Lax Equations and Quantum Groups

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Abstract

We construct Lax equations associated to quantum groups. In particular we derive quantum trace formulas for the Lax matrix L giving the quantum analogue of its usual classical invariants generated by $\text{tr}(L^n)$. These quantities generate an Abelian subalgebra of the quantum group and define Hamiltonian flows that admit quantum Lax representations. This hierarchy of quantum Lax equations is shown to verify zero-curvature relations. In the field theory case we give the lattice construction of these quantum structures.

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1. Introduction

Lax equations are at present the main tool for constructing classical integrable evolution equations and dynamical systems in dimensions less than or equal to two. In the case of Hamiltonian systems their existence lead to integrability in the Liouville sense. A Lax pair [1] is a set of two functions (L, M) on the phase space of the system taking values in some Lie algebra $\mathcal{G}$ such that the evolution equations are given by

$$\frac{dL}{dt} = [M, L] \quad (1)$$

The interest in such Lax-equations is that they allow for an easy construction of constants of motion. They are obtained as the functions of L invariant under the adjoint action of the connected Lie group G having $\mathcal{G}$ as its Lie algebra. In a matrix representation of $\mathcal{G}$ they are given as

$$q_{(n)} = \text{tr} (L^n) \quad (2)$$

It follows easily from eq.(1) that the $q_{(n)}$ are time-independent quantities. In the Hamiltonian framework a beautiful structure arises. For these situations the classical system is described by a phase space which is a group (usually a Lie group), and a Poisson manifold in such a way that the group action is a Poisson morphism [2]. In this description the notion of the classical r -matrix, which solves the classical Yang-Baxter equations (CYBE's), has a natural setting: it is basically the Poisson tensor that gives the proper Lie-Poisson structure compatible with the integrable dynamics, cf. also [3]. The CYBE's arise as sufficient conditions leading to the Jacobi identity for this Poisson structure. Drinfel'd shows that the Lie-Poisson structure on the group G reflects in a bi-algebra structure on its corresponding Lie-algebra $\mathcal{G}$. This is a Lie co-algebra structure on $\mathcal{G}$ such that the dual $\mathcal{G}^*$ of $\mathcal{G}$ carries a natural Lie product compatible with the one on $\mathcal{G}$ itself.

It is well known that whenever one has such a classical r -matrix structure it is possible to exhibit a hierarchy of commuting Hamiltonian flows, generated by the above $q_{(n)}$, and their associated Lax equations.

The purpose of this letter is to show that this scheme extends into the quantum case. Namely, to any quantum group structure described by a constant R -matrix that verifies the quantum Yang-Baxter equation [4,5] we will associate a hierarchy of commuting quantum operators. It should be noted here that our constructions do not need a priori the presence of a spectral parameter. These objects generate an Abelian sub-algebra of the quantum group, and will be shown to define a hierarchy of flows admitting quantum Lax representations. We also exhibit quantum zero-curvature equations that are satisfied by the members of this hierarchy. Finally we give a lattice field theory representation of this construction.

2. Classical r -matrices and Lax Equations

We give in the following a short description of the classical case (for a review see for example [6] and [7]). We start from the fundamental Poisson bracket relation given by the classical r -matrix as

$$\{L_1, L_2\} = [r_{12}, L_1 L_2] \quad (3)$$

The notations are as follows. Any subscript stands for the matrix space the corresponding quantity is acting upon. Hence L_1 is a matrix, acting on a vector space V_1 , whose entries are functions on the phase-space of the system described by (3), and r_{12} is a numerical matrix acting in the tensor product $V_1 \otimes V_2$. Sufficient conditions for eq.(3) to define a Poisson structure are given by the two following sets of constraints on the r -matrix

$$P_{12} r_{12} P_{12} = r_{21} = -r_{12}$$

where P_{12} is the matrix acting in the tensor product $V_1 \otimes V_2$ defined by $P_{ij,kl} = \delta_{ik} \delta_{jl}$ and with the property $P(A \otimes B)P = B \otimes A$ for any matrices A, B , leading to the antisymmetry property of the Poisson bracket. And we have

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{32}, r_{13}] = 0 \quad (4)$$

which is the so-called classical Yang-Baxter equation leading to the Jacobi identity of the Poisson bracket. It is very easy to show that the Poisson

structure (3) admits an Abelian sub-algebra generated by the traces of the powers of L . More explicitly we obtain from eqs.(3,2)

$$\{ q_n, q_m \} = 0 \quad (5)$$

for any positive integers n and m . Hence we may define a set of "times" t_n associated to these commuting Hamiltonian flows and we will denote by ∂_n the derivative with respect to the time t_n such that

$$\{ q_n, f \} = \partial_n f$$

for any function f on the phase space. Now let us define $M^{(n)}$ -matrices associated to these flows as

$$M_1^{(n)} = -n \operatorname{tr}_2 (r_{12} L_2^n)$$

Using the Poisson bracket relations (3) we obtain

$$\partial_n L = [M^{(n)}, L]$$

which is a Lax equation associated to the n^{th} flow. Equivalently we may rewrite this equation as a commutativity equation. If we define the operator

$$U^{(n)} = M^{(n)} - \partial_n$$

we obtain

$$[U^{(n)}, L] = 0 \quad (6)$$

In addition, using the classical Yang-Baxter equation (4) together with (3) it is possible to show that the following zero-curvature relations hold for the hierarchy of $M^{(n)}$ -matrices

$$\partial_m M^{(n)} - \partial_n M^{(m)} + [M^{(n)}, M^{(m)}] = 0 \quad (7)$$

which in turn is just a commutativity relation for the $U^{(n)}$.

This picture admits also a field theoretical setting that we briefly describe now in the case where the dynamical variables are functions defined on a one-dimensional lattice of size N with periodic boundary conditions.

To each site, say (n) , of this lattice we associate a copy of the Poisson algebra (3), with the same r -matrix, and with $L_1(n)$ -matrices depending now explicitly on the corresponding site (n) . We further require these L -matrices to have vanishing Poisson bracket for different sites (ultralocality property). We set

$$\{ L_1(m), L_2(n) \} = \delta_{mn} [r_{12}, L_1(m) L_2(n)]$$

Defining the monodromy matrix as

$$T_1 = L_1(N) \cdots L_1(2) L_1(1)$$

where there is a matrix product on space (1) between the L -matrices, we obtain

$$\{ T_1, T_2 \} = [r_{12}, T_1 T_2]$$

and if we denote

$$q_{(p)} = \operatorname{tr} (T^p) \quad (8)$$

eq.(5) is verified also in this case. Moreover, it is possible to construct local Lax equations associated to such flows. Let

$$M_1^{(p)}(n) = -p \operatorname{tr}_2 (L_2(N) \cdots L_2(n) r_{12} L_2(n-1) \cdots L_2(1) \cdot T_2^{p-1}). \quad (9)$$

We obtain

$$\partial_p L(n) = M^{(p)}(n+1) \cdot L(n) - L(n) \cdot M^{(p)}(n) \quad (10)$$

which is a discretized (in space) version of a zero-curvature equation. Note here that thanks to periodic boundary conditions we have

$$\partial_p T_1 = [M_1^{(p)}(1), T_1]$$

and hence all the $q_{(p)}$ given in eq.(8) are time independent. Remark that the zero-curvature equation (7) holds for all the $M^{(p)}$ defined in (9). If we discretize the time variable, we may also construct Lax equations associated to finite time-shift, instead of time derivative, taking the following form (integrated version of the zero-curvature equation)

$$L(t_p + 1, n) \cdot M^{(p)}(t_p, n) = M^{(p)}(t_p, n+1) \cdot L(t_p, n) \quad (11)$$

We present in the following the quantum analogue of these classical algebraic structures.

3. Quantum Trace Formulas

The purpose of this section is to give the quantized version of relations (2,5).

The concept of a quantum group was defined by Drinfel'd in [8], where he presented in particular a Hopf algebra setting for the quantization prescription of classical integrable systems, see also [9] and [10]. What is crucial in this picture when going to the quantum theory is that the classical bi-algebra structure may be extended to the universal enveloping algebra of $\mathcal{G}$ that becomes a Hopf co-Poisson algebra [2,3]. It is this structure that can be made subject to a q -deformation ($q = e^\hbar$ being the deformation parameter). The resulting "quantum groups" or quantum algebras are by now a subject of intensive studies, see e.g. [8]-[16].

We start from the quantization of the fundamental Poisson bracket relation (3), that now gives the quantum group defining relation

$$R_{12} L_1 \cdot L_2 = L_2 \cdot L_1 R_{12} \quad (12)$$

where the notations for the vector space indices 1, 2 are the same as in the classical case and the dot between the L 's stands now for the (associative) quantum product law. A sufficient condition for the relations (12) to be the only defining relations of the quantum group algebra is that the matrix R has to satisfy the Yang-Baxter equation [4,5]

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} \quad (13)$$

In the classical case the set of commuting quantities following from (3) were given as the invariants of the matrix L generated by the trace of its powers $q_{(n)}$ (2). It is not difficult to see that such expressions have to be modified at the quantum level in order to exhibit an Abelian sub-algebra of

the quantum group defined in (12,13). In particular a direct computation shows that $\text{tr}(L)$ and $\text{tr}(L^2)$ no longer commute in the quantum case. The idea to find the proper definition for the $Q_{(n)}$, the quantum analogue of the classical $q_{(n)}$, may be obtained from the following decomposition of the R matrix due to Drinfel'd. Let us suppose for one moment (just for the sake of getting an insight into this problem) that the R matrix in (12) may be written as [8]

$$R_{12} = F_{21}^{-1} F_{12} \quad (14)$$

where $F_{21} = P_{12} F_{12} P_{12}$ and obeys quadratic relations which ensures that R satisfies the quantum Yang-Baxter relation (13), see [8]. Then it is easy to recast the quantum group relation (12) into the form

$$F_{12} L_1 \cdot L_2 F_{12}^{-1} = F_{21} L_2 \cdot L_1 F_{21}^{-1} \quad (15)$$

Defining now a new product, we will denote by $\ast_F$, between the elements of the algebra generated by the matrix entries of L 's as

$$L_1 \ast_F L_2 = F_{12} L_1 \cdot L_2 F_{12}^{-1},$$

we obtain from (15) that this product is commutative (and associative thanks to the quadratic relations satisfied by F), i.e.,

$$L_1 \ast_F L_2 = L_2 \ast_F L_1.$$

Recalling now that

$$\text{tr}_1 (P_{12} A_1 B_2) = A_2 \cdot B_2$$

where A and B are arbitrary matrices, we have

$$\text{tr}_{12} (P_{12} A_1 B_2) = \text{tr} (A \cdot B)$$

Hence the classical invariant $q_{(2)}$ may be rewritten as

$$q_{(2)} = \text{tr}_{12} (P_{12} L_1 L_2) \quad (16)$$

The product $\ast_F$ being commutative, it is natural to expect the classical expression for the invariant to hold also in the quantum case if we use this $\ast_F$ product between the L matrices. So let us consider the quantity $\hat{q}_{(2)}$,

$$\hat{q}_{(2)} = \text{tr}_{12} (P_{12} L_1 \ast_F L_2)$$

that may be rewritten, using the cyclicity of the trace as

$$\tilde{q}_{(2)} = \text{tr}_{12} (P_{12} R_{12} L_1 \cdot L_2), \quad (17)$$

where the dot product in (17) is now the quantum group product defined in (12). Note that this formula is now given in terms of the R matrix only ; therefore we may use it whether or not this R matrix might be decomposed as in (14). In fact, defining $\tilde{q}_{(1)}$ to be as usual

$$\tilde{q}_{(1)} = \text{tr}_1 (L_1)$$

it is an easy computation, using only (12, 13), to show that

$$\tilde{q}_{(1)} \cdot \tilde{q}_{(2)} = \tilde{q}_{(2)} \cdot \tilde{q}_{(1)}$$

which is just the quantum generalization of (5) we are looking for, here for the case $n = 1, m = 2$. Now we shall remark that the quantity $\tilde{q}_{(2)}$ is obtained from its classical counterpart by simply replacing in (16) P_{12} by what is indeed its quantized version, namely $\hat{R}_{12} = P_{12} R_{12}$. The above considerations carry over to construct all the set of the quantized $Q_{(n)}$. For this purpose let us rewrite the $q_{(n)}$ of (2) as

$$q_{(n)} = \text{tr}_{1 \dots n} (P_{n-1 \ n} \dots P_{12} L_1 \dots L_n)$$

Then we define their quantum analogue $Q_{(n)}$ by

$$Q_{(n)} = \text{tr}_{1 \dots n} (\hat{R}_{n-1 \ n} \dots \hat{R}_{12} L_1 \dots L_n) \quad (18)$$

where the dots between the L 's represent the quantum group product. We have the quantum commutation relation (for all positive integers n and m)

$$Q_{(n)} \cdot Q_{(m)} = Q_{(m)} \cdot Q_{(n)} \quad (19)$$

To prove this result we will need to introduce some notations. Let $V_{\hat{a}} = V_{i_1} \otimes V_{i_2} \otimes \dots \otimes V_{i_n}$, where $\hat{a}$ stands for the ordered set $\hat{a} = (i_1, i_2, \dots, i_n)$. Then we define

$$G_{\hat{a}} = \hat{R}_{i_n \dots i_1 n} \dots \hat{R}_{i_1 i_1} \quad (20)$$

and

$$R_{\alpha \hat{a}} = R_{\alpha i_n} \dots R_{\alpha i_2} R_{\alpha i_1}$$

and similarly,

$$R_{\alpha \hat{a}} = R_{\alpha i_1 \alpha} R_{i_2 \alpha} \dots R_{i_n \alpha}$$

Then it is easy to check the following equalities for any $\hat{a}, \alpha$ and β

$$[G_{\hat{a}}, R_{\alpha \hat{a}}] = [G_{\hat{a}}, R_{\alpha \beta}] = 0 \quad (21)$$

by successive use of eq.(13). Defining now

$$L_{\hat{a}} = L_{i_1} \dots L_{i_n}$$

we have due to the invariance of form of eq.(12) under suitable tensor products (reflecting the co-product action)

$$R_{\alpha \hat{a}} L_{\hat{a}} \cdot L_{\hat{b}} = L_{\hat{b}} \cdot L_{\hat{a}} R_{\alpha \hat{b}} \quad (22)$$

where $\hat{a} = (i_1, \dots, i_n), \hat{b} = (j_1, \dots, j_m)$ and

$$\begin{aligned} R_{\alpha \hat{a}} &= R_{\alpha i_1 \delta} \dots R_{i_n \delta} \\ &= R_{\alpha j_m} \dots R_{\alpha j_1} \end{aligned} \quad (23)$$

is a surface-ordered product of R -matrices which is a representation of the universal $\mathcal{R}$ -matrix in $V_{\hat{a}} \otimes V_{\hat{b}}$ and hence which verifies the quantum Yang-Baxter equation [17]. Using these relations we have ($Q_{(n)} = Q_{\hat{a}}, Q_{(m)} = Q_{\hat{b}}$), compare with eq.(18),

$$Q_{\hat{a}} = \text{tr}_{\hat{a}} (G_{\hat{a}} L_{\hat{a}}) \quad (24)$$

and therefore,

$$\begin{aligned} Q_{\hat{a}} \cdot Q_{\hat{b}} &= \text{tr}_{\hat{a}\hat{b}} (G_{\hat{a}} G_{\hat{b}} L_{\hat{a}} \cdot L_{\hat{b}}), \\ &= \text{tr}_{\hat{a}\hat{b}} (G_{\hat{a}} G_{\hat{b}} R_{\hat{a}\hat{b}}^{-1} L_{\hat{b}} \cdot L_{\hat{a}} R_{\hat{a}\hat{b}}) \end{aligned}$$

Using, following eqs.(21,23), that $R_{\hat{a}\hat{b}}$ commutes with $G_{\hat{a}}$ and $G_{\hat{b}}$, and the cyclicity of the trace, we obtain the announced result (19). Let us note here that we also have the fact that $L_{\hat{a}}$ commutes with $G_{\hat{a}}$ which means that the element $G_{\hat{a}}$ is central in the representation of the dual of the quantum group in $V_{\hat{a}}$.

The quantities given in (18,24) may then be thought of as the quantum invariants of the quantum L matrices. They generate an Abelian sub-algebra of the quantum group. Note that the expressions we have obtained are completely algebraic and hence are valid for any constant R matrix satisfying the quantum Yang-Baxter equation¹. In particular the quantum determinant may be computed in terms of the $Q_{(n)}$. Let us exemplify this point in the most simple case where L is a 2×2 matrix and R is given by (see for example [16])

$$R = e_{11} \otimes e_{22} + e_{22} \otimes e_{11} + q (e_{11} \otimes e_{11} + e_{22} \otimes e_{22}) + (q - q^{-1}) e_{21} \otimes e_{12}$$

e_{ij} being the matrix with entries defined as $(e_{ij})_{ab} = \delta_{ia} \delta_{jb}$. Using the commutation relations (12) we obtain the relation

$$Q_{(2)} = q Q_{(1)}^2 - (q + q^{-1}) \det_q(L) \quad (25)$$

where $\det_q(L)$ is the so-called quantum determinant [19,16]. This is in fact a q -deformed Newton relation.

4. Quantum Lax Equations

The purpose of this section is to give the quantum analogue of the classical Lax equations, associated now to the quantum flows defined by the $Q_{(n)}$, and to exhibit a quantum zero curvature condition for them analogous to eq.(7). To get an insight into this problem let us recast eq.(12) in the following form

$$L_1 \cdot L_2 R_{12}^{-1} = R_{12}^{-1} L_2 \cdot L_1$$

and take the trace on the numerical matrix-space 2. We obtain, using the cyclicity of the trace,

$$L_1 \cdot \hat{U}_1^{(1)} = \hat{U}_1^{(1)} \cdot L_1,$$

¹In the situation where a spectral parameter λ is present and assuming that for a particular value of λ the R -matrix becomes a projector, it is possible to derive a particular case of the above formulas [18,19]. We did not assume any such property here and our result holds true for arbitrary R -matrices satisfying the quantum Yang-Baxter equation.

with

$$\hat{U}_1^{(1)} = \text{tr}_2(L_2 R_{12}^{-1}).$$

If we expand the R -matrix in powers of $\hbar$ when $\hbar \rightarrow 0$, namely, $R_{12} = Id_{12} + \hbar \tau_{12} + O(\hbar^2)$, the above quantum matrix $\hat{U}_1^{(1)}$ becomes

$$\hat{U}_1^{(1)} \rightarrow \text{tr}(L) - \hbar \text{tr}_2(\tau_{12} L_2) + O(\hbar^2).$$

So in the classical limit this operator acts on L as the classical covariant $U_1^{(1)}$ matrix in (6). In addition it contains the quantum invariant $Q_{(1)}$. Defining now the quantum flow associated to $Q_{(n)}$ as $-\hbar \partial_n$ we can write down the quantum $\hat{M}$ -matrix associated to ∂_l as

$$\hbar \hat{M}_1^{(1)} = \hat{U}_1^{(1)} - 1_1 Q_{(1)} \quad (26)$$

which verifies the quantum Lax equation

$$\partial_l L_1 = [\hat{M}_1^{(1)}, L_1], \quad (27)$$

where the product between $\hat{M}$ and L is in the numerical matrix space and in the quantum group algebra. In a similar way as we defined in the previous section the $Q_{(n)}$, it is also possible to construct the quantum $\hat{U}_1^{(n)}$ associated to the n^{th} quantum flow as

$$\hat{U}_1^{(n)} = \text{tr}_a(G_a L_a R_{1a}^{-1}). \quad (28)$$

We have after some algebra, using eqs.(21,22),

$$[\hat{U}_1^{(n)}, L_1] = 0, \quad (29)$$

which is the quantum analogue of (6) in the sense that the classical limit of (29) is indeed (6). The corresponding quantum $\hat{M}$ operator is then

$$\hbar \hat{M}_1^{(n)} = \hat{U}_1^{(n)} - 1_1 Q_{(n)} \quad (30)$$

which verifies the quantum Lax equation

$$\partial_n L_1 = [\hat{M}_1^{(n)}, L_1] \quad (31)$$

Note here that the product used in this formula is in the numerical matrix space (1) and in the quantum group space. Finally let us remark that this hierarchy of quantum flows are indeed commuting and satisfy the following quantum zero curvature equations that may be stated again using eqs.(21,22)

$$\left[\hat{U}_1^{(n)}, \hat{U}_1^{(m)} \right] = 0, \quad (32)$$

or equivalently in terms of the $\hat{M}$ operators,

$$\partial_m \hat{M}^{(n)} - \partial_n \hat{M}^{(m)} + \left[\hat{M}^{(n)}, \hat{M}^{(m)} \right] = 0 \quad (33)$$

which is a quantum zero curvature equation analogous to the classical relation (7).

5. Quantum Lattice Field Theory

The aim of this section is to give a field theoretical extension of the previous quantum mechanical constructions. We will describe it in the case of a discretized space given as a one-dimensional lattice of size N with periodic boundary conditions. As in the classical case, we assign to each site (n) of this lattice a copy $\mathcal{A}_n$ of the quantum group algebra (12), described by a given constant R -matrix satisfying the quantum Yang-Baxter equation, and quantum $L(n)$ -matrices depending, through the field variables, explicitly on the site (n) (and commuting for different sites) as

$$R_{12} L_1(n) \cdot L_2(n) = L_2(n) \cdot L_1(n) R_{12} \quad (34)$$

Then the quantum monodromy matrix

$$T_1 = L_1(N) \otimes \dots \otimes L_1(1)$$

where the tensor products are with respect to the quantum group algebra $\mathcal{A}_n$, satisfies the same equation as L itself, i.e.,

$$R_{12} T_1 \cdot T_2 = T_2 \cdot T_1 R_{12} \quad (35)$$

since the quantum L operators commute for different sites. The corresponding commuting quantum quantities, invariant under space-translations, are given as (we use the same notations as in the preceding sections, i.e., $\hat{a} = (i_1, i_2, \dots, i_p)$)

$$Q_{(p)} = \text{tr}_a (G_a T_a) \quad (36)$$

We may use these quantities to define quantum Hamiltonian flows that admit the following local quantum Lax representations. Let

$$\hat{U}_i^{(p)}(m) = \text{tr}_a (G_a W_{1,a}(m)) \quad (37)$$

where

$$W_{1,a}(m) = W_{1,i_1}(m) W_{1,i_2}(m) \dots W_{1,i_p}(m) \quad (38)$$

and (there is a matrix product on space i_k)

$$W_{1,i_k}(m) = L_{i_k}(N) \otimes \dots \otimes L_{i_k}(m) R_{1,i_k}^{-1} L_{i_k}(m-1) \otimes \dots \otimes L_{i_k}(1) \quad (39)$$

We obtain the local relation

$$\hat{U}_1^{(p)}(m+1) \cdot L_1(m) = L_1(m) \cdot \hat{U}_1^{(p)}(m) \quad (40)$$

Defining now $Q_{(p)}$ as the quantum Hamiltonian associated to the p^{th} flow we have

$$\partial_p L_1(m) = \hat{M}_1^{(p)}(m+1) \cdot L_1(m) - L_1(m) \cdot \hat{M}_1^{(p)}(m) \quad (41)$$

where

$$\hbar \hat{M}_1^{(p)}(m) = \hat{U}_1^{(p)}(m) - 1_1 Q_{(p)} \quad (42)$$

which is just the quantum version of eq.(10)². Note here that in the quantum case the operator $Q_{(p)}$ would be better thought of as an evolution operator associated to a quantum Hamiltonian rather than the Hamiltonian itself. The reason is that the monodromy matrix T is in fact an ordered exponential integral of the local objects L . If we follow this line of reasoning, it is easy to obtain the completely discretized (space and time) quantum Lax equations associated to the quantum evolution operators $Q_{(p)}$. Let

²For the first quantum invariant $Q_{(1)}$, and in the presence of a spectral parameter, an analogous formula was given in [20]. I wish to thank V.E. Korepin for discussion about this point.

us consider all the relevant objects to be functions of the lattice (space) variable (m) and of the discrete time t_p , such that the adjoint action of $Q_{(p)}$ gives a time-shift of one unit; we have

$$L_1(t_p + 1, m) \cdot V_1^{(p)}(t_p, m) = V_1^{(p)}(t_p, m + 1) \cdot L_1(t_p, m) \quad (43)$$

with

$$V_1^{(p)}(t_p, m) = Q_{(p)} \tilde{U}_1^{(p)}(t_p, m) \quad (44)$$

which is now a quantum analogue of the discretized classical zero-curvature equation (11). It is interesting to note that the quantum case is very naturally associated to these lattice equations. In fact eq.(43) may also be reinterpreted as a kind of universal classical lattice Lax equation. Namely it is valid for any representation of the quantum group A , and taking any such matrix representation for the quantum fields contained in L will lead us to particular classical discrete Lax equations. In this sense the quantum R -matrix provides us with the relevant structure constants that generate an integrated or discretized universal Poisson structure compatible with the integrable lattice dynamics. These interesting features deserve more detailed investigations. I hope to come back to these questions in the future. Finally, another remark is in order for the quantum structures we just described. If we had made use not of the numerical R -matrix, but rather of the universal one, it would have been possible to obtain universal quantum Lax equations. The subtleties involved in doing that are linked to the construction of a trace formula in such a case, which is related to the existence of a universal G_a object.

References

- [1] P. D. Lax, Comm Pure Appl. Math. **21** (1969) 467.
- [2] V.G. Drinfel'd, Sov. Math. Dokl. **32** (1985) 254, ibid. **27** (1983) 68.

- [3] M. Semenov-Tian-Shanskii, Funct. Anal. Appl. **17** (1984) 259, Publ. RIMS **21** (1985) 1237.
- [4] C.N. Yang, Phys. Rev. Lett **19** (1967) 1312.
- [5] R.J. Baxter, Ann. Phys. **70** (1972) 193.
- [6] L.D. Faddeev, in *Recent developments in Field Theory and Statistical Mechanics*, eds. R. Stora and J.B. Zuber, (North Holland, Amsterdam, 1983)
- [7] L.D. Faddeev and L. Takhtajan, *Hamiltonian Methods in the Theory of Solitons*, (Springer Verlag, Berlin, 1987)
- [8] V.G. Drinfel'd, *Quantum Groups*, Proc. ICM Berkeley 1986, ed. A.M. Gleason, (AMS, Providence, 1987), p. 798.
- [9] P.P. Kulish and N.Yu. Reshetikhin, J. Sov. Math. **23** (1983) 2435.
- [10] M. Jimbo, Lett. Math. Phys. **10** (1985) 63, ibid. **11** (1986) 247, Commun. Math. Phys. **102** (1986) 537.
- [11] N. Yu. Reshetikhin, *Quantized Universal Enveloping Algebras, the Yang-Baxter Equations and Invariants of Links, I,II*, LOMI Preprints E-4-87, E-17-87, (1987).
- [12] A.N. Kirillov and N. Yu. Reshetikhin, *Representations of the Algebra $U_q(sl(2))$, q -orthogonal polynomials and invariants of links*, LOMI Preprint E-9-88, (1988).
- [13] Yu. I. Manin, *Quantum Groups and Non-commutative Geometry*, Publ. CRM, Université de Montréal (1989).
- [14] M. Rosso, C.R. Acad. Sci. Paris **t304** Série I (1987) 323, ibid. **t305** (1987) 587.
- [15] S. Woronowicz, Publ. RIMS (Kyoto) **23** (1987) 117, Commun. Math. Phys. **111** (1987) 613, Inv. Math. **93** (1988) 35, Commun. Math. Phys. **122** (1989) 125.

- [16] L.D. Faddeev, N.Yu. Reshetikhin and L.A. Takhtajan, *Quantization of Lie Groups and Lie Algebras*, LOMI-Preprint E-14-87.
- [17] J.M. Maillet and F.W. Nijhoff, *Phys. Lett.* **B224** (1989) 389, *ibid.* **B229** (1989) 71.
- [18] P.P Kulish, N. Reshetikhin and E.K. Sklyanin, *Lett. in Math. Phys.* **5** (1981) 393.
- [19] P.P Kulish and E.K. Sklyanin, in *Tvarmine Lectures*, eds. J. Hietarinta and C. Montonen, Springer Lect. Phys. vol.151 (1982).
- [20] A.G. Izergin and V.E. Korepin, *Sov. Phys. Dokl.* **26** (1981) 653.