FREE FIELD REPRESENTATION FOR $\widehat{SU}(2)_k$ WZNW MODELS ON RIEMANN SURFACES

M. Frau

INFN, Sezione di Torino, Cso M. D'Azeglio 46, I-10125 Torino, Italy

A. Lerda ^a

*Center for Theoretical Physics,
Laboratory for Nuclear Science and Department of Physics,
Massachusetts Institute of Technology, Cambridge, MA 02139, USA*

J.G. McCarthy ^b

*Department of Physics,
Brandeis University, Waltham, MA 02254, USA*

S. Sciuto

*Dipartimento di Scienze Fisiche dell' Università di Napoli and
INFN, Sezione di Napoli, Mostra d' Oltremare Pad. 19, I-80125 Napoli, Italy*

and

J. Sidenius

Theory Division, CERN, CH-1211 Geneva 23, Switzerland

Abstract

We give the extension of the free field representation of $\widehat{SU}(2)_k$ WZNW model conformal blocks to an arbitrary Riemann surface. In particular, we avoid the problems associated with the vertex corresponding to the dual of a given representation - namely that it does not transform as the appropriate highest weight field of the current algebra - in a way allowing the sewing procedure to be applied. After analysing the 2-point function on the torus as an illustration of some of the subtleties in this free field representation, we go on to give the genus-two characters as an application.

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The Feigin-Fuchs free field representation [1,2] for the minimal models of the Virasoro algebra (Vir), was recently recast [3] in a "BRST" language which has allowed the extension to higher genus Riemann surfaces [4,5,6,7,8,9]. A similar free field description [10,11,12] and its "BRST" formulation [13] have been given also for the $SU(2)_k$ WZNW model. In this letter we discuss the extension to higher genus of such a model and the related parafermion system [14,15] by means of the sewing procedure [16].

The BRST formulation of the $SU(2)_k$ WZNW model follows almost precisely that of the Vir minimal models and defines a systematic projection by which irreducible representations are obtained from appropriate free field Fock spaces. Each such Fock space has a "dual" with respect to which we may define an inner product. For the Vir minimal models both the Fock space and its dual project under BRST to the same irreducible representation, and thus the corresponding vertices may be used interchangeably. This is no longer the case for $SU(2)_k$, and in fact the current algebra properties of the dual Fock space are not immediately apparent. However, the freedom of using dual vertices was crucial for constructing multiloop conformal blocks of the Vir minimal models [4]. Thus the extension to higher genus is no longer straightforward for $SU(2)_k$.

This problem has been emphasized in Ref. [17] where a solution is given and discussed at tree level. However, the treatment of Ref. [17] requires a prescription to define the inverse of a first-order field and does not seem to extend beyond the sphere. In this letter we give an alternative remedy which completely solves the problem. That is, we construct an operator which maps to the highest weight representation from its dual, and is just the identity when acting on the cohomology class corresponding to the irreducible representation. Incorporating this into the sewing procedure, we are able to give the correlators on higher genus Riemann surfaces.

We begin with a quick review of the Wakimoto free field description of $SU(2)_k$ following Ref. [13]. We present the algebra, the BRST structure which projects out the irreducible representation from a given free field Fock space, and the BRST invariant realization of the chiral vertex operators through which we may construct conformal blocks. After bosonizing, we discuss the restriction to the $SU(2)/U(1)$ coset model (parafermions) [14,15], and verify the fusion rules. This restriction was also considered in [18], but the procedure given there has not been shown to be correct - see the remarks in [19] where the problem is reconsidered. At 1 loop our treatment agrees with the one of Ref. [19], but it is somewhat more streamlined. Moreover it can be applied to generalized parafermions and other coset models [20]. Next we introduce the problem of the conjugate representation within the sewing procedure of the operator formalism [16], and give its resolution as discussed above. The 2-point function at 1 loop serves to exhibit most of the subtleties of this representation, and then we conclude with the example of the 2-loop characters of $SU(2)_k$.

In the Wakimoto construction [10] the $SU(2)_k$ WZNW model is described in terms of one

free boson φ coupled to a background charge $Q = \sqrt{\frac{2}{k+2}}$ and a first order bosonic (β, γ) system of weight $(1,0)$. If we choose the normalization of the fields in such a way that

$$\begin{aligned}\beta(z)\gamma(w) &= \frac{-1}{z-w} + \beta(z)\gamma(w) : \\ \varphi(z)\varphi(w) &= \ln(z-w) + \varphi(z)\varphi(w) : \end{aligned} \quad (1)$$

where

$$\varphi(z) = \hat{q} - \hat{p} \ln z + \sum_{n \neq 0} \frac{\hat{\alpha}_n}{n} z^{-n}, \quad (2)$$

then the currents $J^\alpha(z)$ and the Sugawara energy-momentum tensor $T(z)$ are given by

$$\begin{aligned}J^+(z) &= -\beta(z), \quad J^-(z) = \beta(z)\gamma^2(z) + \sqrt{2(k+2)} \partial\varphi(z)\gamma(z) + k \partial\gamma(z) \\ J^0(z) &= \beta(z)\gamma(z) + \sqrt{\frac{k+2}{2}} \partial\varphi(z) \\ T(z) &= \frac{1}{2} : \partial\varphi(z) \partial\varphi(z) : - \frac{1}{2} \sqrt{\frac{2}{k+2}} \partial^2\varphi(z) - \beta(z) \partial\gamma(z) : \end{aligned} \quad (3)$$

where k is the level of the $SU(2)$ Kac-Moody algebra.

The Fock space F_j , $0 \leq j \leq \frac{k}{2}$, is built by the creation modes of the fields on the vacuum $|j\rangle$ ($SU(2, \mathbb{C})$ invariant vacuum for (β, γ) , and $\beta|j\rangle = -j \sqrt{\frac{2}{k+2}} |j\rangle$) and carries a spin j representation of the $SU(2)_k$ algebra with L_0 -weight $h_j = \frac{j(j+1)}{k+2}$. The highest weight field for such a representation is simply the vertex operator $V_{j,j}(z) = e^{\sqrt{\frac{2}{k+2}} \varphi(z)}$. The other primary fields of the $SU(2)_k$ representation at the weight h_j are obtained by lowering $V_{j,j}$ with J^- and are given by

$$V_{j,m}(z) = : [\gamma(z)]^{j-m} : V_{j,j}(z) \quad (4)$$

where $-j \leq m \leq j$ and $m-j \in \mathbb{Z}$. One can easily check that

$$\begin{aligned}J^+(z) V_{j,m}(w) &= \frac{j+m}{z-w} V_{j,m+1}(w) + \dots \\ J^0(z) V_{j,m}(w) &= \frac{m}{z-w} V_{j,m}(w) + \dots \end{aligned} \quad (5)$$

As shown in Ref. [13], the representation carried by F_j with $2j \in \mathbb{N}$ is reducible. Using the screening operator

$$S(z) = \beta(z) : e^{-\sqrt{\frac{2}{k+2}} \varphi(z)} : \quad (6)$$

whose OPEs with the generators of the algebra are total derivatives, the algebra-invariant intertwiner (or BRST charge) $Q_l : F_j \rightarrow F_{j-1}$ is defined via

$$Q_l = \frac{1}{l} \left(\frac{q^l - 1}{q - 1} \right) \oint_0^{q-1} d\varphi_0 \dots d\varphi_{l-1} S(\varphi_0) \dots S(\varphi_{l-1}) \quad (7)$$

where $l = 2j + 1$, $q = e^{\frac{2\pi i}{L}}$, and $C_{(m)}^{(l-1)}$ are the standard nested contours [3]. In fact a whole BRST complex of such operators may be built

$$\dots F^{(l-1)} = F_{-j-1+(t+2)} \xrightarrow{Q^{(l-1)=Q_L}} F^{(l)} = F_j \xrightarrow{Q^{(l)=Q_L}} F^{(l+1)} = F_{-j-1} \xrightarrow{Q^{(l+1)=Q_L}} F^{(l+2)} = F_{j-(t+2)} \dots \quad (8)$$

where $l = 2j + 1$, $L = k + 2 - l$, and $Q_L Q_L = Q_L Q_L = 0$. It was proven in Ref. [13] that the natural cohomology of this complex is trivial except in the middle Fock space, where it is isomorphic to the irreducible representation $\mathcal{H}_j$,

$$\mathcal{H}_j \simeq \frac{\text{Ker } Q^{[0]}}{\text{Im } Q^{[0]}} \subset F_j \quad (9)$$

Therefore, given any operator O defined on F_j , one can compute its trace over $\mathcal{H}_j$ by means of the Lefschetz principle

$$\text{Tr}_{\mathcal{H}_j} O = \sum_{n=-\infty}^{\infty} (-1)^n \text{Tr}_{F^{(n)}} O^{[n]}, \quad O^{[n]} Q^{[n-1]} = Q^{[n-1]} O^{[n-1]} \quad \text{with } O^{[0]} \equiv O. \quad (10)$$

The screened vertex operator,

$$V_{j,m}^{(j)}(z) \equiv \int_{\mathcal{P}^r} dv_1 \dots dv_r \quad V_{j,m}(z) S(v_1) \dots S(v_r), \quad (11)$$

clearly transforms in the spin j representation. The integration paths, $\mathcal{P}^r$, are given by

$$\mathcal{P}^r = \prod_{i=1}^r \left(\frac{q^{2i_r - i + 1} - 1}{q^{i - 2j - 1} - 1} \right) C^r \quad (12)$$

where C^r are the closed contours around 0 and z introduced in Ref. [6], but with the normalization ¹ used in the second paper of Ref. [4] (see e.g. Eq. (2.17)). In particular the operator $V_{j,m}^{(j+2j-2j)} : F_{j_a} \rightarrow F_{j_b}$ satisfies

$$Q_{j_b+1} V_{j,m}^{(j+2j-2j)}(z) = q^{-j(2j_b+1)} V_{j,m}^{(j-j_a+2j_b)}(z) Q_{j_a+1}, \quad (13)$$

and so it restricts to a map between cohomology classes. Thus it is a representative of the chiral vertex operator $\Phi_{j_b, j_a}^{j_b} : \mathcal{H}_{j_a} \rightarrow \mathcal{H}_{j_b}$. Indeed in Ref. [13] it was checked that $V_{j,m}^{(j+2j-2j)}$ is consistently normalizable, precisely when the fusion rules allow the vertex $\Phi_{j_b, j_a}^{j_b}$. Using Eq. (13), the recursive definition (10) allows the computation of 1-loop conformal blocks.

In order to extend the calculation to higher genus Riemann surfaces, it is convenient to use the bosonized representation of the (β, γ) system [21]

$$\beta = \partial \xi \, e^{i\sigma} = -\partial \chi \, e^{-\chi+i\sigma}, \quad \gamma = \eta \, e^{-i\sigma} = e^{\chi-i\sigma} \quad (14)$$

¹The prefactor in Eq. (12) has been chosen such that $\mathcal{P}^r$ reduce to the standard nested contours, $C_{(j)}$, when the paths around z are shrunk to a point.

where (η, ξ) is a first-order fermionic conjugate system with spin $(1, 0)$, and χ is its bosonization. The free fields σ and χ defined by this representation have *a priori* any momentum with the constraint $i\hat{p}_0 - \hat{p}_\chi \in \mathbb{Z}$ (since β and γ are single valued). However, as discussed in Ref. [21], to recover the original β - γ representation space after bosonizing, we must fix the $Sl(2, \mathbb{C})$ "picture" by the additional constraint $i\hat{p}_0 + \hat{p}_\chi = 0$ and also project onto the subspace of states without the zero-mode ξ_0 - i.e. onto $\text{Ker } \eta_0$ since $\{\eta_0, \xi_0\} = 1$. This was applied to compute the characters in Ref. [22].

Adopting notations similar to those of Ref. [19], we can equivalently describe our system with three bosonic fields ϕ_0, ϕ_1, ϕ_2 , defined as linear combinations ² of φ, σ and χ in such a way that only ϕ_1 is coupled to a background charge $Q = \sqrt{\frac{2}{k+2}}$. With these new fields,

$$\begin{aligned} J^\pm(z) &= \frac{1}{\sqrt{2}} \left[\sqrt{k+2} \, \partial \phi_1(z) \pm i\sqrt{k} \, \partial \phi_2(z) \right] e^{\pm \sqrt{\frac{2}{k}} (i\phi_2(z) - \phi_0(z))} \\ J^0(z) &= -\sqrt{\frac{k}{2}} \, \partial \phi_0(z) \\ T(z) &= \frac{1}{2} \left[(\partial \phi_0(z))^2 + (\partial \phi_1(z))^2 + (\partial \phi_2(z))^2 \right] - \frac{1}{2} \sqrt{\frac{2}{k+2}} \, \partial^2 \phi_1(z). \end{aligned} \quad (15)$$

The screening operator $S(z)$ defined in Eq. (6) now becomes

$$S(z) = \frac{1}{\sqrt{2}} : \left[\sqrt{k+2} \, \partial \phi_1(z) + i\sqrt{k} \, \partial \phi_2(z) \right] e^{-\sqrt{\frac{2}{k+2}} \phi_1(z)} : , \quad (16)$$

while the vertex operator $V_{j,m}$ of Eq. (4) becomes

$$V_{j,m}(z) = : e^{i \left[\sqrt{\frac{2}{k+2}} \phi_1(z) + i\sqrt{k} \phi_2(z) - \sqrt{\frac{2}{k}} \phi_0(z) \right]} : \quad (17)$$

The "physical" fields have $j = 0, \frac{1}{2}, \dots, \frac{k}{2}$; $-j \leq m \leq j$ and $m - j \in \mathbb{Z}$. However in the following we will need more general free field exponential vertices, so we introduce the notation

$$V_{j,m',m}(z) \equiv : e^{i \left[\sqrt{\frac{2}{k+2}} \phi_1(z) + i\sqrt{k} \phi_2(z) - \sqrt{\frac{2}{k}} \phi_0(z) \right]} : \quad (18)$$

The screened vertex operators $V_{j,m,m}^{(r)}$ are defined as in Eq. (11) with $V_{j,m}$ replaced by $V_{j,m',m}$. In particular

$$V_{\frac{k}{2}+1, \pm \frac{k}{2}, 0}(z) \equiv S_\pm(z) \quad (19)$$

are the operators introduced in Ref. [19] where they were shown to be screening operators. We denote by $Q_\pm$ the corresponding charges, i.e. $Q_\pm = \int_0^1 dz S_\pm(z)$. Since S_+ is just the bosonization of η in Eq. (14), $Q_+ = \eta_0$. Further we denote as $F_{j,m',m}$ the Fock space ³ obtained by acting with the creation modes of ϕ_0, ϕ_1 and ϕ_2 on the vacuum $|j, m', m\rangle \equiv \lim_{x \rightarrow 0} V_{j,m',m}(x)|0\rangle$.

² $\phi_0 = i\sqrt{\frac{2}{k}} \sigma - \sqrt{\frac{2}{k+2}} \varphi$; $\phi_1 = -i\sqrt{\frac{2}{k+2}} \sigma + \sqrt{\frac{2}{k+2}} \chi + i\sqrt{\frac{2}{k}} \chi + i\sqrt{\frac{2}{k+2}} \varphi$

³To simplify the notation, we will use the shorthand $F_{j,m}$ for $F_{j,m,m}$ and $|j, m\rangle$ for $|j, m, m\rangle$.

Notice that (anomalous) conservation of momentum of the fields implies that the dual to $F_{j,m',m}$ is the space $F_{-j-1,-m',-m}$. The charges $Q_{\pm}$ are well defined on any Fock space and map $F_{j,m',m}$ into $F_{j+\frac{1}{2},m',\pm\frac{1}{2},m'}$.

As stated above, the original representation space F_j is obtained from the direct sum of the free field Fock spaces $F_{j,m',m}$ with arbitrary integer or half-integer momenta m', m for ϕ_2 and ϕ_0 . Since the constraint $\hat{p}_0 + \hat{p}_k = 0$ now reads $m' = m$, one gets

$$F_j = \bigoplus_{m \in \mathbb{Z} + \frac{1}{2}} \text{Ker } Q_+(F_{j,m}) . \quad (20)$$

Of course $Q_+^2 = 0$, and it has trivial cohomology on all Fock spaces (since η_0 has conjugate ξ_0). Thus, by iteration, we find that for any operator $\mathcal{O}$ defined in $F_{j,m}$

$$\text{Tr}_{|\text{Ker } Q_+(F_{j,m})} \mathcal{O} = \sum_{s=0}^{\infty} (-1)^s \text{Tr}_{j+s\frac{1}{2},m+\frac{1}{2},m} \mathcal{O}_{[s]} \quad (21)$$

$$\sum_{s=-\infty}^{\infty} (-1)^s \text{Tr}_{j+s\frac{1}{2},m+\frac{1}{2},m} \mathcal{O}_{[s]} = 0 \quad (22)$$

where the symbol $\text{Tr}_{j,m}$ denotes the trace over the Fock space $F_{j,m}$ and the operators $\mathcal{O}_{[s]}$ are defined recursively by $\mathcal{O}_{[0]}Q_+ = Q_+\mathcal{O}_{[0]}$ and $\mathcal{O}_{[0]} \equiv \mathcal{O}$. Actually, as we will show, in the cases of interest for calculations of conformal blocks, Eqs. (21) and (22) can be simplified because $\mathcal{O}_{[j]} = \mathcal{O}$ for any s . Furthermore, we note that

$$Q_+Q_n + (-1)^n Q_n Q_+ = 0, \quad Q_+V_{j,m'}^{(r)} = (-1)^{j-m'-r} V_{j,m'}^{(r)} Q_+ . \quad (23)$$

At this stage we are able to give the restriction of this free field representation to that of the Z_k parafermion model [14]. Dropping the ϕ_0 -exponential from $J^{\pm}$ in Eq. (15), we obtain operators ψ_1 and $\psi_1^{\dagger}$ which generate the parafermion algebra. Similarly, the spectrum of conformal dimensions for highest weight parafermion fields is correctly reproduced by Eqs. (17) and (15) without the ϕ_0 contribution. Furthermore, since $S(z)$ is independent of ϕ_0 , Eqs. (16) and (7) define respectively the screening operator and the BRST charge of the parafermion theory. Moreover, the projection onto $\text{ker } Q_+$ can be made on the restricted set ϕ_1 and ϕ_2 , since $S_+(z)$ does not depend on ϕ_0 . Therefore, a free field representation for the parafermion system is built from ϕ_1 and ϕ_2 , and the BRST structure of the $\widehat{SU}(2)_k$ model restricts naturally to this system. The cohomology statement analogous to (9) still holds since in dropping ϕ_0 , we are effectively just taking a fixed isospin "slice" of the $\widehat{SU}(2)_k$ complex. In particular, the screened vertex operators defined as in Eq. (11) are representatives of the chiral vertex operators mapping between cohomology classes.

As a check on the bosonized formalism, we now explicitly verify that the chiral vertex operators defined by the screened vertex operators are normalizable precisely when allowed by the fusion rules. The normalization is given by

$$N_{j,m_0,m_s}^{j_1,m_1} \equiv \langle j_1, m_1 | V_{j,m}^{(j_1+j_2-j_3)} | j_2, m_2 \rangle \quad (24)$$

where $m_0 = m_a + m$. Since the screenings do not depend on ϕ_0 , we may drop ϕ_0 altogether in this calculation. Clearly, by group invariance we may restrict ourselves to the case $m = j$. Substituting Eqs. (11), (16) and (17) into Eq. (24), one obtains

$$N_{j_1 j_2 j_3 m_s}^{j_1 m_1} = q^{\frac{(j_1+j_2-j_3)(j_1+j_2-j_3)}{2}} \prod_{i=1}^{j_1+j_2-j_3} \left[\frac{(m_a - j_a + i - 1)(1 - q^{2i_a - 1})}{(1 - q)} \right. \\ \left. \times \frac{\Gamma(\frac{1}{k+2}) \Gamma(\frac{1-2j_a}{k+2}) \Gamma(\frac{1-2j}{k+2} + 1)}{\Gamma(\frac{1}{k+2}) \Gamma(1 + \frac{1-j_1-j_2-j_3}{k+2})} \right] \quad (25)$$

where $q = e^{\frac{2\pi i}{k+2}}$, from which indeed the well-known parafermion fusion rules [14] follow. By reinstating ϕ_0 we see also that the bosonized formalism gives the correct fusion rules of the $\widehat{SU}(2)_k$ WZNW model [13].

We now turn to the explicit construction of conformal blocks on higher genus Riemann surfaces. In the operator formulation this requires the preparation of tree graphs, in which legs to be sewn into loops lie in dual Fock spaces (see e.g. Refs. [16,4]). The seeds of a problem can be seen already for conformal blocks on the sphere. Here without loss of generality, we can fix one leg to be at the point at infinity and then we simply compute a matrix element of a string of vertex operators. However, the vertex creating the state at infinity, i.e. $\langle -j - 1, -m |$, is $V_{-j-1,-m}^{(0)}$, which does not seem to qualify as a representative of a chiral vertex by the nature of its current-algebra transformation properties. On the torus we can fix again the conjugate leg at infinity, and then we are just required to compute a trace. However at genus two (or higher) there is no longer the freedom to choose such special configurations and the problem has to be faced directly.

The simplest statement of this problem is that, as shown in Fig. 1, on constructing a "bubble" by the 1-loop sewing, the external legs (A and B) have quantum numbers in the physical range and cannot be sewn together to form a second loop. Indeed, following the methods of Refs. [16,4], the 2-point 1-loop vertex of Fig. 1, which upon contraction with two states generates the corresponding amplitude on the torus, has the structure

$$A \langle -j_A - 1, -m_A | B \langle -j_B - 1, -m_B | W_{A,B} ,$$

i.e. it is an operator defined on $F_{j_A, m_A} \otimes F_{j_B, m_B}$. In order to sew together its two legs and form a second loop, we should first take the Fock space conjugate on leg B , i.e.

$$A \langle -j_A - 1, -m_A | W_{A,B}^{\dagger} | -j_B - 1, -m_B \rangle_B ,$$

identify A and B , and then compute a trace. But clearly the charge does not balance. The same problem prevents sewing together two "bubbles" directly. For the Vir minimal models this problem is easily solved by using the dual representation for either one of the two legs and by inserting also a screened dual identity operator for the even genus characters [4]. Here, a dual identity operator is not immediately available; however the problem is completely solved

if we can construct an operator $\tilde{I}_j : F_{-j-1} \rightarrow F_j$, which commutes with all generators of the algebra⁴ and thus reduces to the identity operator on the irreducible representation. Such an operator was introduced in Ref. [17] within the context of the Wakimoto representation. However, as we have already mentioned, it contains the inverse of a first order field and hence it is poorly defined beyond the sphere. As we shall now prove, a well-defined operator with the same properties can be defined in the fully bosonized representation and it is given by

$$\tilde{I}_j(z) = \oint_{\gamma_0} dv_- S_-(v_-) V_{-1,0}^{(k-2j)}(z) \oint_0^z dv_+ S_+(v_+) \quad (26)$$

where γ_0 are defined in Eq. (19) and the symbol $V_{-1,0}^{(k-2j)}$ denotes an integration contour encircling both the point x and the origin.

Using Eq. (5) and the fact that the algebra generators commute with the screening charges, it is clear that $\tilde{I}_j(z)$ has L_0 -weight 0 and also J_0^0 -weight 0. In the same way we find that

$$J^+(w) \tilde{I}_j(z) = \frac{-1}{w-z} \left[\oint_{\gamma_0} dv_- S_-(v_-) V_{-1,1}^{(k-2j)}(z) \oint_0^z dv_+ S_+(v_+) \right] + \dots$$

Now we observe that S_- commutes with $V_{-1,1}^{(k-2j)}$, leaving us with $Q_- Q_+$ on the right which certainly annihilates the vacuum state in F_{-j-1} and thus also all states obtained from it by applying the operators of the current algebra. Hence, J^+ commutes with $\tilde{I}_j$ on the irreducible representation and we are left only to prove that the same is true for J^- . We find that

$$\begin{aligned} J^-(w) \tilde{I}_j(z) &= \frac{-1}{w-z} \left[\oint_{\gamma_0} dv_- S_-(v_-) V_{-1,-1}^{(k-2j)}(z) \oint_0^z dv_+ S_+(v_+) \right] + \dots \\ &\equiv \frac{-1}{w-z} \Psi(z) + \dots \end{aligned} \quad (27)$$

This time S_- does not commute with $V_{-1,-1}^{(k-2j)}$; however, we may follow the same line of argument used in Ref. [17]. In fact, from Eq. (5) it is easy to check that

$$\begin{aligned} J^+(w) \Psi(z) &= 0 + \dots, \quad J^0(w) \Psi(z) = \frac{-1}{w-z} \Psi(z) + \dots, \\ J^-(w) \Psi(z) &= \frac{-2}{w-z} \left[\oint_{\gamma_0} dv_- S_-(v_-) V_{-1,-2}^{(k-2j)}(z) \oint_0^z dv_+ S_+(v_+) \right] + \dots \end{aligned} \quad (28)$$

These equations show that Ψ is a null operator (cf. Ref. [17]) which therefore decouples in any conformal block for physical fields. Hence we can conclude that $\tilde{I}_j$ effectively commutes with all the currents J^a 's.

We remark that Eq. (26) can be rewritten as

$$\begin{aligned} \tilde{I}_j(z) &= \oint_{\gamma_0} dv_- S_-(v_-) V_{-1,0}^{(k-2j)}(z) Q_+ + V_{-1,0}^{(k-2j)}(z) Q_- Q_+ \\ &= V_{-\frac{1}{2}-\frac{k}{2},0}^{(k-2j)}(z) Q_+ + V_{-1,0}^{(k-2j)}(z) Q_- Q_+ \end{aligned} \quad (29)$$

where we have used the notation introduced in Eq. (18). As noted above, the $Q_- Q_+$ term vanishes on physical states and can be dropped. Thus in the following, we will use only the first

term of Eq. (29). Since the BRST charges Q_i do not depend on the field ϕ_0 , the operator $\tilde{I}_j$ transforms under BRST as the vertex $V_{-\frac{1}{2}-\frac{k}{2}}^{(k-2j)}$ (cf. Eq. (13) with $j \rightarrow -\frac{k}{2}$, $m \rightarrow -\frac{k}{2}$, $j_A \rightarrow -\frac{k}{2} - j$ and $j_B \rightarrow j$). Moreover since Q_+ commutes with $V_{-\frac{1}{2}-\frac{k}{2}}^{(r)}$ and $Q_+^2 = 0$, one has also $Q_+ \tilde{I}_j = \tilde{I}_j Q_+ = 0$. Therefore the insertion of $\tilde{I}_j$ in the neck between two handles does not introduce any spurious state.

As an illustration of the above, we will write down the general 2-loop characters of the $SU(2)_k$ WZNW model. But first, to exhibit several subtleties of this representation, we consider the 2-point 1-loop amplitude $A_{j_A j_B}^{(j)}$ represented in Fig. 1. This, together with the tadpole $A_{j_A}^{(j)}$ of Fig. 2, is the building block for multiloop correlation functions as presented in Fig. 3. In this configuration, momentum conservation for ϕ_0 and ϕ_1 requires that the m_i 's on the legs joining two "bubbles" must vanish; thus for our purposes it is enough to consider $m_A = m_B = 0$ in Fig. 1. Then, following steps similar to those described in Ref. [4] for the Vir minimal models, Fig. 1 is represented before sewing by the $U(1)$ -valued operator

$$O_{j_A j_B}(\tau, w) = W_{j_A,0}^{(r_A)} W_{j_B,0}^{(r_B)} P^{A,B}(\tau) e^{2\pi i j_0^0 w} \quad (30)$$

where $r_A = j_A - j + \bar{j}$ and $r_B = j_B + j - \bar{j}$. In Eq. (30) $P^{A,B}(\tau)$ (e.g. $e^{2\pi i r_L \tau}$)⁴ is the sewing operator dependent on the modulus τ , and $W_{j,m}^{(r)}$ is the screened W -Vertex [4] which generalizes the vertex operator $V_{j,m}^{(r)}$ for the emission of arbitrary excited states. It satisfies a generalized BRST equation (see for instance Eq. (4.2) of Ref. [4]) in the case of the Vir minimal models) but for the configurations we have chosen, it is enough to consider simply Eq. (13) with V replaced by W (since W will always be contracted with physical states [9]). The loop is computed by taking the trace of $O_{j_A j_B}$ over $\mathcal{H}$; according to Eq. (10). From Eq. (13), we have for $n \in \mathbb{Z}$

$$\begin{aligned} O_{j_A j_B}^{[2n]} &= W_{j_A,0}^{(r_A)} W_{j_B,0}^{(r_B)} P^{A,B}(\tau) e^{2\pi i j_0^0 w} \equiv \mathcal{W}_0(\tau, w) \\ O_{j_A j_B}^{[2n+1]} &= q^{-\theta} W_{j_A,0}^{(r_A)} W_{j_B,0}^{(r_B)} P^{A,B}(\tau) e^{2\pi i j_0^0 w} \equiv q^{-\theta} \mathcal{W}_1(\tau, w) \end{aligned} \quad (31)$$

where $r'_A = r_A + 2j - 2\bar{j}$, $r'_B = r_B - 2j + 2\bar{j}$ and $\theta = j_A(2j + 1) + j_B(2\bar{j} + 1)$. Further, in the bosonized description we have to project onto the $\text{Ker } \eta_0$ as in Eq. (21). One easily checks from Eqs. (23) and (31) that $O_{[i]}^0 = O_{[i]}^0 \forall i \in \mathbb{Z}$. Therefore, from Eqs. (10), (20) and (21) we finally obtain

$$\begin{aligned} A_{j_A j_B}^{(j)}(\tau, w) &\equiv \text{Tr } O_{j_A j_B}(\tau, w) \Big|_{\mathcal{H}}, \\ &= \sum_{\gamma=0}^1 (-q^{-\theta})^\gamma \left[\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (-1)^n \text{Tr}_{j_A \gamma + (n+\frac{k}{2})(k+2), m+\frac{k}{2}, m} \mathcal{W}_\gamma(\tau, w) \right] \end{aligned} \quad (32)$$

where $j^{\gamma=0} = j$ and $j^{\gamma=1} = -j - 1$. The operators $\mathcal{W}_\gamma$ in Eq. (32) contain the exponential of a quadratic form in the loop momenta, and the trace over their zero modes essentially factorizes.

⁴No twist operator is needed in the configuration of Fig. 1, differently from that of Ref. [4].

Thus it is useful to introduce the lattice sum

$$\Gamma_j(\xi_1, \xi_2, \xi_3) \equiv \sum_{m=-\infty}^{\infty} \sum_{n=-j \in \mathbb{Z}} \Gamma_{j,m}(\xi_1, \xi_2) e^{2\pi i k(\frac{m}{2})} e^{2\pi i \frac{n}{2} \xi_3} \quad (33)$$

where $\Gamma_{j,m}$ is the parafermion contribution given by

$$\begin{aligned} \Gamma_{j,m}(\xi_1, \xi_2) &= \sum_{n=-\infty}^{\infty} \sum_{j=0}^{\infty} (-1)^n e^{2\pi i n \left[(k+2)(\frac{k}{k+2} + n + \frac{1}{2})^2 - k(\frac{n}{2} + \frac{1}{2})^2 \right]} \\ &\quad \times e^{2\pi i \left[(\frac{k}{k+2} + n + \frac{1}{2}) \xi_1 - (\frac{n}{2} + \frac{1}{2}) \xi_2 \right]} \end{aligned} \quad (34)$$

We will show that $\Gamma_{j,m}$ and $\Gamma_{j,m+k}$ give the same contribution inside the conformal blocks. Therefore Eq. (33) can be rewritten as

$$\Gamma_j(\xi_1, \xi_2, \xi_3) = \sum_{m=0}^{k-1} \sum_{n=-\infty}^{\infty} \sum_{m-j \in \mathbb{Z}} \Gamma_{j,m}(\xi_1, \xi_2) e^{2\pi i n k(\frac{m}{2} + n)} e^{(\frac{m}{2} + n) \xi_3} \quad (35)$$

The remaining trace over the non-zero modes in Eq. (32) is easily computed and can be conveniently summarized by a $(2 + \tau_A + \tau_B)$ -point 1-loop vertex as in Ref. [16]. For simplicity we label the legs of such a vertex with capital indices $I, L, \dots$, whereas lower case indices $i, l, \dots$ in the range $(1, \dots, \tau_A + \tau_B)$ are used to label the screening legs. When the external legs A and B are saturated with highest weight states and the screening legs are saturated with the states $i\partial\phi_2^{(j)}(0) - 1, 0, 0$, the corresponding 1-loop amplitude is given by

$$A_{j,AB}^{(j)}(\tau, w) \Big|_{h,w} = \sum_{m=-\infty}^{\infty} \sum_{m-j \in \mathbb{Z}} \tilde{A}_{j,AB}^{(j,m)}(\tau) e^{2\pi i n k(\frac{m}{2})} e^{2\pi i \frac{n}{2} w} \quad (36)$$

$$\tilde{A}_{j,AB}^{(j,m)}(\tau) = \prod_i \frac{\partial}{\partial y_i} \left(\mathcal{A}_{j,AB}^{(j,m)}(\tau | y_i) \right) \Big|_{y_i=0} \quad (37)$$

$$\begin{aligned} \mathcal{A}_{j,AB}^{(j,m)}(\tau | y_i) &= \left(\frac{e^{\frac{\pi i}{2}}}{\eta(\tau)} \right)^3 \sum_{\gamma=0}^1 (-q^{-\theta})^\gamma \int_{\mathcal{P}_\gamma} \prod_i dz_i \Gamma_{j,m}(\xi_1, \xi_2) \\ &\quad \times \prod_{I < L} E(z_I, z_L)^{2p_I p_L} \prod_i \sigma(z_i) \frac{1}{\sqrt{k+2} p_i} \prod_{i < l} \exp \left\{ -y_i y_l \frac{\partial}{\partial z_i} \frac{\partial}{\partial z_l} [\ln E(z_i, z_l)] \right\} \end{aligned} \quad (38)$$

In this formula $\eta(\tau)$ is the Dedekind function, $E(z, w)$ is the prime form, $\sigma(z)$ is the $g/2$ -differential with no zeroes and no poles ($g = 1$ for the torus and in fact $\sigma = 1$ in the coordinates implied by our choice of sewing operator, $e^{2\pi i \tau L_0}$) and p_I is the ϕ_1 -momentum of leg I , i.e. $p_A = \frac{1}{\sqrt{k+2}}$, $p_B = \frac{1}{\sqrt{k+2}}$, $p_i = \frac{1}{\sqrt{k+2}}$. The arguments ξ_1 and ξ_2 of the lattice sums $\Gamma_{j,m}$ are

$$\xi_1 = \frac{-\sqrt{k+2}}{\pi i} \sum_I p_I \ln z_I + \tau, \quad \xi_2 = \frac{\sqrt{2k}}{2\pi i} \sum_i \frac{y_i}{z_i} \quad (39)$$

In Eq. (38) the indices $I, L, \dots$ take values in the ordered set $\{A, 1, \dots, \tau_A, B, \tau_A + 1, \dots, \tau_B\}$ for $\gamma = 0$, and in the ordered set $\{A, 1, \dots, \tau'_A, B, \tau'_A + 1, \dots, \tau'_B\}$ for $\gamma = 1$, as dictated by the appropriate radial ordering of the operators W in Eq. (31) (and the contours $\mathcal{P}_\gamma$ change accordingly).

At this point it is interesting to consider in more detail the 1-loop structure embodied in Eq. (32). The sum over m is the trace over the zero mode of ϕ_0 , and thus over the isospin of the $SU(2)_k$ representation. Indeed, since the screening is independent of ϕ_0 , a given term $m = M$ in this sum is exactly the contribution from all states in the irreducible spin- j representation with isospin M . As discussed above this factorizes into the parafermion contribution, and that of the free scalar ϕ_0 . In particular the $U(1)$ -valued character $\chi_j(\tau, w)$ (for which $j_A = j_B = 0$ and hence $j = \bar{j}$, $\tau_A = \tau_B = 0$) is given by

$$\begin{aligned} \chi_j(\tau, w) &\equiv \frac{e^{\frac{\pi i \tau}{k+2}}}{(\eta(\tau))^3} \sum_{\gamma=0}^1 (-1)^\gamma \Gamma_{j,\gamma}(\tau, 0, w) \\ &= \sum_{m=-\infty}^{\infty} c_j^m(\tau) \left(e^{2\pi i \tau \frac{m^2}{2}} e^{2\pi i \frac{m}{2} w} \right) = \sum_{m=0}^{k-\frac{1}{2}} c_j^m(\tau) \Theta_{2m}^k(\tau, \frac{w}{k}) \end{aligned} \quad (40)$$

where

$$c_j^m(\tau) = \frac{e^{\frac{\pi i \tau}{k+2}}}{(\eta(\tau))^3} \sum_{\gamma=0}^1 (-1)^\gamma \Gamma_{j,\gamma,m}(\tau, 0)$$

are the Kac-Peterson string functions [23]. In writing the last equality of Eq. (40) (see e.g. Ref. [13], from which we take the definition of the Θ function) we have used the shift invariance, $c_j^m = c_j^{m+k}$. This may be proven directly. Using Eq. (22), the expression for c_j^m can be split into four absolutely convergent pieces. Then, since

$$\text{Tr}'_{j+(k+2)(n+\frac{1}{2}), m+k\frac{1}{2}} e^{2\pi i \tau \tilde{L}_0} = \text{Tr}'_{-j-1+(k+2)(-n-\frac{1}{2}), m+k\frac{1}{2}} e^{2\pi i \tau \tilde{L}_0} \quad (41)$$

(where Tr' denotes the trace over only the non-zero modes of ϕ_1 and ϕ_2 and $\sim$ the restriction to the parafermion system), it is easy to check that terms in the difference $c_j^m - c_j^{m+k}$ cancel pairwise. The proof of the shift invariance of the parafermionic contribution in Eq. (36),

$$\tilde{A}_{j,AB}^{j,m}(\tau) = \tilde{A}_{j,AB}^{j,m+k}(\tau) \quad (42)$$

goes along the same lines as for the string functions if

$$\text{Tr}'_{j+(k+2)(n+\frac{1}{2}), m+k\frac{1}{2}} \tilde{W}_0 = q^{-\theta} \text{Tr}'_{-j-1+(k+2)(-n-\frac{1}{2}), m+k\frac{1}{2}} \tilde{W}_1 \quad (43)$$

We have proven this equality when $\mathcal{W}_\gamma$ are the operators appropriate for the 1-point functions (which is sufficient for the 2-loop characters computed below) and also in a few other non-trivial cases⁵.

⁵The proof of Eq. (43) is easiest after transforming to the cylinder ($z = e^{2\pi i \tau}$) where the relative phase $q^{-\theta}$ between the two terms in the sum over γ cancels.

Let us now turn to the 2-loop $U(1)$ -valued character $Z_{j_1, j_2}^{(j)}(\vec{w})$ depicted in Fig. 4 as the sewing of two tadpoles. As stressed above, to obtain a non-vanishing result the operator $\tilde{I}_j$ defined in Eq. (29) must be inserted, so Fig. 4 corresponds to

$$Z_{j_1, j_2}^{(j)}(\vec{w}) = \left(\text{Tr} \left|_{\mathcal{H}_{j_1}} \right. W_{j_0}^{(j)} P(\tau_1) e^{2\pi i \tau_1^2 \omega_1} \right) \tilde{I}_j(z_0) \left(\text{Tr} \left|_{\mathcal{H}_{j_2}} \right. W_{j_0}^{(j)} P(\tau_2) e^{2\pi i \tau_2^2 \omega_2} \right) \quad (44)$$

where $P(\tau_1)$ and $P(\tau_2)$ are the sewing operators inserted to form the two loops, and z_0 is an arbitrary point on the Riemann surface (the dependence on z_0 of course drops out in the final result). Using the first term of Eq. (29) for $\tilde{I}_j$, we can rewrite Eq. (44) as a $(k+2)$ -point 2-loop vertex. Of these $(k+2)$ legs, j correspond to the screenings operators of the first loop, $(k-2j+2)$ correspond to the operator $V_{\frac{k}{2}-\frac{j}{2}, 0}^{(k-2j)}(z_0)Q_+$, and the remaining j legs correspond to the screenings of the second loop. Then when this $(k+2)$ -point 2-loop vertex is saturated with the states $|\frac{k}{2}, -\frac{j}{2}, 0\rangle_0, i\partial\phi_2^{(j)}(0) = -1, 0, 0$, and $|\frac{k}{2}+1, \frac{j}{2}, 0\rangle_{k+1}$, from the results of Ref. [16], Eq. (44) can be rewritten as

$$Z_{j_1, j_2}^{(j)}(\vec{w}) = \prod_{i=1}^k \frac{\partial}{\partial y_i} \left(Z_{j_1, j_2}^{(j)}(\vec{w}|y_i) \right) \Big|_{y_i=0}, \quad (45)$$

$$\begin{aligned} Z_{j_1, j_2}^{(j)}(\vec{w}|y_i) = & \mathcal{D}^3 \sum_{\{j_\mu, \mu=0,1\}} \prod_{\mu=1}^2 (q^{-\theta_\mu})^{\gamma_\mu} \int_{\mathcal{P}_{\gamma_\mu}} \prod_{i=1}^k dz_i \oint d z_{k+1} \sigma(z_0)^{\frac{1}{k+2}} \sigma(z_{k+1}) E(z_0, z_{k+1})^k \\ & \times \prod_{i=1}^k \left[E(z_0, z_i)^{\frac{1}{k+2}} E(z_i, z_{k+1})^{-1} \sigma(z_i)^{\frac{1}{k+2}} \prod_{i' < i} E(z_i, z_{i'})^{\frac{1}{k+2}} \Gamma_{j_1, j_2}(\vec{\xi}_1, \vec{\xi}_2, \vec{w}) \right. \\ & \left. \times \prod_{i' < i} \exp \left[-y_i y_{i'} \frac{\partial^2}{\partial z_i \partial z_{i'}} \ln E(z_i, z_{i'}) \right] \prod_{i=1}^k \exp \left[\sqrt{\frac{k}{2}} y_i \frac{\partial}{\partial z_i} \ln \frac{E(z_0, z_i)}{E(z_{k+1}, z_i)} \right] \right]. \quad (46) \end{aligned}$$

Here $\mathcal{D}$ is related to the non-zero mode determinant of the scalar Laplace operator on the genus-two Riemann surface, $\theta_\mu = -j(2j_\mu + 1)$, $j_\mu^{\gamma_\mu=0} = j_\mu$, $j_\mu^{\gamma_\mu=1} = -j_\mu - 1$ for $\mu = 1, 2$ and the integration contours are schematically depicted in Fig. 5. Finally, using the shift invariance of the 1-point functions at 1 loop (42), the 2-loop lattice sum Γ_{j_1, j_2} is given by

$$\Gamma_{j_1, j_2}(\vec{\xi}_1, \vec{\xi}_2, \vec{\xi}_3) = \sum_{\substack{m_1, m_2=0 \\ m_\mu - j_\mu \in \mathbb{Z}}}^{k-\frac{j}{2}} \sum_{n_1, n_2=-\infty}^{\infty} \Gamma_{j_1, m_1; j_2, m_2}(\vec{\xi}_1, \vec{\xi}_2) e^{2\pi i \left[\tau_{1\nu} k \left(\frac{m_\mu}{2} + n_\mu \right) + \left(\frac{m_\mu}{2} + n_\mu \right) \xi_1^\nu \right]} \quad (47)$$

where $\tau_{\mu\nu}$ is the period matrix and the sum over μ and ν in the exponent is understood, and

$$\begin{aligned} \Gamma_{j_1, m_1; j_2, m_2}(\vec{\xi}_1, \vec{\xi}_2) = & \sum_{n_1, n_2=-\infty}^{\infty} \sum_{z_1, z_2=0}^{y_1+y_2} (-1)^{z_1+z_2} \\ & \times e^{2\pi i \tau_{\mu\nu} \left[(k+2) \left(\frac{j_1^2}{4} + n_\mu + \frac{z_1}{2} \right) \left(\frac{j_2^2}{4} + n_\nu + \frac{z_2}{2} \right) - k \left(\frac{m_\mu}{2} + \frac{z_1}{2} \right) \left(\frac{m_\nu}{2} + \frac{z_2}{2} \right) \right]} \\ & \times e^{2\pi i \left[\left(\frac{j_1^2}{4} + n_\mu + \frac{z_1}{2} \right) \xi_1^\mu - \left(\frac{m_\mu}{2} + \frac{z_1}{2} \right) \xi_2^\mu \right]}. \quad (48) \end{aligned}$$

The vector arguments $\vec{\xi}_1$ and $\vec{\xi}_2$ to be used in Eq. (46) are

$$\begin{aligned} \xi_1^\mu &= \frac{1}{\pi i} \left[k \int_{z_0}^{z_1} du \omega^\mu(u) + (k+2) \int_{z_0}^{z_{k+1}} du \omega^\mu(u) - 2 \sum_{i=1}^k \int_{z_0}^{z_i} du \omega^\mu(u) \right] - 2 \left(\Delta^\mu + \frac{1}{2} \right) \\ \xi_2^\mu &= \frac{1}{2\pi i} \left[-k \int_{z_0}^{z_1} du \omega^\mu(u) + k \int_{z_0}^{z_{k+1}} du \omega^\mu(u) + \sqrt{2} k \sum_{i=1}^k y_i \omega^\mu(z_i) \right]. \quad (49) \end{aligned}$$

[We refer to Ref. [16] for notations, explicit expressions and properties of the Abelian differentials ω^μ and the vector of Riemann constants Δ^μ .] Notice that Eqs. (46), (47) and (49) are the obvious extension of the 1-loop results (38), (35) and (39) respectively.

We end with a few remarks. First, it is clear that the prescription given above allows us to write down the integral representation for any $SU(2)_k$ conformal block on any Riemann surface. Further, since the operators S , $S_\pm$, and $\tilde{I}_j$ are constructed from only the fields ϕ_1 and ϕ_2 , the restriction of these results to the parafermion conformal blocks obtained by dropping ϕ_0 is immediate. Moreover, since the conformal blocks of the $N=2$ minimal series can be written as linear combinations of factorized (parafermion and $c=1$ rational boson) blocks, the extension of our results to the $N=2$ minimal models looks very promising. Finally, it is interesting that $S_\pm$ played such an important role in the construction. This suggests that they too might be related to the underlying structure of the model in a way analogous to how S appears [24]. In any case, a result similar to that obtained in the present paper is required before the free field representation of larger WZNW models [11, 12, 24] may be extended to higher genus.

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Figure Captions

Fig. 1: 2-point 1-loop amplitude; the dashed line represents the sewing operator.

Fig. 2: Tadpole.

Fig. 3: The general N -point multiloop conformal block obtained by sewing tadpoles and “bubbles”.

Fig. 4: The 2-loop character; shorter legs represent screening operators.

Fig. 5: Integration paths for genus-2 characters.

References

- [1] B.L. Feigin and D.B. Fuchs, Moscow preprint (1983); see also *Representations of the Virasoro Algebra*, in *Representations of Infinite - Dimensional Lie Groups and Lie Algebras*, Gordon and Breach, New York to appear.
- [2] V.I.S. Dotsenko and V.A. Fateev, Nucl. Phys. B240 [FS12] (1984) 312 and Nucl. Phys. B251 [FS13] (1985) 691.
- [3] G. Felder, Nucl. Phys. B317 (1989) 215; Erratum, Nucl. Phys. B324 (1989) 548.
- [4] M. Frau, A. Lerda, J.G. McCarthy and S. Sciuto, Phys. Lett. B228 (1989) 205 and MIT preprint CTP-1803 (1989) to appear in Nucl. Phys. B.
- [5] J. Bagger and M. Goulian, Nucl. Phys. B330 (1990) 488.
- [6] G. Felder and R. Silvotti, Phys. Lett. B231 (1989) 411.
- [7] S.-J. Sin, Berkeley preprint UCB-PTH-89/20 (1989).
- [8] G. Cristofano, G. Maiella, R. Musto and F. Nicodemi, Phys. Lett. B237 (1990) 379.
- [9] S. Sciuto, Naples preprint (1989) to appear in the Festschrift in honour of J.J. Giambiagi.
- [10] M. Wakimoto, Comm. Math. Phys. 101 (1986) 605;
A.B. Zamolodchikov, talk given at the Montreal Summer School (June 1988).
- [11] B.L. Feigin and E.V. Frenkel, Russ. Math. Surv. 43 (1989) 221; in *Physics and Mathematics of Strings*, V.G. Knizhnik Memorial Volume, eds. L. Brink et al., World Scientific, Singapore 1990; *Representations of affine Kac-Moody algebras, bosonization and resolutions*, Moscow preprint, April '89; Comm. Math. Phys. 128 (1990) 161.
- [12] A. Gerasimov, A. Marshakov, A. Morozov, M. Olshanetskii and S. Shatashvili, ITEP preprints ITEP-89-70, ITEP-89-72, ITEP-89-74 (1989).
- [13] D. Bernard and G. Felder, Comm. Math. Phys. 127 (1990) 145.
- [14] A.B. Zamolodchikov and V.A. Fateev, Sov. Phys. JETP 62 (1985) 215.
- [15] D. Gepner and Z. Qiu, Nucl. Phys. B285 [FS19] (1987) 423.
- [16] P. Di Vecchia, F. Pezzella, M. Frau, K. Hornfeck, A. Lerda and S. Sciuto, Nucl. Phys. B322 (1989) 317.
- [17] V.I.S. Dotsenko, NORDITA preprint NORDITA-89-54/P (1989).
- [18] J. Distler and Z. Qiu, Cornell preprint CLNS 89/911 (1989)
K. Ito, Nucl. Phys. B332 (1990) 566.
D. Nemeschansky, USC preprint USC-89-012 (1989).
- [19] T. Jayaraman, K.S. Narain and M.H. Sarmadi, CERN preprint CERN-TH 5552/89 (1989).
- [20] P. Bouwknegt, J.G. McCarthy and K. Pilch, MIT preprint CTP-1861 (1990).
- [21] D. Friedan, E. Martinez and S. Shenker, Nucl. Phys. B271 (1986) 93.
- [22] P. Bouwknegt, A. Ceresole, J.G. McCarthy and P. van Nieuwenhuizen Phys. Rev. D39 (1989) 2971;
C.R. Preitschopf, Maryland preprint UMD-EPP-88-206 (1988);
C. Thorn, Phys. Rep. 175 (1989) 1;
D.Z. Freedman and K. Pilch, Int. J. Mod. Phys. A4 (1989) 5553.
- [23] V.G. Kac and D.H. Peterson, Adv. Math. 53 (1984) 125.
- [24] P. Bouwknegt, J.G. McCarthy and K. Pilch, Phys. Lett. B234 (1990) 297 and MIT preprint CTP-1797 (1989) to appear in Comm. Math. Phys.

FIG. 5

FIG. 3