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ZERO CENTRAL DENSITY SOLUTIONS OF THE ANISOTROPIC JEANS EQUATION AND NEUTRINO DARK MATTER

Jes Madsen

Abstract

Zero central density solutions of the spherically symmetric, anisotropic Jeans equation are shown to exist for a wide range of conditions when the transverse velocity dispersion exceeds the radial one. Explicit solutions are described for generalized polytropes with distribution functions like ordinary polytropes multiplied by a power of angular momentum, and for systems with constant phase-space density in velocity ellipsoids (only the first kind of solutions are time-independent).

The constant phase-space density systems obey mass-radius relations of the type $MR^3 = \text{constant}$, and so do the generalized polytropes *regardless of the polytropic index n* , provided that the maximum value of the phase-space density is fixed. This constraint follows naturally in collisionless systems.

In most cases the anisotropic solutions are more massive than their isotropic counterparts for fixed radius. One consequence of this is a weakening of phase-space constraints on the mass of fermionic dark matter, e.g. neutrinos.

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Zero central density solutions of the anisotropic Jeans equation and
neutrino dark matter.

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ABSTRACT.

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The constant phase-space density systems obey mass-radius relations of the type $MR^3 = \text{constant}$, and so do the generalized polytropes regardless of the polytropic index n , provided that the maximum value of the phase-space density is fixed. This constraint follows naturally in collisionless systems.

In most cases the anisotropic solutions are more massive than their isotropic counterparts for fixed radius. One consequence of this is a weakening of phase-space constraints on the mass of fermionic dark matter, e.g. neutrinos.

1. INTRODUCTION.

For reasons to be described in Section II most treatments of the anisotropic Jeans equations for collisionless, spherical systems assume the velocity distribution to approach isotropy in the center. If however the transverse velocity dispersion exceeds the radial one, a class of solutions exist with anisotropy all the way to the center, but with central density zero. Such systems are studied in the present paper. The results have a bearing on the problem of dark matter in galactic halos, and they may also have applications in stellar dynamics.

The paper is organized as follows : In Section II the anisotropic, spherically symmetric Jeans equation is discussed and recent work by Ralston (1989a) is extended to demonstrate the possibility of solutions with zero central density in a wide range of cases where the transverse velocity dispersion exceeds the radial one. In Section III the equation is solved for an anisotropic, degenerate equation of state and in Section IV for generalized polytropes obtained from the distribution function for ordinary polytropes by multiplication with a power of angular momentum. It is shown how both kinds of systems have mass-radius relations of the form $MR^3 = \text{constant}$, for the generalized polytropes under the extra (but for collisionless systems natural) assumption of fixed maximal phase-space density. Anisotropic systems can be much more massive than isotropic systems with the same radius. Applications to spheres of self-gravitating fermions, such as massive neutrinos, are

described in Section V, where it is argued, that the results weaken neutrino mass limits derived from phase-space considerations. Conclusions are drawn in Section VI.

II. THE ANISOTROPIC SPHERICALLY SYMMETRIC JEANS EQUATION.

Consider a nonrotating, spherical configuration of collisionless particles. The particles exhibit no net velocity flow and the azimuthal and polar velocity dispersions are equal. These symmetries can be expressed in a spherical coordinate system as

$$\langle v_r \rangle = \langle v_\theta \rangle = \langle v_\phi \rangle = 0 \quad (1)$$

and

$$\langle v_\theta^2 \rangle = \langle v_\phi^2 \rangle = (1-\alpha)\langle v_r^2 \rangle. \quad (2)$$

Angular brackets denote averages over the particle distribution, and $\alpha \leq 1$ is a parameter measuring the anisotropy of the velocity distribution. For $\alpha=1$ orbits are purely radial; isotropy corresponds to $\alpha=0$, and for $\alpha \rightarrow -\infty$ orbits become purely circular. In equilibrium the collisionless gas is described by the Jeans equation

$$\frac{dP_r(r)}{dr} + 2\alpha(r)\frac{P_r(r)}{r} = -\frac{GM(r)}{r^2}\rho(r), \quad (3)$$

where $P_r \equiv \rho\langle v_r^2 \rangle$ is the radial component of the pressure tensor, ρ is the density of collisionless particles, and $M(r)$ is the total mass interior to radius r .

For an isotropic system with $\alpha=0$ equation (3) is equivalent to the usual equation of hydrostatic equilibrium, which can be solved in the fluid case given suitable boundary conditions and an equation of state, $P(\rho)$.

The lack of a preferred direction at $r=0$ implies that solutions of the Jeans equation with *finite, non-zero* central density must be isotropic at the center. This explains why $\alpha(0)=0$ is taken for granted in most of the published studies of equation (3). However as recently pointed out by Ralston (1989a) $P_r(0)=0$ is another possible boundary condition. This permits solutions to equation (3) with the interesting property that the central density is *zero*. It will be demonstrated below, that such solutions are almost generic features of the anisotropic Jeans equation when $\alpha(0)<0$, and specific examples with constant α will be described in detail.

Before studying the structure of the anisotropic solutions it is worthwhile to look at the argument for central isotropy for a general $\alpha(r)$.

Expanding $P_r(r)$, $\alpha(r)$ and $\rho(r)$ around $r=0$ and inserting the results in equation (3) leads to the relation

$$(1+2\alpha(0))P_r'(0)+2\alpha'(0)P_r(0)+\frac{2}{r}\alpha(0)P_r(0) \rightarrow 0 \quad \text{for } r \rightarrow 0. \quad (4)$$

If $P_r(0) \neq 0$ the existence of solutions requires isotropy at $r=0$ ($\alpha(0)=0$), and at the same time $P_r'(0)+2\alpha'(0)P_r(0)=0$. However $P_r(0)=0$ is also a possibility, in which case either $P_r'(0)=0$ or $\alpha(0)=-\frac{1}{2}$. This was noted from a similar argument by Ralston (1989a), who in general expected $P_r'(0)=0$ in this case.

The discussion above assumed the derivatives of ρ , P_r and α to be finite. In fact solutions with $P_r(0)=0$ are more general than this. As long as the force density $GM\rho/r^2$ approaches zero more rapidly than $2\alpha\frac{P_r}{r}$, equation (3) has the asymptotic form $\frac{dP_r}{dr} \rightarrow -2\alpha\frac{P_r}{r}$, giving a solution $P_r(r) \sim r^{-2\alpha(0)}$, if α can be expanded in a power series.

Thus $P_r(0)=0$ for $\alpha(0)<0$. For $\alpha(0)<-\frac{1}{2}$ these solutions obey the restriction $P_r'(0)=0$. For $\alpha(0)=-\frac{1}{2}$ the solution has $P_r'(0)=\text{constant}$, and for $-\frac{1}{2}<\alpha(0)<0$ solutions have divergent P_r' . Unphysical solutions ($P_r(0) \rightarrow \infty$) result if $\alpha(0)>0$, whereas $\alpha(0)=0$ is the only possibility for finite, non-zero $P_r(0)$.

Generally $P_r \rightarrow 0$ implies that $\langle v_r^2 \rangle \rightarrow 0$ and/or $\rho \rightarrow 0$. Here we shall concentrate on the latter possibility, which will be relevant both for generalized polytropes, and in the case of a polytrope-like equation of state, $P_r \sim \rho^\gamma$. For given values of γ and $\alpha(0)$ it can be shown that solutions with the asymptotic form $\rho \sim r^{-2\alpha(0)/\gamma}$ exist for $0 < \gamma < \frac{2\alpha(0)}{1+\alpha(0)}$ ($\alpha(0)<-1$); $0 < \gamma$ ($-1 \leq \alpha(0) \leq 0$), where the restrictions on γ are needed to ensure a sufficiently rapid decrease of the force density as $r \rightarrow 0$.

III. SOLUTIONS FOR A DEGENERATE, ANISOTROPIC EQUATION OF STATE.

For non-relativistic particles with constant density in velocity ellipsoids, integration over the ellipsoids shows that

$$P_r = K \frac{\rho^{5/3}}{(1-\alpha)^{2/3}}, \quad (5)$$

where K is a constant involving the particle mass m and the number density in phase space N in the combination $K = \frac{1}{5} \left[\frac{3}{4\pi} \right]^{2/3} m^{-8/3} N^{-2/3}$. This equation of state

was first studied by Madsen and Epstein (1985) in an attempt to limit the neutrino mass needed to explain dark matter around galaxies. In this section we shall discuss the properties of equilibrium systems described by equation (5) with constant, negative α . It should be noted already, that these systems in general are time-dependent, but nevertheless have some interesting properties. Examples of steady-state solutions to equation (3) will be given in Section IV. (Time-independent spherical solutions of the collisionless Boltzmann-equation can be written in terms of a distribution function which depends only on energy and angular momentum; this is not in general possible in the case of equation (5), or in the case of the isotropic "maximally compact sphere" introduced by Madsen and Epstein (1984). The equation-of-state approaches used in these papers give equilibrium configurations at some instant of time, but the description may be adequate over at least a particle crossing time scale. To obtain firm lower mass limits on dark matter particles one needs the most conservative bounds that can be consistent with the data, and that is what the equation-of-state approaches were constructed to give).

Substituting equation (5) in equation (3) along with the relations $\rho \equiv \rho_0 \theta^{3/2}$ and $r = a\xi$, where $a \equiv (5K/(1-\alpha)^{2/3} \rho_0^{1/3} 8\pi G)^{1/2}$, leads to a "generalized Lane-Emden equation" of the form

$$\frac{d^2\theta}{d\xi^2} + (2 + \frac{4}{5}\alpha) \frac{1}{\xi} \frac{d\theta}{d\xi} + \frac{4\alpha}{5} \frac{1}{\xi^2} \theta + \theta^{3/2} = 0, \quad (6)$$

which is equivalent to

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left[\frac{4}{5} \alpha \xi \theta + \xi^2 \frac{d\theta}{d\xi} \right] = -\theta^{3/2}. \quad (7)$$

For an isotropic gas with $\alpha=0$ this is the well-known equation of hydrostatic equilibrium for a non-relativistic degenerate gas, e.g. a neutron star. In this case one may choose ρ_0 to be the central density, and the solution has $\theta(0)=1$, dropping to $\theta=0$ at $\xi=\xi_1=3.65375$.

To find the exact solutions for $\alpha<0$ it is useful to choose ρ_0 to be the maximal density, which is reached at some radius r_0 . Thus solutions to equations (6) or (7) begin at $\theta(0)=0$, reach a maximum $\theta(\xi_0)=1$, and extend to $\theta(\xi_1)=0$.

The mass interior to radius r is

$$M(r) = -4\pi a^3 \rho_0 \left[\frac{4}{5} \alpha \xi \theta + \xi^2 \theta' \right], \quad (8)$$

and the total mass M is given by

$$M = 4\pi a^3 \rho_0 \xi_1^2 \left| \theta'(\xi_1) \right| \quad (9)$$

where primes denote differentiation with respect to ξ .

The total mass is related to the total radius, R , by

$$MR^3 = 4\pi a^6 \rho_0 \xi_1^5 \left| \theta'(\xi_1) \right| \\ = 4\pi \left[\frac{5K}{8\pi G} \right]^3 (1-\alpha)^{-2} \xi_1^5 \left| \theta'(\xi_1) \right| \quad (10)$$

Apart from the α -dependence this is the usual scaling relation for $n=3/2$ polytropes.

Physically acceptable solutions must have a force density approaching zero in the center. Thus the right-hand-side of equation (3) must approach zero, corresponding to $\theta' \rightarrow -\frac{4}{5}\alpha\frac{\theta}{\xi}$, so θ is proportional to $\xi^{-\frac{4}{5}\alpha}$ near the center. Therefore $\theta' \rightarrow \infty$ for $-5/4 < \alpha < 0$, $\theta' \rightarrow \text{constant}$ for $\alpha = -5/4$, and $\theta' \rightarrow 0$ for $\alpha < -5/4$. The density gradient diverges for $-5/6 < \alpha < 0$, since $\rho \sim \theta^{3/2} \sim \xi^{-\frac{6}{5}\alpha}$ near the center, and the pressure gradient diverges for $-1/2 < \alpha < 0$.

Figure 1 shows some typical examples of $\theta(\xi)$ -solutions. Important properties of these and other solutions are summarized in Table 1.

Figure 2 clearly illustrates similarities and differences between anisotropic and isotropic systems. The density profiles in Figure 2a are plotted for solutions with the same total mass, M . Equation (10) has been used to scale the radii in a way so that the isotropic system extends to unit radius. In Figure 2b density profiles are plotted for systems with the same radius, R .

Apart from a "hole in the middle", the anisotropic solutions approach the isotropic one when α goes to zero. The higher the degree of anisotropy is, the more the system concentrates its mass in a shell-like structure at large radius. The total extent (mass) of the system for fixed M (R) increases infinitely for $\alpha \rightarrow -\infty$, has a minimum near $\alpha = -2/3$, and increases slightly for $\alpha \rightarrow 0$. For $-2 < \alpha < 0$ the deviation from the isotropic radius (mass) is less than 5% (14%). This is also apparent from Figure 3, which shows MR^3 as given by equation (10).

IV. GENERALIZED POLYTROPES.

The degenerate structures described in Section III are solutions to the anisotropic Jeans equation (3) at a given time, but they are not in general time-independent. One way of assuring a steady-state solution is to start from a

distribution function $f(E, L)$, where E and L denote the energy per unit mass and the magnitude of angular momentum per unit mass respectively (see Binney and Tremaine 1987 for a review). In particular we shall study generalized polytropes with distribution function

$$f(E, L) = HL^{-2\alpha}(\Phi_1 - E)^{n-3/2}, \quad (11)$$

where H , n and α are constants ($n > 1/2$, $\alpha < 1$), and Φ_1 is the value of the gravitational potential at the surface of the system. (The dimension of f is mass/(length $\times$ velocity)³). From now on we shall use the relative potential $\psi = \Phi_1 - \Phi$, and the relative energy $\varepsilon = \Phi_1 - E = \psi - v^2/2$, where v is the particle speed. Thus

$$f(\varepsilon, L) = HL^{-2\alpha} \varepsilon^{n-3/2}. \quad (12)$$

It is easy to show, that the constant, α , is in fact the anisotropy parameter introduced in equation (2). This is a property of any distribution of the form $f(\varepsilon, L) = L^{-2\alpha} f_0(\varepsilon)$ with $\alpha < 1$ (see Problem 4-19 in Binney and Tremaine 1987). Thus generalized polytropes have constant anisotropy throughout. For $\alpha = 0$ one obtains the well-studied isotropic polytropes.

Generalized polytropes were introduced by Hénon (1973), and later discussed by Barnes, Goodman, and Hut (1986), in both cases with particular emphasis on their dynamical stability for $n \lesssim 1.2-1.8$, and mainly with emphasis on preferentially radial orbits ($\alpha > 0$). Below we shall concentrate on systems with $\alpha < 0$, since these are prototypes of the zero central density solutions discussed in Section II. We shall consider a range of n -values. In particular we shall investigate the interesting consequences of imposing a maximum phase-space density on the systems.

First let us state some properties of generalized polytropes. Integrating equation (12) over all velocities one finds the spatial density

$$\rho = K \psi^{n-\alpha} r^{-2\alpha}, \quad (13)$$

where $K = 2^{3/2-2\alpha} \pi^{3/2} H \Gamma(1-\alpha) \Gamma(n-1/2) / \Gamma(n+1-\alpha)$. Note that systems with $\alpha > 0$ have divergent densities at $r=0$.

Introducing dimensionless variables ξ and θ by $r = b\xi$ and $\psi = \psi_0 \theta$, with $\psi_0 = \psi(0)$, and $b^{2-2\alpha} = (4\pi G K)^{-1} \psi_0^{1-\alpha-n}$, Poisson's equation takes the form

$$\theta'' + \frac{2}{\xi} \theta' = -\xi^{-2\alpha} \theta^{n-\alpha}. \quad (14)$$

Primes denote differentiation with respect to ξ . Boundary conditions are $\theta(0)=1$, $\theta'(0)=0$, and $\theta(\xi_1)=0$, where $R = b\xi_1$ is the radius of the system.

Most global physical properties of the systems are determined by ξ_1 and $|\theta'(\xi_1)|$. Thus the total mass is

$$M = G^{-1} b \psi_0 \xi_1^2 |\theta'(\xi_1)|, \quad (15)$$

the surface potential is

$$\Phi_1 = -GM/R = -\psi_0 \xi_1 |\theta'(\xi_1)|, \quad (16)$$

and the total energy of the system is

$$E_{\text{tot}} = -\frac{3-2\alpha}{10-2n-6\alpha} G^{-1} b \psi_0^2 \xi_1^3 |\theta'(\xi_1)|^2 = -\frac{3-2\alpha}{10-2n-6\alpha} \frac{GM^2}{R}. \quad (17)$$

It follows immediately from the latter equation, that bound systems with finite mass and radius are possible for $n < 5-3\alpha$. All of these results are well-known for $\alpha=0$, and were first derived by Hénon (1973) for $\alpha \neq 0$.

The radial pressure component is $P_r = \rho \psi / (1+n-\alpha)$, and when combined with equation (13) it behaves as described in section II. (Kent and Gunn, 1982, who briefly discuss generalized polytropes as models for the Coma cluster, have an extra factor of $3/2-\alpha$ in their expression for P_r . This seems to be erroneous).

Some solutions of equation (14) are shown in Figure 4 and Table 2.

An important property of collisionless evolution is the non-increase of the maximum phase-space density. If the prior distribution function is known, as in the case of massive neutrinos forming galactic halos, this constrains the final state of the system. As a simple illustration we shall investigate generalized polytropes with a phase-space density $f(\varepsilon, L)$ which takes a maximum f_{max} at scaled radius $\xi = \xi_m$ (or which in the special case $\alpha=0$, $n=3/2$ has $f = f_{\text{max}}$ throughout).

The maximal value of the phase-space density at radius r is found from equation (12) as

$$f_{\text{max}}(r) = \left[\frac{-2\alpha}{n-\alpha-3/2} \right]^{-\alpha} \left[\frac{n-3/2}{n-\alpha-3/2} \right]^{n-3/2} H r^{-2\alpha} \psi^{n-\alpha-3/2} \\ \equiv B(\alpha, n) H b^{-2\alpha} \psi_0^{n-\alpha-3/2} \xi^{-2\alpha} \theta^{n-\alpha-3/2} \quad (18)$$

(for $\alpha \leq 0$, $n \geq 3/2$; there is no maximum for $\alpha > 0$ and/or $n < 3/2$). The radius where this expression is maximized has to be found numerically, and the maximum is then proportional to $\xi_m^{-2\alpha} \theta(\xi_m)^{n-\alpha-3/2}$. Assuming $f_{\text{max}}(r_m) = f_{\text{max}}$ and introducing $C(n, \alpha)$ by

$$C(n, \alpha) = 2^{2\alpha} \left[\frac{\Gamma(n-\alpha+1)}{\Gamma(1-\alpha)\Gamma(n-1/2)} \right]^2 \xi_1^2 |\theta'(\xi_1)| \xi_m^{-2\alpha} \theta(\xi_m)^{2n-2\alpha-3} B^2 \quad (19)$$

finally leads to a mass-radius relation of the form

$$MR^3 = \frac{C(n, \alpha)}{27\pi^2 G^3 f_{\max}^2}. \quad (20)$$

In other words, generalized polytropes with $n \approx 3/2$ obey a mass-radius relation similar to that for an isotropic, $n=3/2$ -polytrope, regardless of the value of n , if the maximum phase-space density of the system is forced to have a given value. As noted above, this is not as artificial as it may sound, since collisionless evolution naturally leads to an upper bound on the phase-space density.

The density structure of some generalized polytropes with fixed radius R or mass M are shown in Figures 5 and 6. Note in particular the pronounced shell-structure for large negative α . Figure 7 gives $C(n, \alpha)$ for a few n -values. C is an increasing function of n for small $|\alpha|$, and a decreasing function of n for large $|\alpha|$. It diverges for $\alpha \rightarrow -\infty$.

Generalized polytropes are steady-state solutions to the collisionless Boltzmann-equation. Whether they are stable is not known in general, but systems in the isotropic limit are certainly stable for $n > 3/2$, since they have $df/dE < 0$. Also all generalized polytropes with $n \approx 2$ should be stable to radial perturbations according to "Antonov's Fourth Law" as stated in Binney and Tremaine (1987, p.308). Hénon (1973) and Barnes, Goodman and Hut (1986) have studied the stability of generalized polytropes with numerical simulations. Hénon concentrates on cases with $n < 3/2$, but Barnes, Goodman and Hut include models with n up to 1.8 and α down to -4.5. Their results indicate that a certain reorganization of mass takes place for $\alpha < -1$, so that the final shell-structure becomes less pronounced than found above. They also discuss the instability of an infinitely thin shell (the limit $\alpha \rightarrow -\infty$). It is reasonably clear that shell-like systems may develop non-radial instabilities because all orbits have similar periods, making azimuthal clumping of particles easier, but the exact relation between α , n , and stability is hard to judge. Furthermore no numerical experiments have been performed for $n > 1.8$.

V. AN EXAMPLE : ANISOTROPIC SPHERES OF MASSIVE NEUTRINOS.

To illustrate the results obtained in Sections III and IV we shall now look at neutrino spheres, e.g. dark neutrino halos around galaxies or clusters of galaxies.

It is well-known that there are problems in fulfilling the lower-mass-bound on neutrino dark matter given by Tremaine and Gunn (1979) for several well-observed galaxies. However, the limit assumes that the neutrinos are distributed in an isotermal sphere. Other isotropic distributions give other limits, and *anisotropy* in the velocity distribution is another way of weakening the bound. The latter possibility

was first discussed by Madsen and Epstein (1985), and more recently Ralston (1989a, 1989b) has attacked the problem based on the collisionless Boltzmann-equation. Ralston (1989a) also discusses the possibility of "hollow" anisotropic halos, where the density maximum is not at the center, and he argues for a weakening of the Tremaine-Gunn limit.

Below we consider the "global" properties (mass-radius relations) of the zero central density configurations of Section III and IV. The structures described are not meant as specific models for the dark matter, but they demonstrate some general properties of anisotropic spheres, and possibilities for weakening the Tremaine-Gunn limit.

For one flavor of massive neutrino and corresponding antineutrino the number density in phase space is $N=2gF/h^3$, where g is the number of helicity states ($1 \leq g \leq 2$) and F is the occupation number. According to the Pauli exclusion principle $F \leq 1$, but cosmological neutrinos obey a stricter rule, namely $F \leq 0.5$ (e.g. Tremaine and Gunn 1979). The softest equation of state possible for fixed α is thus constructed by choosing $F=0.5$. In fact the median value of F for cosmological neutrinos decoupling while relativistic is 0.06 (Madsen and Epstein 1984; see also Madsen 1990, where non-relativistic decoupling and the case of bosons are discussed as well), so any real neutrino halos will have a harder equation of state and thus be spatially more extended for fixed M . This leads to the need for a larger neutrino mass if a given mass is to be explained within a given radius.

The constant K in equation (5) becomes

$$K = 7.2321 \times 10^{32} m^{8/3} (2gF)^{-2/3} \quad (21)$$

in cgs-units (with the neutrino mass m in eV/ c^2). The corresponding value of the radius scale-factor, a , is

$$a = 475.97 \text{ kpc} (1-\alpha)^{-1/3} \rho_0^{1/6} m^{4/3} (2gF)^{-1/3}, \quad (22)$$

where $\rho_{0,24} \equiv \rho_0 / 10^{-24} \text{ g cm}^{-3}$.

The mass-radius relation (equation (10)) becomes

$$MR^3 = 2.1583 \times 10^{24} M_0 \text{ kpc}^3 m^{-8} (2gF)^{-2} (1-\alpha)^{-2} \xi^{5/3} |\theta'(\xi_1)|, \quad (23)$$

where M_0 is the solar mass. The factor $(1-\alpha)^{-2} \xi^{5/3} |\theta'(\xi_1)|$ is shown in Figure 3. Thus the radius of the isotropic system used for normalization in Figure 2a becomes

$$R_0 = 35.183 \text{ kpc} \left[\frac{10^{10} M_0}{M} \right]^{1/3} \left[\frac{30 \text{ eV}}{m} \right]^{8/3} (2gF)^{-2/3}. \quad (24)$$

Similarly for the generalized polytropes one finds a mass-radius relation

$$MR^3 = 1.2215 \times 10^{24} M_\odot \text{ kpc}^3 m^{-8} (2gF)^{-2} C(n, \alpha). \quad (25)$$

($C(n, \alpha)$ is shown in Figure 7).

Equations (23) and (25) agree (as indeed they should) for $\alpha=0$, $n=3/2$, the case of an isotropic, non-relativistic, degenerate gas. An important observation is, that for most choices of $\alpha < 0$ MR^3 exceeds the isotropic value, and in fact MR^3 diverges when orbits approach circularity. For the generalized polytropes even the isotropic systems have larger values of MR^3 for $n > 3/2$ than for $n = 3/2$.

One consequence of this is, that *anisotropic systems with given radius can be (much) more massive than the isotropic, degenerate counterparts for equal maximal phase-space density and particle mass*. Most of this mass is concentrated in the outer parts of the system, which is nice if one wants to fit the outer parts of galaxy rotation curves without spoiling the inner parts, which are often well explained by the luminous matter. This opens possibilities for a weakening of the phase-space constraints on fermionic dark matter particle masses, since a main problem with low-mass fermions as dark matter has been to obtain sufficient spatial concentration. (See also Ralston 1989a and 1989b).

VI. DISCUSSION.

Zero central density solutions have been shown to exist for the anisotropic, spherically symmetric Jeans equation whenever the transverse velocity dispersion dominates over the radial dispersion near the center. Explicit solutions were derived for systems with constant phase-space density, and for generalized polytropes, and the scaling of the solutions was exemplified with massive neutrinos.

Apart from being interesting in their own right the solutions may have a bearing on the observed shell-like structures around elliptical galaxies, and they are important for attempts to solve the dark matter problem.

Being time-dependent, the degenerate structures discussed in Section III are perhaps less interesting than steady state solutions like the generalized polytropes of Section IV, but they do serve to illustrate the zero central density solutions in a setting, which may be intuitively easier to grasp than the $f(\epsilon, L)$ -approach.

The generalized polytropes are time-independent, and as such perhaps of more interest, in particular if they are also stable. As discussed in Section IV the question of stability is only settled for some ranges of parameters. The extra assumption of a given maximal phase-space density was imposed to obtain the mass-radius

relation $MR^3 = \text{constant}$. Such an assumption is natural for dark matter fermions, e.g. neutrinos, but it is a general consequence of any dissipationless evolution, for example in a stellar gas forming an elliptical galaxy. The mass-radius scaling can be calculated if the initial phase-space distribution has a known maximum.

For a fixed stellar/particle mass anisotropic systems can account for a larger total mass within a given radius than the isotropic solution. Most of this mass is concentrated in the outer parts of the system, which in the case of dark matter may help fit the outer parts of galaxy rotation curves without influencing the inner parts, that are dominated by the luminous matter. It is of course far from obvious that massive neutrinos should form generalized polytropes or degenerate halos dominated by transverse velocity dispersion. In fact radial anisotropy in the outermost regions seems more likely according to most dissipationless collapse simulations. Therefore it has not been attempted to make detailed fits to observed galaxy rotation curves with proper compensation for the luminous matter contribution etcetera. But the results are important illustrations of ways to compress self-gravitating neutrinos (and other collisionless particles) in smaller spatial regions than previously thought possible.

NOTE ADDED: After this work was finished I received an interesting preprint by Ralston and Smith (1990), extending the ideas on hollow fermion halos first described by Ralston (1989a). Most of the structures they study have non-zero central density, but the density maximum occurs at finite radius. Thus the structures are shell-like. For reasonable choices of parameters the authors reproduce flat rotation curves, and they conclude that existing fermion lower mass bounds are overly restrictive, and in certain cases inapplicable. A review and comparison of phase-space constraints and lower mass-limits for fermion and boson dark matter is in preparation (Madsen 1990, in preparation).

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TABLE 1
Properties of some solutions to equation (6).

α	ξ_1	$ \theta'(\xi_1) $	$(1-\alpha)^2 \xi_1^3 \theta'(\xi_1) $
0	3.65375	0.20330	132.38
-0.001	3.65159	0.20420	132.31
-0.01	3.64057	0.21004	131.68
-0.1	3.60594	0.25058	126.25
-0.5	3.68334	0.38308	115.42
-5/6	3.82159	0.47478	115.14
-1	3.90082	0.51660	116.64
-5/4	4.02735	0.57511	120.36
-2	4.43855	0.72600	138.97
-5	6.22650	1.1069	287.72
-10	9.29372	1.4061	805.72

TABLE 2

Properties of some solutions to equation (14)

$ \alpha $	ξ_1	$ \theta'(\xi_1) $	$C(n, \alpha)$
n=1.5	0	3.65375	0.20330
			2.340×10^2
	0.01	3.64441	0.20543
			2.429×10^2
	0.1	3.55948	0.22471
			3.357×10^2
	0.2	3.46599	0.24635
			4.693×10^2
	0.5	3.21020	0.31151
			1.131×10^3
	1	2.88173	0.41689
n=2.0			3.670×10^3
	2	2.47576	0.60499
			2.071×10^4
	5	1.98289	1.0007
			4.666×10^5
	10	1.72168	1.3555
			8.012×10^6
	20	1.55545	1.6734
			1.798×10^8
	0	4.35287	0.12725
			1.013×10^3
n=2.5	0.01	4.33264	0.12911
			8.810×10^2
	0.1	4.16084	0.14600
			6.097×10^2
	0.2	3.98998	0.16510
			5.795×10^2
	0.5	3.57883	0.22355
			7.956×10^2
	1	3.11817	0.32071
			1.760×10^3
	2	2.60579	0.50064
n=3.0			7.419×10^3
	5	2.03397	0.89866
			1.321×10^5
	10	1.74612	1.2715
			2.070×10^6
	20	1.56724	1.6143
			4.418×10^7
	100	1.39139	2.0745
			7.718×10^{10}
	0	6.89685	0.042430
			1.349×10^4
n=4.0	0.01	6.81866	0.043597
			1.108×10^4
	0.1	6.20948	0.054519
			4.900×10^3
	0.2	5.68382	0.067436
			3.182×10^3
	0.5	4.64821	0.10977
			2.114×10^3
	1	3.73882	0.18669
			2.574×10^3
	2	2.91666	0.34286
n=5.0			6.603×10^3
	5	2.14562	0.72742
			7.721×10^4
	10	1.79743	1.1208
			1.015×10^6
	20	1.59141	1.5037
			1.969×10^7
	100	1.39594	2.0415
			3.217×10^{10}
	0	14.9714	8.0182×10^{-3}
			3.145×10^5
n=5.0	0.01	14.5426	8.5193×10^{-3}
			2.408×10^5
	0.1	11.6329	1.3633×10^{-2}
			6.303×10^4
	0.2	9.62085	2.0473×10^{-2}
			2.676×10^4
	0.5	6.60809	4.6765×10^{-2}
			8.244×10^3
	1	4.67934	1.0282×10^{-1}
			5.504×10^3
	2	3.31905	2.3201×10^{-1}
n=5.0			8.726×10^3
	5	2.27183	5.9025×10^{-1}
			6.854×10^4
	10	1.85220	9.9016×10^{-1}
			7.681×10^5
	20	1.61641	1.4020
			1.349×10^7
	100	1.40052	2.0092
			2.070×10^{10}
	0	∞	$\dots$
			$\dots$
n=5.0	0.01	582.01	5.0933×10^{-6}
			2.980×10^{10}
n=5.0	0.02	288.51	2.0664×10^{-5}
			3.170×10^9

0.1	55.476	5.544×10^{-4}	1.182×10^7
0.5	11.2629	1.4269×10^{-2}	5.415×10^4
1	6.26026	0.05066	1.494×10^4
2	3.85831	0.15288	1.345×10^4
10	1.91077	0.87612	6.686×10^5
100	1.40513	1.9774	1.546×10^{10}

FIGURE CAPTIONS

Figure 1 Solutions of equation (6) for the values of α indicated.

Figure 2a Density profiles for equal-mass degenerate systems with different degrees of anisotropy. Radii are scaled in units of the radius of the isotropic solution.

Figure 2b Density profiles for equal-radius degenerate systems with different degrees of anisotropy.

Figure 3 The combination $(1-\alpha)^2 \xi_1^3 |\theta'(\xi_1)|$ entering the mass-radius relation (equation (10)) is shown as a function of α .

Figure 4 Solutions of equation (14) for two values of α . Full lines: $n=1.5$. Short dashed lines: $n=2$. Long dashed lines: $n=3$. Short-long dashed line: $n=5$.

Figure 5a Density profiles for equal-mass generalized polytropes with α -values as indicated and $n=1.5$.

Figure 5b As Figure 5a, but for $n=3$.

Figure 6a Density profiles for equal-radius generalized polytropes with α -values as indicated and $n=1.5$.

Figure 6b As Figure 6a, but for $n=3$.

Figure 7 The function $C(n, \alpha)$ defined in equation (19).

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Figure 4

Figure 5a

Figure 7

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Figure 6b

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