot can be constructed from any toral knot of the form $n = 1$ and m arbitrary. For example, Fig. 7b shows the case in which $n = 1$ and $m = 1$. From (3.30), one finds

$$V_{1,m}^1 = \frac{1}{2i} \sqrt{\frac{2}{k+2}} t^{\frac{3}{2}m} (t - t^{-1}). \quad (3.32)$$

We observe the effect discussed above (eq. (3.27)) due to framing. Notice that for $SU(2)$ and the fundamental representation the conformal weight (2.68) is $h = \frac{3}{4(k+2)}$ and therefore the factor appearing in (3.32) is of the form (3.27). Comparing (3.31) and (3.32), it is clear that the effect due to framing for a toral knot n, m is a factor $t^{\frac{3}{4}(k+2)mn}$.

Another fact to be taken into account before comparing our vacuum expectation value (3.29) with Jones' expression for its polynomial is the orientation. One may finish by obtaining the Jones polynomial for the mirror image knot if our convention to label the direction of the α -cycle with respect to the direction of the β -cycle is the opposite as in [14]. This is indeed the case and therefore we must change the sign of n before making contact with Jones' result.

Finally, one must take care of the normalization. The Jones polynomial is defined in such a way that it is 1 for the unknot. Therefore, one must divide the vacuum expectation value (3.29) by (3.31). Notice that the polynomial (3.29) has as roots 1 and -1 if n and m are coprime and therefore this ratio is always a polynomial.

Taking into account all these considerations, we define the following poly-

mial,

$$J_{n,m}(t) = \frac{V_{-n,m}^1 t^{\frac{3}{4}mn}}{V_{1,0}^1}. \quad (3.33)$$

Notice that the sign in the exponential is positive because there are two minus signs, one from diframing, and another from $n \rightarrow -n$. In this way what appears in the numerator of (3.33) is the vacuum expectation value corresponding to the mirror image of a toral knot in S^3 in the standard framing. Using (3.29) and performing some manipulations, one finds,

$$J_{n,m}(t) = \frac{t^{\frac{1}{2}(n-1)(m-1)}}{1 - t^2} (1 - t^{n+1} - t^{m+1} + t^{n+m}), \quad (3.34)$$

which is exactly the form obtained by Jones in [14]. Notice that if we do not take into account the difference in the notation to label n and m we would have obtained

$$\begin{aligned} \tilde{J}_{n,m}(t) &= \frac{V_{n,m}^1 t^{-\frac{3}{4}mn}}{V_{1,0}^1} \\ &= \frac{t^{-\frac{1}{2}(n+1)(m-1)}}{1 - t^2} (1 - t^{-n+1} - t^{m+1} + t^{m-n}), \end{aligned} \quad (3.35)$$

which must correspond to the Jones polynomial (3.34) of the mirror image knot. It is well known [14] that the Jones polynomial corresponding to the mirror image of a given knot is obtained by changing $t \rightarrow 1/t$ in the Jones polynomial of the knot. One can check the consistency of our discussion verifying that, indeed,

$$\tilde{J}_{n,m}(t) = J_{n,m}\left(\frac{1}{t}\right). \quad (3.36)$$

To end this section let us remark that $V_{n,m}^A$ in (3.26) is always a polynomial in the variable t defined in (3.30). Furthermore, for lens spaces this is also true since the T transformations (2.66) also contain powers of t . To compute $V_{n,m}^A$ from (3.26) is in general rather tedious but straightforward. The advantage of associating an operator to a knot instead of a vacuum expectation value is that it

contains all the information present in the vacuum expectation value in a rather compact form. Furthermore, they are simpler to compute from the point of view of CS gauge theory. It is very important to extend this association of knots to operators to knots other than the toral ones. This would require the construction of the formalism presented in this work for higher genus.

4. Conclusions

In this paper we have developed the operator formalism of CS gauge theories with gauge group $SU(N)$. We have envisaged the powerfulness of the formalism obtaining an explicit representation of the Wilson line operators in the Hilbert space of the theory. A basis of this Hilbert space has the same structure as the set of characters of RCFT associated to a Kac-Moody algebra with group $SU(N)$. The construction of knot operators has permitted us to obtain an explicit representation of the Verlinde operators. The proof of this statement has been made in sect. 3. Knot operators may be of some use in dealing with some problems in RCFT. For example, these operators have a very simple behavior under modular transformations. This fact may be useful in constructing modular invariant partition functions.

It is worth remarking that our construction of the operator formalism for CS theory involves a choice of measure for the functional integrals present in the formulation. We have made the choices (2.18) and (2.32) for the cases of genus 0 and 1 respectively, which seem to us to be the most natural ones. We would like to point out, however, that other choices are possible. For example, one could have taken in (2.18) and (2.32) something involving only $dud\bar{u}$ and $dud\bar{u}du_a d\bar{u}_a^\dagger$, respectively. Following the same steps of our construction one finds that the Hilbert space now involves the Weyl-symmetrized combination of level k theta functions and not the Weyl-antisymmetrized combination of level $k+N$ ones which we found in (2.58). It is however well known [32] that this just corresponds to a change of basis. Other choices would lead to other bases of the Hilbert space. We have preferred to take the measures as in (2.18) and (2.32) to obtain a basis which makes the contact with the characters of RCFT in the most natural form, and because it provides a simpler operator representation.

We have only considered in this paper the case in which no Wilson lines are cut by the two-dimensional surface Σ . The more general situation in which the Wilson lines intercept Σ has been studied using our formalism in [23]. In

this case the wave functionals can also be represented by means of a functional integral similar to (2.14) and it can be easily shown that they satisfy a Gauss law with δ -function sources on the right-hand side that depend on the representations carried by the Wilson lines. When M_1 is a solid ball the solution of the Gauss law can be easily obtained. It can be proved that the wave functionals evaluated at $A_z = 0$ can be identified with the conformal blocks of the WZW theory. The Ward identities satisfied by the blocks (including the Knizhnik-Zamolodchikov equations [33]) follow from the Gauss law of the theory.

We have restricted ourselves in this work to the case of a gauge group $SU(N)$. The extension of the formalism to an arbitrary simply-laced group is straightforward. It should be also interesting to consider the case of non-simply laced ones. Another issue which should be addressed is the extension of the formalism to the case of genus 2 and higher. This could allow us to obtain the form of the operators corresponding to graphs in addition to general knot operators. An explicit representation of these operators could be of some use in the problem of classifying RCFT. In addressing this last problem one should consider also the issue of coset constructions within this formalism.

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APPENDIX

A1 GROUP-THEORETICAL CONVENTIONS.

In this first section of the appendix we will summarize our conventions for the group $SU(N)$. The $N^2 - 1$ generators of $SU(N)$ are chosen antihermitian and in the fundamental representation are normalized as follows:

$$\text{Tr}(T^i T^j) = -\delta^{ij} \quad (\text{A1})$$

The rank of $SU(N)$ is $r = N - 1$. The r fundamental roots will be denoted by $\alpha_{(i)}$, $i = 1, \dots, r$. They are chosen to have length $\sqrt{2}$, $\alpha_{(i)}^2 = 2$, and the following Cartan matrix, $g_{ij} = \alpha_{(i)} \cdot \alpha_{(j)}$,

$$g_{ij} = \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & \dots & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & \dots & \dots & 0 \\ 0 & -1 & 2 & -1 & \dots & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix} \quad (\text{A2})$$

The fundamental roots generate over the integers $\mathbf{Z}$ an r -dimensional lattice called the root lattice that we shall denote by Λ_R . In this r -dimensional space, the fundamental roots can be taken to form a basis, the root basis. Any vector x has in this basis components x^i given by:

$$x = \sum_{i=1}^r x^i \alpha_{(i)}. \quad (\text{A3})$$

The fundamental weights $\lambda^{(i)}$, $i = 1, \dots, r$, are defined by

$$2 \frac{\alpha_{(i)} \cdot \lambda^{(j)}}{\alpha_{(i)} \cdot \alpha_{(i)}} = \delta^{ij}, \quad (\text{A4})$$

which in the case of $SU(N)$, since the square length of the fundamental roots is

2, becomes

$$\alpha_{(i)} \cdot \lambda^{(j)} = \delta_i^j, \quad (\text{A5})$$

i.e., the fundamental weights are the duals to the fundamental roots. The fundamental weights generate over $\mathbb{Z}$ an r -dimensional lattice called the weight lattice which will be denoted by Λ_W . The lattices Λ_R and Λ_W are dual to each other and $\Lambda_R \in \Lambda_W$. The r -dimensional basis expanded by the fundamental weights is called the Dynkin basis. Any vector x has in this basis components x_i given by:

$$x = \sum_{i=1}^r x_i \lambda^{(i)}. \quad (\text{A6})$$

The matrix $g^{ij} = \lambda^{(i)} \cdot \lambda^{(j)}$ is the inverse of the Cartan matrix, $g^{ij} \cdot g_{jk} = \delta_k^i$, and turns out to be

$$g^{ij} = \text{Inf}\{i, j\} - \frac{ij}{N}. \quad (\text{A7})$$

Notice that we have denoted the components of a vector in the root basis with superindices (see (A3)) and the ones in the Dynkin basis with subindices (see (A6)). This notation is rather convenient since one has

$$x^i = g^{ij} x_j, \quad x_i = g_{ij} x^j, \quad (\text{A8})$$

as follows from (A5).

The irreducible representations of $SU(N)$ are characterized by highest weights Λ . Highest weights can be written uniquely as a linear combination of fundamental weights with non-negative integer coefficients h_i ,

$$\Lambda = \sum_{i=1}^r h_i \lambda^{(i)}. \quad (\text{A9})$$

The set of weights of an irreducible representation of highest weight Λ will be denoted as M_Λ . To build this set one may use the following rule: if a weight

$\mu \in M_\Lambda$ has the k^{th} Dynkin component greater than zero (i.e., $\mu_k > 0$), then the vectors obtained by subtracting $t\alpha_{(k)}$ ($t = 1, \dots, \mu_k$) to μ are also elements of M_Λ . One can start applying this rule to Λ and then to the successive weights to build the different elements of M_Λ . The multiplicities of each weight can be obtained using the Freudenthal formula [34].

The Weyl group is generated by r reflections σ_i , $i = 1, \dots, r$, on weight space

$$x \in \Lambda_W, \quad \sigma_i(x) = x - \alpha_{(i)}(\alpha_{(i)} \cdot x). \quad (\text{A10})$$

It divides the weight lattice Λ_W into $N!$ factorial domains. The fundamental domain or Weyl chamber is chosen to be the one containing all the weights $x \in \Lambda_W$ such that

$$\alpha_{(i)} \cdot x = x_i \geq 0. \quad (\text{A11})$$

The Weyl character for an irreducible representation of highest weight Λ is defined as

$$\text{ch}_\Lambda(a) = \sum_{\mu \in M_\Lambda} e^{a \cdot \mu}, \quad (\text{A12})$$

where $a = a_i \lambda^{(i)}$. The Weyl character satisfies the equation [34],

$$\text{ch}_\Lambda(a) = \frac{\sum_{w \in W} \epsilon(w) e^{w(\Lambda + \rho) \cdot a}}{\sum_{w \in W} \epsilon(w) e^{w(\rho) \cdot a}}, \quad (\text{A13})$$

known as the Weyl character formula.

A2 LEVEL l THETA FUNCTIONS.

Level l theta functions play a fundamental role in the construction of the Hilbert space carried out in sect. 2. They are defined as follows:

$$\Theta_{l,p}(\tau, a) = \sum_{\nu \in \Lambda_R} \exp\{i\pi\tau l(\nu + \frac{p}{l})^2 + 2\pi i l(\nu + \frac{p}{l}) \cdot a\}. \quad (\text{A14})$$

These functions are well defined for $\text{Im}\tau > 0$, which makes the sum convergent. We will consider the case in which p belongs to Λ_W . The theta functions (A14) satisfy several properties. The first one, which is rather manifest from their definition is the following. Under a shift of p by a vector in $l\Lambda_R$ (A14) remains unchanged,

$$\Theta_{l,p+l\alpha}(\tau, a) = \Theta_{l,p}(\tau, a), \quad \alpha \in \Lambda_R, \quad (\text{A15})$$

which means that the index p in $\Theta_{l,p}(\tau, a)$ lives in the domain, $p \in \Lambda_W/l\Lambda_R$. Another important property is the following. Let us consider m and n as two vectors in Λ_R , $m, n \in \Lambda_R$. Then,

$$\Theta_{l,p}(\tau, a + m + n\tau) = e^{-ixl\tau n^2 - 2\pi i l n \cdot a} \Theta_{l,p}(\tau, a). \quad (\text{A16})$$

Notice that the prefactor is independent of m and p . This fact is essential in searching for the solution to (2.52) leading to (2.53).

Of particular importance in our analysis are the Weyl-antisymmetrized combinations of level l theta functions. Let us define

$$\Theta_{l,p}^A(\tau, a) = \sum_{w \in W} \epsilon(w) \Theta_{l,w(p)}(\tau, a), \quad (\text{A17})$$

where $\epsilon(w)$ is the signature of the permutation corresponding to the element w of the Weyl group W . These functions satisfy the property

$$\Theta_{l,p}^A(\tau, a) = \epsilon(w) \Theta_{l,w(p)}^A(\tau, a), \quad (\text{A18})$$

i.e., they are Weyl-antisymmetric. This property implies some relations among the antisymmetrized level l theta functions. The standard choice of independent

weights $p \in \Lambda_W$ for these functions is the following,

$$p_i > 0, \quad \sum_{i=1}^r p_i < l, \quad (\text{A19})$$

which become a total of $(l-1)!/(l-N)!(N-1)!$. We will denote the domain of the weights satisfying (A19) as the fundamental chamber $\mathcal{F}_l$.

Finally we will recall the behavior of the level l theta functions under modular transformations. The modular group is generated by two transformations S and T . The transformation S is defined as follows:

$$\begin{aligned} \tau &\rightarrow -\frac{1}{\tau}, \\ a &\rightarrow \frac{a}{\tau}, \end{aligned} \quad (\text{A20})$$

while T is defined by:

$$\begin{aligned} \tau &\rightarrow \tau + 1, \\ a &\rightarrow a. \end{aligned} \quad (\text{A21})$$

Under these generating transformations the level l theta functions turn out to transform as

$$\Theta_{l,p}\left(-\frac{1}{\tau}, \frac{a}{\tau}\right) = \frac{1}{\text{Vol}(\Lambda_R)} \left(\frac{\tau}{il}\right)^{\frac{l}{2}} e^{\frac{i\pi l}{2} a^2} \sum_{q \in \Lambda_W/l\Lambda_R} e^{-2\pi i \frac{p \cdot q}{l}} \Theta_{l,q}(\tau, a), \quad (\text{A22})$$

and,

$$\Theta_{l,p}(\tau + 1, a) = e^{i\pi \frac{p^2}{l}} \Theta_{l,p}(\tau, a). \quad (\text{A23})$$

In (A22) $\text{Vol}(\Lambda_R)$ is the volume of the fundamental cell of the lattice Λ_R . For $SU(N)$, $\text{Vol}(\Lambda_R) = \sqrt{N}$.

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FIGURE CAPTIONS

- Fig. 1. A Heegaard splitting: the three-manifold M is built as the connected sum of the two three-manifolds M_1 and M_2 , joined along their common boundary Σ .
- Fig. 2. The wave functionals $\Psi_1[A_{\vec{x}}]$ and $\Psi_2[A_{\vec{x}}]$ are obtained by integrating over gauge configurations on M_1 and M_2 respectively.
- Fig. 3. A solid ball can be regarded as the superposition of a continuous set of concentric spherical layers. For the inner layers it is possible to choose a gauge invariant measure for the gauge fields. On the contrary, this is not possible on the boundary.
- Fig. 4. Homology cycles for the torus.
- Fig. 5. In the $S^2 \times S^1$ three-manifold the S^1 direction can be taken as time. The partition function becomes the trace of the time evolution operator which can be normalized to 1 for a topological field theory.
- Fig. 6. A Wilson line lying on the boundary of a solid torus can be continuously moved to the interior of the handlebody.
- Fig. 7. a) Knot of type 1,0 on the surface of a solid torus; b) Knot of type 1,1 on the surface of a solid torus.
- Fig. 8. a) Ribbon of type 1,0 on the surface of a solid torus; b) Ribbon of type 1,1 on the surface of a solid torus.
- Fig. 9. a) Framed unknot in S^3 in the standard framing; b) Framed unknot in S^3 with one twist.

FIGURE 1

FIGURE 2

FIGURE 3

FIGURE 4

FIGURE 5

a)

b)

FIGURE 7

a)

b)

FIGURE 8

FIGURE 6

b)

a)

FIGURE 9