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ON BRAID STATISTICS

R. Cappuccio

Istituto Nazionale di Fisica Nucleare, Sezione di Pisa

and

E. Guadagnini*

CERN - Geneva

Abstract

It is shown how the action of the braid group generators can be interpreted as a combination of permutations and colored braid group transformations. Explicit solutions, constructed in terms of ordinary Lie algebras and in which the diagonal action of the Lie group belongs to the centralizer, are presented.

* *On leave from Dipartimento di Fisica dell'Università di Pisa and INFN, Sezione di Pisa*

The Artin braid group B_n of n strings represents some kind of generalization of the permutation group P_n on n objects. One can give [1] the following presentation of B_n ; it is generated by $n - 1$ elements (generators) $\{g_1, g_2, \dots, g_{n-1}\}$ satisfying the relations:

$$g_i g_j = g_j g_i \quad , \quad |i - j| > 1, \quad (1)$$

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \quad , \quad i < n - 1. \quad (2)$$

A well-known graphical representation of B_n exists. For instance, one can associate to g_i the picture shown in Fig.1 and the product of two generators, say $g_2^{-1} g_1$, can be represented as in Fig.2.

Eqs.(1) and (2) define an equivalence relation between the product of generators and, consequently, between the associated graphs. In particular, two diagrams are equivalent iff they are related by smooth deformations in R^3 performed by keeping the two planes, to which the endpoints of the strings belong, fixed. So, conditions (1) and (2) give just a realization of ambient isotopy; for this reason the braid group plays an important role in knot theory.

The three-dimensional graphs associated with the elements of B_n can be understood as world-line pictures of n -point particles lying on a plane and moving each around the others as the time goes on. The final positions of the n particles must coincide with the initial positions up to a permutation; therefore, the braid group has also a direct application in the context of $(2 + 1)$ -dimensional particle physics [2,3,4]. Three-dimensional models with anyons [5] are, perhaps, the most familiar examples. Representations of B_n appear also in conformal field theory [6] and, more generally, in two-dimensional integrable models [7,8]. In the three-dimensional non-Abelian Chern-Simons theory [9], the strings on which the braid group acts are precisely Wilson-line operators in some definite representation of the gauge group [10]. Finally, the braid group may also enter, for topological reasons, into the study of generalised kinematics of extended objects in higher dimensions.

Now, one can make abstraction of the three-dimensional context which is at the origin of the braid group and pose the problem of its realizations in general terms. Independently of the geometrical or dynamical origin of the braiding symmetry, one can consider the problem of finding how the wave functions of a system of n identical "particles" behave under braid transformations. The different types of irreducible representations realized on the wave functions can be interpreted as different kinds of generalized or braid statistics. Within this program, the explicit construction of the

operator realizations of the braid transformations represents in fact the main problem [11]. Part of the difficulties that one finds are due to the fact that, apart from a few exceptional cases, a simple dynamical interpretation of the braiding symmetry is still missing.

The universal R-matrix approach to the construction of braid representations [7,12] is certainly powerful; unfortunately, the underlying quantum group structure [13] does not have a transparent physical meaning. On the other hand, recent developments [14,15] on this subject show that new possible formulations in terms of ordinary Lie algebras could lead to a deeper understanding of the matter.

Some results concerning the construction of braid representations are reported in the present paper. The strategy pursued here is based essentially on two points: first, disentangle somehow the action of the permutation group out of B_n ; second, maintain the standard Lie algebra structure. Both of these requirements are based on physical considerations.

We will show how the first point can be fulfilled and that, under certain conditions, the problem of finding B_n representations reduces to finding C_n representations, where C_n is the colored braid group. Combining this last result with the second point mentioned above, we are led to consider the construction of C_n representations in the framework of ordinary Lie algebras. We present here a general solution for the first two non-trivial cases, namely for C_3 and C_4 representations associated with an arbitrary standard Lie algebra (in arbitrary representation).

Before entering into details, let us consider what is expected to come out on general grounds. Let us recall that, defining $X_n = \{(x_1, x_2, \dots, x_n) \in R^{2n} : x_i \neq x_j \text{ for } i \neq j\}$, the braid group coincides with the fundamental group of $Y_n = X_n / P_n$. On the other hand, if one does not identify the elements of X_n under the action of the permutation group, then one finds $\pi_1(X_n) = C_n$, where C_n is the colored (or pure) braid group. So, by factorizing the action of the permutation group, one selects the colored braid group. A possible way to do that is to decompose B_n into equivalence classes by right multiplication with elements of $C_n \subset B_n$. The equivalence classes are in one-to-one correspondence with the elements of P_n . Consequently, left multiplication by a generic element of B_n modifies the equivalence classes according to the action of B_n on P_n and the elements of a given class by the action of C_n .

In a slightly different context, this kind of construction is well known in physics: if B_n is replaced by a symmetry group G and C_n by a subgroup H of G , then one finds the Coleman-Wess-Zumino [16] construction for non-linear realizations of G with exact subgroup H . What is peculiar in our case is the fact that G/H , equipped with

the natural linear structure, can be identified with the regular representations of P_n .

In practice, one can proceed as follows. Each element of $b \in B_n$ defines a permutation $\bar{b} \in P_n$ of the n end-points of the strings. The map π from B_n to P_n

$$\pi : b \mapsto \pi(b) = \bar{b} \quad (3)$$

represents a homomorphism of B_n into P_n . Indeed, P_n also has a presentation in terms of $(n-1)$ generators $(t_1, t_2, \dots, t_{n-1})$ with defining relations

$$t_i t_j = t_j t_i, \quad |i - j| > 1, \quad (4)$$

$$t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}, \quad i < n-1, \quad (5)$$

$$t_i^2 = 1, \quad i = 1, 2, \dots, n-1. \quad (6)$$

Condition (6) makes P_n finite (ord $P_n = n!$) and shows that π is only a homomorphism.

One has

$$\text{Ker } \pi = C_n, \quad (7)$$

where the colored braid group C_n is, by definition, the subgroup of B_n whose elements send an ordered (or colored) set on n points in itself preserving the ordering (or coloring). The map π is not invertible in B_n , but it can be inverted in a subset Ξ of B_n . The choice of Ξ is not unique, of course. For convenience, Ξ can be chosen in the following manner: given a particular form of all the elements of P_n in terms of the generators t_i , the set Ξ is simply obtained by substituting t_i with the generators g_i of B_n in the expression of the elements of P_n . It is easy to verify that this is a good choice for Ξ .

From eqs.(3) and (7), it follows that any element $b \in B_n$ can be written in a unique way as

$$b = \xi \cdot \gamma. \quad (8)$$

where $\xi \in \Xi$ and $\gamma \in C_n$. Indeed, $\xi = \pi^{-1}(\bar{b}) \in \Xi$ and $\gamma = \xi^{-1} \cdot b$.

Consider now the action of B_n on ξ by left multiplication. It is sufficient to concentrate ourselves on the generators; one has

$$g_i \cdot \xi = \xi' \cdot \gamma. \quad (9)$$

where $\xi' \in \Xi$ and $\gamma \in C_n$, both depending on ξ and g_i .

As an example, let us consider B_3 . The six elements of P_3 are

$$\{1, t_1, t_2, t_1 t_2, t_2 t_1, t_1 t_2 t_1\}$$

and therefore the elements of Ξ are taken to be

$$\{1, g_1, g_2, g_1 g_2, g_2 g_1, g_1 g_2 g_1\}.$$

The group C_3 is generated by three elements

$$\gamma_{12} = g_1^2, \quad \gamma_{13} = g_1^{-1} g_2^2 g_1, \quad \gamma_{23} = g_2^2, \quad (10)$$

and the relations described by eq.(9) are

$$\begin{aligned} g_1 \cdot 1 &= g_1 & g_1 \cdot g_1 g_2 &= g_2 \cdot g_1 g_2 g_1 g_2^{-1} \\ g_1 \cdot g_1 &= 1 \cdot \gamma_{12} & g_1 \cdot g_2 g_1 &= g_1 g_2 g_1 \\ g_1 \cdot g_2 &= g_1 g_2 & g_1 \cdot g_1 g_2 g_1 &= g_2 g_1 \cdot \gamma_{23} \end{aligned} \quad (11)$$

and

$$\begin{aligned} g_2 \cdot 1 &= g_2 & g_2 \cdot g_1 g_2 &= g_1 g_2 g_1 \\ g_2 \cdot g_1 &= g_2 g_1 & g_2 \cdot g_2 g_1 &= g_1 \cdot \gamma_{13} \\ g_2 \cdot g_2 &= 1 \cdot \gamma_{23} & g_2 \cdot g_1 g_2 g_1 &= g_1 g_2 \cdot \gamma_{12} \end{aligned} \quad (12)$$

Coming back to the general situation, the result of Ref.[16] can be formulated in terms of the "master set" $M = \{(\xi, \Psi)\}$, where $\xi \in \Xi$ and Ψ belongs to a vector space V on which a linear representation C of C_n is defined. The B_n representation realized on the elements of M is given by [16]

$$g_i : (\xi, \Psi) \mapsto (\xi', \Psi') = C(\gamma) \Psi. \quad (13)$$

where ξ' and γ are defined in eq.(9).

Given any linear B_n representation, one can easily construct [16] the master set M and recover eq.(13). Vice-versa, given the master set M , one can in principle construct B_n representations, following the line suggested in [16], provided the appropriate decomposing matrices are known.

In the present paper we will consider a further possibility. Since Ξ has a finite number of elements, a linear B_n representation can immediately be obtained from M . Indeed, by equipping M with the natural linear structure, eq.(13) defines a linear B_n representation in a vector space of dimension $D = (n!) \cdot \dim V$. This representation, called the master representation, has actually the direct physical interpretation illustrated below.

Consider a wave function $\phi_1(y_1, y_2, \dots, y_n)$ of n identical "particles", where y_i (with $i = 1, 2, \dots, n$) denotes the set of variables (or "orbital" variables) of the i -th particle. In general, ϕ_1 is not invariant under particle permutations; even more, we do not want to impose on ϕ_1 any a priori fixed behaviour under P_n . So, let us assume that the wave functions obtained from ϕ_1 by applying the elements of P_n are linear independent and give rise to a linear space $\mathcal{L}$ of dimension $n!$. By definition, $\mathcal{L}$ is a vector space on which the regular representation of P_n is realized.

The wave functions $\{\phi_t, t \in P_n\}$, obtained from ϕ_1 under the action of P_n , can be taken as a basis in $\mathcal{L}$; there is, then, a one-to-one correspondence between the elements of P_n and hence of $\Xi \subset B_n$ and a basis in $\mathcal{L}$. How to obtain a master representation is clear now. It is sufficient to require that, in addition to the "orbital" degrees of freedom, each particle has also "colour" degrees of freedom. Assume that the colour states of the i -th particle belong to a vector space V_i on which a representation R of a Lie group G is defined. The colour states on the n particles belong to $V = V_1 \otimes V_2 \otimes \dots \otimes V_n$. Then, the tensor product $\mathcal{L} \otimes V$ defines a master representation of B_n provided a linear C_n representation is realized on V .

Since π defines a bijective correspondence between the elements of Ξ and those of P_n , the transformation $\xi \mapsto \xi'$ in eq.(13) can be interpreted as the result of a permutation. Therefore, from eq.(13) it follows that the action of any element of B_n can be understood as a combination of a permutation and a colored braid group transformation. This provides the solution of the first problem mentioned before.

Of course, the master set construction does not produce directly B_n representations, because one has to find now C_n representations. Hopefully, however, this last problem has more chance of being solved in quite general terms than the original one. As an example we present here a solution for the first two non-trivial cases: C_3 and C_4 .

It is perhaps useful to recall first some properties of C_n . With the natural identification of C_n as a subgroup of B_n , the $n(n-1)/2$ generators $\{\gamma_{ij}\}$ of C_n can be written as follows:

$$\gamma_{ij} = g_1^{-1} \dots g_{j-2} g_{j-1}^2 g_{j-1} g_{j-2} \dots g_i, \quad 1 \leq i < j \leq n. \quad (14)$$

The relations satisfied by $\{\gamma_{ij}\}$ can easily be obtained by using eqs.(1,2).

For $i < j < k < l$ one finds:

$$\gamma_{ij} \gamma_{kl} = \gamma_{kl} \gamma_{ij}, \quad \gamma_{il} \gamma_{jk} = \gamma_{jk} \gamma_{il}, \quad (15)$$

for $i < k < j$ one gets:

$$\gamma_{ik} \gamma_{ij} \gamma_{kj} = \gamma_{kj} \gamma_{ik} \gamma_{ij}, \quad (16)$$

$$\gamma_{ij} \gamma_{kj} \gamma_{ik} = \gamma_{ik} \gamma_{ij} \gamma_{kj}, \quad (17)$$

and, finally, for $i < k < l < j$ one has:

$$\gamma_{ik} \gamma_{il} \gamma_{ij} \gamma_{kj} = \gamma_{kj} \gamma_{ik} \gamma_{il} \gamma_{ij}. \quad (18)$$

Let us recall also that, since the centre of B_n is generated by $(g_1 g_2 \dots g_{n-1})^n$, the natural inclusion $B_2 \subset B_3 \subset \dots \subset B_n$ shows that

$$\{\gamma_{12}, \gamma_{13}, \gamma_{23}, \gamma_{12} \gamma_{13} \gamma_{23}, \gamma_{12} \gamma_{13} \gamma_{23} \gamma_{14} \gamma_{34}, \dots\}$$

is a set of commuting elements.

In the case of C_3 , only the conditions (16) and (17) have to be satisfied, whereas for C_4 all the relations (15-18) give non-trivial constraints. Let $c = R^a R^a$ be the quadratic Casimir operator constructed with the generators R^a of the R -representation of the group G . We are interested in finding a representation of C_3 on $V_1 \otimes V_2 \otimes V_3$ such that the commuting elements γ_{12} and $\gamma_{12} \gamma_{13} \gamma_{23}$ are represented by

$$\gamma_{12} \mapsto e^{i\lambda \Omega_{12}}, \quad (19)$$

$$\gamma_{12} \gamma_{13} \gamma_{23} \mapsto e^{i\lambda(\Omega_{12} + \Omega_{13} + \Omega_{23})}, \quad (20)$$

where

$$\Omega_{ij} = R_i^a R_j^a \quad (21)$$

acts non-trivially on $V_i \otimes V_j$ and λ can be considered as a free parameter. In several physical systems [6,7,8] the commuting elements have precisely the form shown in

eqs.(19) and (20) and the value of λ is often given by $\frac{2\pi}{k}$ where k is an integer. A simple solution of eqs.(16,17,19,20) is given by

$$\begin{aligned} \gamma_{12} &\mapsto e^{i\lambda\Omega_{12}} , \\ \gamma_{13} &\mapsto e^{-i\lambda\Omega_{12}} e^{i\lambda(\Omega_{12}+\Omega_{13})} , \\ \gamma_{23} &\mapsto e^{i\lambda\Omega_{23}} . \end{aligned} \quad (22)$$

For the representation of C_4 , in addition to eqs.(19,20), we require

$$\gamma_{12} \gamma_{13} \gamma_{23} \gamma_{14} \gamma_{24} \gamma_{34} \mapsto e^{i\lambda(\Omega_{12}+\Omega_{13}+\Omega_{23}+\Omega_{14}+\Omega_{24}+\Omega_{34})} . \quad (23)$$

An explicit solution is given by

$$\begin{aligned} \gamma_{12} &\mapsto e^{i\lambda\Omega_{12}} , \\ \gamma_{13} &\mapsto e^{-i\lambda\Omega_{12}} e^{i\lambda(\Omega_{12}+\Omega_{13})} , \\ \gamma_{23} &\mapsto e^{i\lambda\Omega_{23}} , \\ \gamma_{24} &\mapsto e^{i\lambda(\Omega_{14}+\Omega_{24}+\Omega_{34})} e^{-i\lambda\Omega_{23}} e^{-i\lambda\Omega_{12}} , \\ \gamma_{34} &\mapsto e^{-i\lambda(\Omega_{14}+\Omega_{24}+\Omega_{34})} e^{i\lambda\Omega_{12}} , \\ \gamma_{14} &\mapsto e^{i\lambda(\Omega_{14}+\Omega_{24}+\Omega_{34})} e^{i\lambda\Omega_{23}} . \end{aligned} \quad (24)$$

Eqs.(22) and (24) define C_3 and C_4 representations for any group G and any representation R . Note that if one is interested in C_3 and C_4 only, the representations realized on V_1 , V_2 , V_3 and V_4 can be chosen to be different (and arbitrary). In any case, the diagonal action of G on V belongs to the centralizer of C_3 and C_4 and, consequently, it also belongs to the centralizer of the associated master representations of B_3 and B_4 .

With respect to the usual R -matrix construction, the master representations associated with eqs.(22,24) present some complementarity aspects. More precisely, in the former approach the inclusion $B_n \subset B_{n+1}$ is straightforward, whereas a change in the choice of the representation presents difficulties. In fact, the R -matrix method cannot be used directly for producing C_n representations when different strings are associated with different representations of G . On the contrary, in the latter case the form of the solutions (22,24) is universal as far as the group G and its representations are concerned, whereas the inclusion $B_n \subset B_{n+1}$ is not completely straightforward.

Summarizing, the master set construction presented here can be considered as a starting point for exploring some properties of the braid group representations. This method, combined with the standard R -matrix approach, could hopefully suggest new developments in both the construction of B_n (and C_n) representations and the understanding of the interplay between Hopf (or quasi-Hopf) and standard Lie algebras.

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FIGURE CAPTIONS

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Fig.1 Graphical representation of g_1 .

Fig.2 Graphical representation of $g_2^{-1} g_1$.

Fig. 1

Fig. 2