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A Theorem on the Zeta-Function of Argument 4

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ABSTRACT

The conjecture that $\zeta(4)$ satisfies

$$\zeta(4) = \frac{36}{17} \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \cdot \frac{1}{n^4}$$

is proved analytically.

It has been a long-standing conjecture [1] that $\zeta(4)$ - the Riemann zeta-function of argument 4 - could be expressed by an infinite series involving the binomial coefficients $\binom{2n}{n}$, namely

$$\zeta(4) = \frac{36}{17} \sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n} \cdot n^4} \quad (1)$$

The ζ -function, and in a wider sense the polylogarithms are intimately connected with perturbative calculations both in QED and QCD. But, in addition, this paper is dealing with the so-called Log-sine integrals which play an important rôle in various domains of physics and the technique given below, which converts them in infinite series with $\binom{2n}{n}$ binomial coefficients, applies specifically to these integrals. For instance the Clausen integrals enter this class and are widely used in electrical engineering, in repeated structures, cavity resonators, wave guides, slotted lines, domain pattern in alloys, etc. However, unlike $\zeta(4)$, Clausen integrals are not known analytically and must be tabulated. The quickly convergent series given in the following are especially useful for their calculation at angles very close to 180 degrees.

The rigorous proof of (1) can be sketched as follows. The general definition of the Log-sine integrals is usually [2] given by

$$Ls_n^{(m)}(\theta) = - \int_0^\theta \theta^m \text{Log}^{n-m-1} |2 \sin \frac{\theta}{2}| d\theta \quad (2)$$

and $Ls_n(\theta)$ is often used for $Ls_n^{(0)}(\theta)$. Similar expressions involving $\cos \frac{\theta}{2}$ and $\text{tg} \frac{\theta}{2}$ may easily be deduced by elementary trigonometric transformations. Taking $z = 1 - e^{i\theta}$, the real part of the polylogarithm of order 4, $Li_4(z)$, is, when $\theta = \frac{\pi}{3}$,

$$R.P.Li_4(e^{-i\frac{\pi}{3}}) = \frac{1}{4} Ls_4^{(1)}(\frac{\pi}{3}) - \frac{\pi}{12} \cdot Ls_3(\frac{\pi}{3}) - \frac{\pi^4}{15552} \quad (3)$$

On the other hand, $R.P.Li_4(e^{-i\frac{\pi}{3}})$ is, by definition, the Glaisher Gl_4 function of argument $\frac{\pi}{3}$:

$$R.P.Li_4(e^{-i\frac{\pi}{3}}) = Gl_4(\frac{\pi}{3}) = \frac{91\pi^4}{19440} \quad (4)$$

With the known value $Ls_3(\frac{\pi}{3}) = -\frac{7\pi^3}{108}$, (3) and (4) give, for $Ls_4^{(1)}(\frac{\pi}{3})$,

$$Ls_4^{(1)}(\frac{\pi}{3}) = - \int_0^{\pi/3} \theta \cdot \text{Log}^2(2 \sin \frac{\theta}{2}) d\theta = -\frac{17\pi^4}{6480} \quad (5)$$

In the integral in (5), the changes of variable $t = 2 \sin \frac{\theta}{2}$, then $u^2 = t$ and integrating term by term the expansion in u , gives in a straightforward way

$$\int_0^{\pi/3} \theta \cdot \text{Log}^2(2 \sin \frac{\theta}{2}) d\theta = \frac{1}{2} \cdot \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)} \cdot \frac{1}{n^4} \quad (6)$$

which, combined with the analytical value in (5) provides the result announced in (1), since $\zeta(4) = \frac{\pi^4}{90}$.

The case of $\zeta(4)$ has been treated at length because it was known as a conjecture [1] but, using the same procedure for other quantities expressible in terms of the Log-sine type integrals, one has some new results of the same nature, namely

$$G = \frac{3}{8} \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \cdot \frac{1}{(2n+1)^2} - \frac{\pi}{8} \cdot \text{Log}(2 - \sqrt{3}) \quad (7)$$

for the Catalan's constant $G = .915965594177...$ and

$$C_{MS} = \frac{2}{5} \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \cdot \frac{3^n}{n(2n-1)} + \frac{\pi\sqrt{3}}{15} \text{Log } 3 \quad (8)$$

for the MS-scheme constant in QCD, $C_{MS} = 1.17195361936...$

Beyond these specific results, more general considerations can be made.

First in considering that for $n > 4$, the $\zeta(n)$ do not seem to have such simple expansions with binomial coefficients $\binom{2n}{n}$: the reason appears to lie in the existence of a closed group of transformations valid only for polylogarithms of low orders, 4 being the upper limiting case.

Second, it seems that only irrational quantities can be expressed in terms of such expansions (this is at least the case for $\zeta(2), \zeta(3), \zeta(4), \pi\sqrt{3}$, etc.) and it is tempting to conjecture that the possibility for a numerical quantity to be expressible by rational expansions involving the $\binom{2n}{n}$ is a criterion of irrationality. The Brouncker proof of the irrationality of π rests on rather similar considerations.

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REFERENCES

1. See, for instance: F. Le Lionnais, "Les Nombres Remarquables", Ed. Hermann, Paris (1986).
2. L. Lewin, "Dilogarithms and Associated Functions", Ed. Mac Donald, London (1958).