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Beltrami Differentials, Conformal Models and their Supersymmetric Generalizations

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Abstract

Conformally invariant couplings of two-dimensional field theories to gravity can be formulated either in a Riemannian surface or in a Riemannian manifold (metric) framework. After a detailed presentation of the Riemannian surface approach (and comparison with the metric formalism), we develop its supersymmetric generalization. Super Beltrami differentials are introduced without any reference to metric (vielbein) structures and superconformal models are studied in this framework. The main goal of our work consists of the construction of local (and supersymmetric) field theories defined on arbitrary (super-) Riemann surfaces and exhibiting the (super-) holomorphic factorization in a manifest way.

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Chapter 1

Introduction

Beltrami differentials parametrize conformal classes of 2-dimensional metrics [1]. Therefore, it is natural to use these quantities as basic variables whenever a 2-dimensional field theory is coupled to gravity in a conformally invariant way. This viewpoint has been taken and pursued by various authors, both for the bosonic and supersymmetric theories (e.g. see [2]-[6] and [7] [8]). One of the most remarkable and satisfying aspects of this approach is that it renders the so-called factorization [9] manifest at all stages of the gauge-fixing procedure. Further motivation is provided in references [2]-[4] and a comparison with other approaches is made at the end of reference [2] (see also our chapter 4).

In a Euclidean framework, Beltrami coefficients can be introduced in two different, although equivalent ways. First, in the Riemannian manifold approach, they parametrize conformal classes of metrics [2]-[4]. Second, in the Riemannian surface formalism, they parametrize complex structures [5] [6]. In the first approach, one has the possibility to consider Weyl rescalings of the metric, but for the second, the only symmetries to remain are the reparametrizations. In both cases, the Beltrami differentials change only under these symmetries, their transformation laws being identical in both cases. The fact of not having to deal with Weyl rescalings in the second case appears to be a great advantage, in particular for supersymmetric theories where Weyl rescalings generalize to inhomogeneous transformations in superspace [10].

In supersymmetry, one has to face an additional (related) complication in that the supervielbeins are constrained by Wess-Zumino-type conditions on the torsion [10] [11] [12]. Quantization in superspace requires an explicit solution of these conditions which consists of expressing the constrained supervielbeins in terms of prepotential superfields. This provides a Beltrami parametrization in superspace [13] [14]. Although the supervielbein formulations could be successfully applied to various problems [15]-[18] [8], their current form appears to be unsatisfactory and/or incomplete. For instance, in references [13] [14], the holomorphic factorization is not realized for the basic transformation laws and the variation of the so-called supersymmetry compensator is not explicitly known. On the other hand, in references [8], the factorization for the variations of the Beltrami superfields was achieved following a certain pre-gauge-fixing, but so far an explicit expression for the action of matter fields in terms of these variables has not been given (nor has its quantization been investigated). Furthermore, the anomaly problem in superspace has only been addressed in a very particular case [14].

In this paper, we propose a Riemannian surface approach to the formulation and quantization of superconformal models. As a first step, we give a general definition of super Beltrami differentials on super Riemann surfaces and show that a natural choice of variables leads to the holomorphic factorization of symmetry transformations¹. Subsequently, we introduce superconformal fields and apply our results to the study of superconformal models.

The article is organized as follows. In chapter 2, we develop the Riemannian surface approach to the bosonic theory while stressing both the differences and relations with the metric formalism. We use the standard mathematical definitions, but we should mention that slightly different definitions and geometric interpretations of Beltrami coefficients are often considered in the physics literature. Therefore, some comments on the relations to these viewpoints have been included in section 2.5. In chapter 3, we study the conformally invariant coupling of scalar superfields to $(1,1)$ supersymmetry and we present the Riemannian surface approach to this theory. Following the general discussion, we show that the projection to component fields leads to simple and physically transparent space-time results. In conclusion, the geometric interpretation of Beltrami superfields and the supersymmetric generalization of quasi-conformal mappings are commented upon. Appendices A and B are devoted to a derivation of some basic results of the metric formalism and Appendix C summarizes the Riemannian surface approach to the $(1,0)$ supersymmetric theory.

¹This has been pointed out before in the metric approach [8].

Chapter 2

Bosonic theory

This chapter is based on two brief reports [5] which focused on Ward identities and their anomalous breaking. For various other aspects not treated here, we refer to [6]. The underlying scheme is that of local field theory in the Lagrangian formalism.

2.1 General framework

The arena we will work on is a Riemann surface M , i.e. a connected, topological 2-manifold which is equipped with a complex structure (or, equivalently, a real, smooth $(= C^\infty)$, connected and oriented 2-manifold which is equipped with a conformal class of metrics [1])¹. Roughly speaking, a *complex structure* is a collection of complex local coordinates satisfying the following property: if $(Z, \bar{Z})$ and $(Z', \bar{Z}')$ are two sets of such local coordinates for the same region of M , then both are related by a holomorphic (=complex analytic) coordinate transformation.

As discussed in appendix A.1, conformal classes of metrics on 2-manifolds are parametrized by Beltrami coefficients $\mu(z, \bar{z})$: these are smooth complex-valued functions with specific transformation properties to be given below. In order to formulate the equivalent of this parametrization for complex structures, we compare the Beltrami parametrization of the metric, $ds^2 \propto |dz + d\bar{z}\mu|^2$, with the metric written in terms of isothermal coordinates, $ds^2 \propto |dZ|^2$ and draw the following conclusion. If $(z, \bar{z})$ denotes a reference system of holomorphic coordinates (corresponding to a reference complex structure), then the holomorphic coordinates $(Z, \bar{Z})$ corresponding to the complex structure parametrized by μ are given by the relation

$$dZ = \lambda(z, \bar{z}) [dz + d\bar{z}\mu(z, \bar{z})] \quad \text{and c.c.} \quad (2.1)$$

Here, λ and μ are smooth complex-valued functions and the condition $d(dZ) = 0$ is equivalent to λ satisfying the differential equation

$$(\bar{\partial} - \mu\partial)\lambda = (\partial\mu)\lambda \quad (2.2)$$

Thus, λ is to be viewed as an *integrating factor* for the relation $d(dZ) = 0$. (For $\mu \neq 0$, it is not possible to set $\lambda \equiv 1$.)

¹Whenever integration over M is considered, the surface is assumed to be compact in order to avoid divergence problems of integrals.

Locally, Z and $\bar{Z}$ can be given by smooth functions :

$$(z, \bar{z}) \longrightarrow (Z(z, \bar{z}), \bar{Z}(z, \bar{z}))$$

The induced variation of the differential dZ then reads

$$\begin{aligned} dZ &= dz(\partial Z) + d\bar{z}(\bar{\partial} Z) \\ &= (\partial Z) [dz + d\bar{z} \frac{\bar{\partial} Z}{\partial Z}] \end{aligned}$$

where $\partial \equiv \partial_z$ and $\bar{\partial} \equiv \partial_{\bar{z}}$. Henceforth, we recover the parametrization (2.1) with $\lambda = \partial Z$ and $\mu = (\bar{\partial} Z)/(\partial Z)$. This last relation is equivalent to $Z(z, \bar{z})$ satisfying the *Beltrami equation*

$$(\bar{\partial} - \mu\partial)Z = 0 \quad (2.3)$$

The function μ is usually referred to as *Beltrami coefficient* or *Beltrami differential* [1], the reason for this last terminology being as follows. In the foregoing and in most of what follows, we have suppressed the indices of μ and λ which are explicitly given by:

$$dZ = [dz + d\bar{z}\mu_z] \lambda_z$$

Accordingly, the functions μ and λ transform under complex analytic coordinate transformations $z \rightarrow z'(z)$ in such a way that the differentials $\mu_z dz + d\bar{z}\partial_z$ and $\lambda_z dz$ are invariant:

$$\mu_z^z = \mu_{z'}^{z'} \frac{\partial z'}{\partial z} \quad , \quad \lambda_z = \lambda_{z'} \frac{\partial z'}{\partial z} \quad (2.4)$$

This explains the abusive terminology Beltrami ‘differential’ for the function μ .

The basis $(dZ, d\bar{Z})$ of the cotangent space can be expressed in matrix form as

$$(dZ, d\bar{Z}) = (dz, d\bar{z}) \cdot M \cdot Q \quad (2.5)$$

where the matrices M and Q contain the Beltrami coefficients and integrating factors, respectively :

$$M = \begin{pmatrix} 1 & \mu \\ \mu & 1 \end{pmatrix} \quad Q = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}$$

The tangent vectors which are dual to $dZ, d\bar{Z}$ are then given by

$$\begin{pmatrix} \partial_z \\ \bar{\partial}_z \end{pmatrix} = Q^{-1} \cdot M^{-1} \cdot \begin{pmatrix} \partial \\ \bar{\partial} \end{pmatrix} = \begin{pmatrix} \lambda^{-1}(1 - \mu\bar{\mu})^{-1}(\partial - \mu\bar{\partial}) \\ \bar{\lambda}^{-1}(1 - \mu\bar{\mu})^{-1}(\bar{\partial} - \mu\partial) \end{pmatrix} \quad (2.6)$$

Thus, the Beltrami equation (2.3) is equivalent to the basic relation $\partial_z Z = 0$.

Before proceeding further, we should make a comment about the solutions of the differential equation (2.2). The problem of finding solutions of this equation can be rephrased as follows. Given the reference holomorphic coordinates $(z, \bar{z})$ and the Beltrami coefficient $\mu(z, \bar{z})$, is it possible to find holomorphic coordinates $(Z, \bar{Z})$ such that the corresponding differentials are related by (2.1) ?

Dividing the condition (2.2) by λ , we get

$$(\bar{\partial} - \mu \partial) \ln \lambda = \partial \mu .$$

This is a linear, inhomogenous differential equation for $\ln \lambda$ and its solutions are only determined up to solutions of the associated homogenous equation:

$$0 = (\bar{\partial} - \mu \partial) \ln \bar{\lambda} \propto \partial \bar{\mu} \ln \bar{\lambda} . \quad (2.7)$$

The general solution of this equation is given by a holomorphic function $\ln \bar{\lambda} = f(Z)$ and therefore a particular solution of (2.2) can always be multiplied by $\bar{\lambda} = \exp f(Z)$.

For $\mu = 0$, equations (2.3) and (2.2) reduce to $\partial \bar{Z} = 0 = \partial \lambda$, i.e. Z and λ become holomorphic functions.

It is instructive to compare with the metric approach to Beltrami differentials which is summarized in appendix A.1. Although the differential dZ in eq.(2.1) has the same form as the zweibein (A.2) associated to the two-dimensional metric, the properties and interpretations of the factors in both expressions are quite different. The conformal factor ρ contained in the zweibein is unconstrained, but transforms under Weyl rescalings. By contrast, the integrating factor λ of dZ satisfies the differential equation (2.2) and Weyl rescalings are not present in this case. Moreover, both variables behave inequivalently under reparametrizations. To check this point, let us consider an infinitesimal change of the coordinates $(z, \bar{z})$ generated by a vector field

$$\xi \cdot \partial \equiv \xi^z(z, \bar{z}) \partial_z + \xi^{\bar{z}}(z, \bar{z}) \partial_{\bar{z}} .$$

The induced variation of $Z(z, \bar{z})$ is described by the Lie derivative along $\xi \cdot \partial$:

$$\begin{aligned} \delta Z &= L_{\xi \cdot \partial} Z = i_{\xi \cdot \partial} (dZ) = i_{\xi \cdot \partial} \left([dz + d\bar{z} \mu_z] \lambda_z^z \right) \\ &= [\xi^z + \xi^{\bar{z}} \mu_z] \lambda_z^z \\ &\equiv c^z \lambda_z^z . \end{aligned}$$

The corresponding transformation laws of μ and λ follow by evaluating the variation of dZ in two different ways:

$$\delta(dZ) = d(\delta Z) = d(c^z \lambda_z^z) = dz \partial_z (c^z \lambda_z^z) + d\bar{z} \partial_{\bar{z}} (c^z \lambda_z^z)$$

and

$$\delta(dZ) = d\bar{z} (\delta \mu_z^z) \lambda_z^z + [dz + d\bar{z} \mu_z^z] (\delta \lambda_z^z) .$$

By comparing the different coefficients of dz and $d\bar{z}$, one finds

$$\begin{aligned} \delta \mu_z^z &= (\partial_z - \mu_z^z \partial_z) c^z + (\partial_z \mu_z^z) c^z \\ \delta \lambda_z^z &= \partial_z (c^z \lambda_z^z) = c^z (\partial_z \lambda_z^z) + (\partial_z c^z) \lambda_z^z . \end{aligned} \quad (2.8)$$

The variations of $\bar{\mu}$ and $\bar{\lambda}$ follow by complex conjugation. Note that μ and λ only transform with c , i.e. Becchi's reparametrization [2]

$$\begin{aligned} c^z &= \xi^z + \xi^{\bar{z}} \mu_z^z \\ c^{\bar{z}} &= \xi^{\bar{z}} + \xi^z \mu_z^z \end{aligned} \quad (2.9)$$

ensures the *holomorphic factorization* of the variations of μ [2]-[4] and λ [5].

To simplify the notation in the following, we usually suppress the indices and write $\xi(\bar{\xi})$ for $\xi^z(\xi^{\bar{z}})$ and $c(\bar{c})$ for $c^z(c^{\bar{z}})$.

The use of the integrating factor equation (2.2) allows one to rewrite the transformation law of λ as

$$\delta \lambda = (\xi \cdot \partial) \lambda + (\partial \xi + \mu \partial \bar{\xi}) \lambda .$$

This variation has the same form as the one of the conformal factor ρ in the metric approach (see eq.(A.3)). For the latter, the holomorphic factorization cannot be achieved, because it does not satisfy any differential equation. Thus, despite their formal resemblance, the factors λ and ρ are very different in many respects.

The algebra of the infinitesimal symmetry transformations (2.8) can easily be determined by turning ξ and $\bar{\xi}$ into ghost fields and replacing the variation δ by the BRS differential s . (The operator s acts as an antiderivation from the left and the gradation is given by adding the ghost number to the form degree.) The nilpotency of s requires

$$0 = s^2 Z = s(c\lambda) = [sc - c\partial c] \lambda ,$$

and thereby

$$sc = c\partial c . \quad (2.10)$$

From $s^2 Z = 0$ and $sd + ds = 0$, we conclude that

$$0 = d(s^2 Z) = s^2(dZ) = dz(s^2 \lambda) + d\bar{z}[(s^2 \lambda)\mu + \lambda(s^2 \mu)] ,$$

henceforth $s^2 \lambda = 0 = s^2 \mu$. Consequently, eqs.(2.8) (with δ replaced by s) define a differential algebra with generator c and structure relations (2.10). This algebra is compatible with holomorphic coordinate transformations $z \rightarrow z'(z)$ in the sense that s commutes with such changes of coordinates under which c transforms as $c^z = c^{z'} \partial z' / \partial z$ and λ, μ as in eq.(2.4). Therefore, the local expressions (2.8) and (2.10) can be patched together holomorphically for different coordinate systems and they provide a global description on the Riemann surface.

2.2 Scalar fields

Let us now come to the coupling of a scalar field X to a conformal class of metrics. In terms of holomorphic coordinates $(Z, \bar{Z})$, the action functional for X reads [5]

$$2i S_{\text{inv}} = \int dZ \wedge d\bar{Z} (\partial_z X) (\partial_{\bar{z}} X) . \quad (2.11)$$

Expressing $(Z, \bar{Z})$ in terms of the reference coordinates $(z, \bar{z})$ by means of eqs.(2.5)(2.6), we get

$$2i S_{\text{inv}} = \int dz \wedge d\bar{z} \frac{1}{1 - \mu \bar{\mu}} (\bar{\partial} - \mu \partial) X (\partial - \bar{\mu} \bar{\partial}) X , \quad (2.12)$$

which result coincides with the expression of the metric approach, eq.(A.5). For the sake of locality, it is crucial that the action does not explicitly depend on the integrating factor λ which represents a non-local function of μ by virtue of eq.(2.2).

Although the holomorphic factorization is realized for the geometric quantities μ and λ , it is not for the matter field X :

$$sX = (\xi \cdot \partial) X = \frac{1}{1 - \mu\bar{\mu}} [c(\partial - \bar{\mu}\bar{\partial}) + \bar{c}(\bar{\partial} - \mu\partial)] X . \quad (2.13)$$

The functional (2.11) is invariant under holomorphic changes of coordinates,

$$(Z, \bar{Z}) \longrightarrow (Z'(Z), \bar{Z}'(\bar{Z})) ,$$

and thereby represents a globally well-defined expression for any compact Riemann surface. A fortiori, this action is globally well-defined when expressed in terms of Beltrami differentials transforming in the given way.

More generally, one can consider a collection $X^1, \dots, X^d$ of scalar fields parametrizing a d -dimensional Riemannian manifold N with metric (G_{ij}) :

$$\begin{aligned} \vec{X} : M &\longrightarrow N \\ (z, \bar{z}) &\longmapsto \vec{X}(z, \bar{z}) = (X^1(z, \bar{z}), \dots, X^d(z, \bar{z})) . \end{aligned}$$

The associated action,

$$2i S_{\text{inv}}(X^1, \dots, X^d) = \sum_{i,j=1}^d \int dZ \wedge d\bar{Z} (\partial_Z X^i)(\partial_{\bar{Z}} X^j) G_{ij}(X^1, \dots, X^d) , \quad (2.14)$$

describes the (Euclidean version of the) motion of a string moving in the space N . The previous remarks concerning the functional (2.11) also apply to the more general expression (2.14).

2.3 Conformal fields and bc-systems

In the following, we will introduce conformal fields and the associated functionals, the so-called bc-systems. Like (2.11), these actions are naturally defined on any Riemann surface and may be viewed as universal objects. A particular bc-system will be identified with the gauge-fixing functional (A.12) for the bosonic string (moving in a flat background space).

Let $\mathcal{B}$ and $\mathcal{C}$ denote globally defined tensors on the Riemann surface [19] which are locally given in terms of holomorphic coordinates $(Z, \bar{Z})$ by

$$\mathcal{B}_{Z\dots Z}(Z, \bar{Z}) dZ \otimes \dots \otimes dZ , \quad \mathcal{C}^{Z\dots Z}(Z, \bar{Z}) \partial_Z \otimes \dots \otimes \partial_Z . \quad (2.15)$$

Under a holomorphic (=conformal) change of coordinates, their components transform as

$$\begin{aligned} \mathcal{B}_{Z'\dots Z'}(Z, \bar{Z}) &= \mathcal{B}_{Z'\dots Z'}(Z', \bar{Z}') \left(\frac{\partial Z'}{\partial Z} \right)^q , \\ \mathcal{C}^{Z'\dots Z'}(Z, \bar{Z}) &= \mathcal{C}^{Z'\dots Z'}(Z', \bar{Z}') \left(\frac{\partial Z}{\partial Z'} \right)^q , \end{aligned} \quad (2.16)$$

where q is the number of Z -indices. In the language of conformal field theory [9], this means that $\mathcal{B}_{Z\dots Z}$ and $\mathcal{C}^{Z\dots Z}$ are conformal fields of weight q .

To obtain a globally well-defined action for a system of $\mathcal{B}$ and $\mathcal{C}$ fields, one relies on the fact that every Riemann surface admits a natural linear connection that is locally represented by ∂_Z when acting on tensors of the form $\mathcal{C}^{Z\dots Z}$. Thus, a natural, conformally invariant action is given by

$$2i S_{\text{bc}} = \int dZ \wedge d\bar{Z} \left\{ \mathcal{B}_{Z\dots Z} \partial_Z \mathcal{C}^{Z\dots Z} + \text{c.c.} \right\} , \quad (2.17)$$

where $\mathcal{B}$ and $\mathcal{C}$ carry q and $q-1$ indices, respectively.

We now go over to the reference coordinates $(z, \bar{z})$ related to $(Z, \bar{Z})$ by (2.5)(2.6) and rescale the fields with q indices according to

$$b_{z\dots z} \equiv (\lambda_z^Z)^q \mathcal{B}_{Z\dots Z} , \quad \mathcal{C}^{z\dots z} \equiv c^{z\dots z} (\lambda_z^{\bar{Z}})^q \quad \text{and c.c.} \quad (2.18)$$

Here, $\mathcal{B}$ and $\mathcal{C}$ depend on $(Z, \bar{Z})$ while b, c and λ depend on $(z, \bar{z})$. Then, suppressing all indices,

$$\begin{aligned} 2i S_{\text{bc}} &= \int dz \wedge d\bar{z} \det(MQ) \left\{ (\lambda^{-q} b) \partial_Z (c \lambda^{q-1}) + \text{c.c.} \right\} \\ &= \int dz \wedge d\bar{z} \left\{ \lambda^{1-q} b (\bar{\partial} - \mu \partial) (c \lambda^{q-1}) + \text{c.c.} \right\} \\ &= \int dz \wedge d\bar{z} \left\{ b \left[(\bar{\partial} - \mu \partial) + (q-1)(\partial\mu) \right] c + \text{c.c.} \right\} , \end{aligned} \quad (2.19)$$

where we used the integrating factor equation (2.2) to pass to the last line. Obviously, (2.19) is independent of λ and represents a local functional. When restricted to $q=2$, it coincides with the gauge-fixing functional (A.12) associated to the action (2.12) :

$$\begin{aligned} 2i S_{\text{gf}} &= \int dZ \wedge d\bar{Z} \left\{ \mathcal{B}_{ZZ} (\partial_Z \mathcal{C}^Z) + \mathcal{B}_{Z\bar{Z}} (\partial_{\bar{Z}} \mathcal{C}^{\bar{Z}}) \right\} \\ &= \int dz \wedge d\bar{z} \left\{ b_{zz} \left[\bar{\partial} - \mu \partial + \partial\mu \right] c^z + \text{c.c.} \right\} \\ &= \int dz \wedge d\bar{z} \left\{ b_{zz} (s\mu_z^z) + \text{c.c.} \right\} . \end{aligned} \quad (2.20)$$

Thus, the gauge-fixing action (A.12) can be viewed as a particular case of bc-system, an interesting result which is still lacking further understanding.

Like (2.11), the action (2.17) formally coincides with a conformal gauge (isothermal coordinates) expression on a Riemannian manifold; the Beltrami differentials are required to express it in terms of local quantum fields b and c acted upon by the diffeomorphism group (see next section).

The bc-system also allows for Thirring-type couplings and it can be coupled to a chiral current constructed from X [5] :

$$2i S_{\text{int}} = \int dZ \wedge d\bar{Z} \left\{ \mathcal{B}_{Z\dots Z} \mathcal{J}_{\mathcal{B}}(X) \mathcal{C}^{Z\dots Z} + \text{c.c.} \right\} . \quad (2.21)$$

Here, $\mathcal{B}$ and $\mathcal{C}$ carry q and $q-1$ indices, respectively, and $\mathcal{J}_{\mathcal{B}}$ transforms as $\mathcal{J}_{\mathcal{B}} = \mathcal{J}_{\mathcal{B}}(\partial\bar{Z}/\partial\bar{Z})$. From (2.18) and

$$j_z \equiv (1 - \mu\bar{\mu}) \lambda_z^z \mathcal{J}_{\mathcal{B}} , \quad (2.22)$$

one immediately gets the following local expression :

$$2i S_{\text{int}}[X, b, c] = \int dz \wedge d\bar{z} \left\{ b_{z\dots z} j_{\bar{z}}(X) c^{z\dots z} + \text{c.c.} \right\} . \quad (2.23)$$

2.4 Diffeomorphisms and associated Ward identities

The transformation laws of μ and λ under a finite diffeomorphism

$$(z, \bar{z}) \longrightarrow (z'(z, \bar{z}), \bar{z}'(z, \bar{z}))$$

follow from $dz' = dz(\partial z') + d\bar{z}(\bar{\partial} z')$ and

$$\begin{aligned} dZ &= [dz' + d\bar{z}'\mu_{z'}(z', \bar{z}')]\lambda_{z'}(z', \bar{z}') \\ &= [dz + d\bar{z}\mu_z(z, \bar{z})]\lambda_z(z, \bar{z}), \end{aligned}$$

the result being given by

$$\mu_z^z(z, \bar{z}) = \frac{\bar{\partial} z' + \bar{\partial} \bar{z}'\mu_{z'}(z', \bar{z}')}{\partial z' + \partial \bar{z}'\mu_{z'}(z', \bar{z}')}, \quad \lambda_z^z(z, \bar{z}) = [\partial z' + \partial \bar{z}'\mu_{z'}(z', \bar{z}')] \lambda_{z'}^z(z', \bar{z}'). \quad (2.24)$$

The corresponding transformations of the quantum fields $b_{z...z}$, $c^{z...z}$ (with q indices each) and of j_z may be deduced from the previous equation and the relations (2.18) (2.22), e.g.

$$\begin{aligned} B_{z...z} &= b_{z...z} (\lambda_z^z)^{-q} = b_{z...z} [\partial z' + \partial \bar{z}'\mu_{z'}(z', \bar{z}')]^{-q} (\lambda_{z'}^z)^{-q} \\ &= b_{z'...z'} (\lambda_{z'}^z)^{-q}. \end{aligned}$$

In this way, one finds

$$\begin{aligned} b_{z...z}(z, \bar{z}) &= [\partial z' + \partial \bar{z}'\mu_{z'}(z', \bar{z}')]^q b_{z'...z'} \\ c^{z...z}(z, \bar{z}) &= c^{z'...z'} [\partial' z + \partial' \bar{z}\mu_z^z]^q = c^{z'...z'} [\partial z' + \partial \bar{z}'\mu_{z'}(z', \bar{z}')]^{-q} \\ j_z(z, \bar{z}) &= \frac{(\partial z')(\bar{\partial} \bar{z}') - (\bar{\partial} z')(\partial \bar{z}')}{\partial z' + \partial \bar{z}'\mu_{z'}(z', \bar{z}')} j_{z'}. \end{aligned} \quad (2.25)$$

The matter field X behaves like a scalar, $X(z, \bar{z}) = X(z', \bar{z}')$.

Now, the infinitesimal transformation laws immediately follow from the finite ones by a first order expansion, $z' \simeq z + \xi(z, \bar{z})$, $\bar{z}' \simeq \bar{z} + \bar{\xi}(z, \bar{z})$:

$$\begin{aligned} \delta_\xi \mu &= (\bar{\partial} - \mu \partial + \partial \mu) [\xi + \bar{\xi} \mu] \\ \delta_\xi \lambda &= (\xi \cdot \partial) \lambda + (\partial \xi + \mu \partial \bar{\xi}) \lambda \\ \delta_\xi b_{z...z} &= (\xi \cdot \partial) b_{z...z} + q (\partial \xi + \mu \partial \bar{\xi}) b_{z...z} \\ \delta_\xi c^{z...z} &= (\xi \cdot \partial) c^{z...z} - q (\partial \xi + \mu \partial \bar{\xi}) c^{z...z} \\ \delta_\xi j_z &= (\xi \cdot \partial) j_z + (\bar{\partial} \bar{\xi} - \mu \partial \bar{\xi}) j_z \\ \delta_\xi X &= (\xi \cdot \partial) X. \end{aligned} \quad (2.26)$$

By construction, these variations represent symmetries of the actions for X , b and c , eqs.(2.12)(2.19)(2.23). This was first noticed by C.Becchi [2] for the quantum action of the bosonic string, eqs.(A.11)(A.12).

Three remarks concerning the symmetry algebra (2.26) are in order.

(i) The algebra of the ξ -variations is given by the Lie bracket of vector fields (and is therefore field-independent!):

$$[\delta_{\xi_1}, \delta_{\xi_2}] = -\delta_{\xi_{12}} \quad \text{with} \quad \xi_{12} \cdot \partial \equiv [\xi_1 \cdot \partial, \xi_2 \cdot \partial]. \quad (2.27)$$

(ii) Assuming $s\xi = 0 = s\bar{\xi}$, the ξ -transformations of $b_{z...z}$, $c^{z...z}$ and X commute with the corresponding s -operations, eqs.(A.13) (up to equations of motion in the case of $b_{z...z}$).

(iii) Assume that μ transforms as in (2.26) and decompose the quantum fields c^z and $c^{\bar{z}}$ as follows:

$$\begin{aligned} c^z &= \gamma^z + \gamma^{\bar{z}} \mu_z^z \\ c^{\bar{z}} &= \gamma^{\bar{z}} + \gamma^z \mu_z^{\bar{z}}. \end{aligned} \quad (2.28)$$

Then, the given ξ -variations of c^z and $c^{\bar{z}}$ are equivalent to the fact that the vector field $\gamma \cdot \partial$ transforms with the Lie bracket,

$$\delta_\xi(\gamma \cdot \partial) = [\xi \cdot \partial, \gamma \cdot \partial]. \quad (2.29)$$

For the supersymmetric theory, this provides an useful alternative derivation of the ξ -variations of the c -fields (from which it is also obvious that the underlying algebra is given by the Lie bracket).

In conclusion, we derive the Ward identities generated by the diffeomorphisms and discuss their formal relation with the operator product expansions of conformal field theory. The Ward operators acting on functionals $Z_c[\mu, \bar{\mu}]$ have the form

$$\mathcal{W}(\xi \cdot \partial) = \int dz d\bar{z} \left[\delta_\xi \mu \frac{\delta}{\delta \mu} + \text{c.c.} \right] \equiv \int dz d\bar{z} [\xi(z, \bar{z}) \mathcal{W}(z, \bar{z}) + \text{c.c.}]$$

and the anomalous Ward identity reads [2]

$$\frac{1}{1 - \mu \bar{\mu}} (W - \bar{\mu} \bar{W}) Z_c[\mu, \bar{\mu}] = - \left[\bar{\partial} - \mu \partial - 2(\partial \mu) \right] \frac{\delta}{\delta \mu} Z_c[\mu, \bar{\mu}] = k \partial^3 \mu \quad \text{and c.c.}$$

Here, the r.h.s. corresponds to the anomaly (A.15) and Z_c denotes the generating functional of connected Green's functions. Using the distributional relation

$$\partial_{\bar{z}} \frac{1}{z - w} = \pi \delta^{(2)}(z - w) \quad (z, w \in \mathbb{C}),$$

we can integrate the Ward identity on the complex plane with the following result:

$$-\frac{\delta Z_c}{\delta \mu(z)} = \frac{1}{2\pi i} \int dw d\bar{w} \mu(w) \left\{ -k \partial_w^3 \frac{1}{z - w} + \left[2 \partial_w \frac{1}{z - w} - \frac{1}{z - w} \partial_w \right] \frac{\delta Z_c}{\delta \mu(w)} \right\}.$$

The derivatives of $(z - w)^{-1}$ occurring on the r.h.s. can be evaluated, but only in a formal way, because the resulting functions $(z - w)^{-2}$ and $(z - w)^{-4}$ do not represent well-defined distributions:

$$-\frac{\delta Z_c}{\delta \mu(z)} = \frac{1}{2\pi i} \int dw d\bar{w} \mu(w) \left\{ \frac{-6k}{(z - w)^4} + \frac{2}{(z - w)^2} \frac{\delta Z_c}{\delta \mu(w)} + \frac{1}{z - w} \partial_w \frac{\delta Z_c}{\delta \mu(w)} \right\}.$$

Henceforth, we get the formal result

$$-\pi \frac{\delta^2 Z_c}{\delta \mu(w) \delta \mu(z)} \Big|_{\mu=0} = \frac{-6k}{(z-w)^4} + \frac{2}{(z-w)^2} \frac{\delta Z_c}{\delta \mu(w)} \Big|_{\mu=0} + \frac{1}{z-w} \partial_w \frac{\delta Z_c}{\delta \mu(w)} \Big|_{\mu=0} ,$$

which may be compared with the operator product expansion of the components $T = \delta Z_c / \delta \mu$ of the energy-momentum tensor [9] :

$$T(z) T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2} T(w) + \frac{1}{z-w} \partial_w T(w) + \text{regular terms} .$$

The expansion for the product of T and b (or c) can also be reproduced along these lines [6].

2.5 Geometric interpretation

The Beltrami differentials discussed in the previous sections represent the components of ‘vector-valued 1-forms’ $\mu_x^*(z, \bar{z}) d\bar{z} \otimes \partial_x$ (and c.c.). They correspond to globally defined objects on the Riemann surface M and parametrize the space $\mathcal{M}(M)$ of all complex (=conformal) structures on M .

For any smooth, oriented 2-manifold M which is compact and of genus $g > 1$, the space $\mathcal{M}(M)$ defines a principal fibre bundle over Teichmüller space $\mathcal{T}(M)$ [20]. The typical fibre of this bundle is given by the group $\mathcal{D}_0(M)$ of diffeomorphisms of M which are homotopic to the identity. The base space $\mathcal{T}(M)$ is homeomorphic to $\mathbb{R}^{6g-6}$ and may be parametrized by real local coordinates m^k ($k = 1, \dots, 6g - 6$).

This is the standard mathematical definition and interpretation of Beltrami coefficients to which we also adhere in this paper. Another viewpoint which is often taken in the physics literature is the following. Since $\mathcal{M}(M)$ is a principal fibre bundle over $\mathcal{T}(M)$, one can consider smooth local sections from $\mathcal{T}(M)$ in $\mathcal{M}(M)$. Now, let $\{\mu^k(\mu; z, \bar{z}); k = 1, \dots, 6g-6\}$ denote a tangent vector field to a family of such local sections transforming equivariantly under $\mathcal{D}_0(M)$ [21] :

$$\begin{aligned} s\mu^k &= c(\partial\mu^k) - (\partial c)\mu^k \\ sc &= c\partial c . \end{aligned} \quad (2.30)$$

Then, $\{\mu^k\}$ projects down on $\mathcal{T}(M)$ to a linear combination of tangent vectors $\partial/\partial m^k$ (associated to a local coordinate system $\{m^k\}$ of $\mathcal{T}(M)$), see the figure. It is the latter which are often referred to as Beltrami differentials in the string literature.

Chapter 3

Supersymmetric theory

In this chapter, we will discuss super Beltrami differentials and superconformal models for (1,1) supersymmetry. The generalization to (p, q) supersymmetry [22] should be realizable along the same lines and, in appendix C, we summarize the results for the (1, 0) case. To keep supersymmetry manifest, all considerations will be carried out in superspace and the projection to ordinary space will only be performed at the end.

In the (1,1) case, superspace is locally parametrized by labels $(z, \bar{z}, \theta, \bar{\theta})$ with $z, \bar{z}$ as before and $\theta, \bar{\theta}$ two complex conjugate Grassmannian variables. The definition of two-dimensional superconformal transformations can be motivated as follows [23]. In the supervielbein (“metric”) approach to supergravity, one can introduce super-Weyl transformations [10] and prove the existence of super-isothermal coordinates [10] [23]. Then, the supercoordinate transformations which preserve the superconformally flat form of the vielbeins are precisely the superconformal transformations [23].

3.1 General framework

Super Riemann surfaces (SRS) have recently been defined in a rigorous way within the different categories of supermanifolds (see [24] [25]). A SRS is a $(2+2)$ -dimensional supermanifold which is equipped with a superconformal structure. Roughly speaking, this means that it admits local coordinates $(Z, \bar{Z}, \Theta, \bar{\Theta})$ and that any two sets of such coordinates, $(Z, \bar{Z}, \Theta, \bar{\Theta})$ and $(Z', \bar{Z}', \Theta', \bar{\Theta}')$ are related by a superconformal transformation, i.e. a transformation satisfying the following two conditions :

$$(i) \quad Z' = Z'(Z, \Theta) \quad , \quad \Theta' = \Theta'(Z, \Theta) \quad \text{and c.c.}$$

or, equivalently,

$$D_\Theta Z' = 0 = D_\Theta \Theta' \quad \text{and c.c.} \quad (3.1)$$

$$(ii)$$

$$D_\Theta Z' = \Theta'(D_\Theta \Theta') \quad \text{and c.c.} \quad (3.2)$$

Here, the vector fields $\partial_Z, \partial_{\bar{Z}}, D_\Theta \equiv \partial_\Theta + \Theta \partial_Z, D_{\bar{\Theta}} \equiv \partial_{\bar{\Theta}} + \bar{\Theta} \partial_{\bar{Z}}$ define the canonical basis of the super tangent space. The graded Lie brackets between these fields are given

by

$$[D_\Theta, D_\Theta] = 2\partial_Z \quad \text{and c.c.} \quad (3.3)$$

(all others = 0) .

Conditions (3.1)(3.2) imply that D_Θ transforms homogeneously under superconformal transformations (just as ∂_Z transforms homogeneously under conformal mappings in the bosonic theory) :

$$\begin{aligned} D'_\Theta &= e^W D_\Theta & \text{and c.c.} \\ \partial'_Z &= e^{2W} [\partial_Z + (D_\Theta W) D_\Theta] & \text{and c.c.} \end{aligned} \quad (3.4)$$

Here,

$$e^W \equiv (D_\Theta \Theta')^{-1} = (\partial_Z Z' + \Theta' \partial_Z \Theta')^{-1/2} ,$$

where the last equality follows by applying D_Θ to eq.(3.2). By complex conjugate (c.c.) expressions, we always mean those obtained by replacing Z, Θ by $\bar{Z}, \bar{\Theta}$ without modifying the order in products.

The vectors $\partial_Z, \partial_{\bar{Z}}, D_\Theta, D_{\bar{\Theta}}$ are dual to the 1-forms

$$\begin{aligned} e^Z &\equiv dZ + \Theta d\Theta & \text{and c.c.} \\ e^\Theta &\equiv d\Theta & \text{and c.c.} \end{aligned} \quad (3.5)$$

The latter span the cotangent space and equations (3.3) are equivalent to

$$\begin{aligned} 0 &= de^Z + e^\Theta e^\Theta & \text{and c.c.} \\ 0 &= de^\Theta & \text{and c.c.} \end{aligned} \quad (3.6)$$

These structure relations represent the supersymmetric generalization of the condition $d(dZ) = 0$.

For later reference, we specify the superconformal transformations of e^Z and e^Θ :

$$\begin{aligned} e^{Z'} &= e^{-2W} e^Z & \text{and c.c.} \\ e^{\Theta'} &= e^{-W} [e^\Theta - e^Z (D_\Theta W)] & \text{and c.c.} \end{aligned} \quad (3.7)$$

In the following, we will generalize the discussion of section 2.1 to the supersymmetric case. The parametrization of superconformal structures in terms of Beltrami differentials can be motivated from the metric approach as indicated in the bosonic case, but, for completeness and conceptual clarity, we will present a direct derivation here. To start with, we assume that the superconformal coordinates $(Z, \bar{Z}, \Theta, \bar{\Theta})$ have been obtained from a reference system of superconformal coordinates $(z, \bar{z}, \theta, \bar{\theta})$ by a smooth¹ change of coordinates:

$$(z, \bar{z}, \theta, \bar{\theta}) \longrightarrow (Z(z, \bar{z}, \theta, \bar{\theta}), \bar{Z}(z, \bar{z}, \theta, \bar{\theta}), \Theta(z, \bar{z}, \theta, \bar{\theta}), \bar{\Theta}(z, \bar{z}, \theta, \bar{\theta})) \quad (3.8)$$

¹By "smooth", we mean the appropriate generalization of C^∞ to the category of supermanifolds one is working in.

Expressing the one-forms (3.5) in terms of the corresponding basis of the reference coordinate system, one finds

$$\begin{aligned} e^Z &= e^z E_z^Z + e^\theta E_\theta^Z + e^{\bar{z}} E_{\bar{z}}^Z + e^{\bar{\theta}} E_{\bar{\theta}}^Z \\ e^\Theta &= e^\theta E_\theta^\Theta + e^{\bar{z}} E_{\bar{z}}^\Theta + e^z E_z^\Theta + e^{\bar{\theta}} E_{\bar{\theta}}^\Theta \end{aligned}$$

with

$$\begin{aligned} E_z^Z &= \partial_Z + \Theta \partial_\Theta & , & & E_\theta^Z &= \partial_Z + \Theta \partial_\Theta \\ E_{\bar{z}}^Z &= D_\Theta \bar{Z} - \Theta D_\Theta \bar{\Theta} & , & & E_{\bar{\theta}}^Z &= D_\Theta \bar{Z} - \Theta D_\Theta \bar{\Theta} \\ E_z^\Theta &= \partial_\Theta & , & & E_\theta^\Theta &= \partial_\Theta & , & & E_{\bar{z}}^\Theta &= D_\Theta \bar{\Theta} & , & & E_{\bar{\theta}}^\Theta &= D_\Theta \bar{\Theta} \end{aligned} \quad (3.9)$$

Here and in the sequel, we use the shorthand notation

$$\partial \equiv \partial_z & , & \bar{\partial} \equiv \partial_{\bar{z}} & , & D \equiv D_\theta &= \partial_\theta + \theta \partial_z & , & \bar{D} \equiv D_{\bar{\theta}} &= \partial_{\bar{\theta}} + \bar{\theta} \partial_{\bar{z}} .$$

Furthermore, we will refer to $(z, \bar{z}, \theta, \bar{\theta})$ as the 'small' or 'reference' coordinates and to $(Z, \bar{Z}, \Theta, \bar{\Theta})$ as the 'capital' coordinates.

According to eq.(3.7), the 1-form e^Z transforms homogeneously under a superconformal transformation (3.1)(3.2), and therefore we can obtain variables H^z which are inert under these transformations by factorizing E_z^Z in e^Z :

$$e^Z = [e^z + e^z H_z^z + e^\theta H_\theta^z + e^{\bar{z}} H_{\bar{z}}^z] \Lambda_z^Z \quad (3.10)$$

with

$$\Lambda_z^Z \equiv E_z^Z \quad \text{and} \quad H_a^z \equiv \frac{E_a^Z}{E_z^Z} \quad \text{for} \quad a = z, \theta, \bar{\theta} . \quad (3.11)$$

The coefficients in e^Θ do not transform homogeneously,

$$E_a^{\Theta'} = e^{-W} [E_a^\Theta - E_a^Z (D_\Theta W)] \quad \text{for} \quad a = z, \theta, \bar{\theta} ,$$

but some particular combinations of them do. From the latter, we again construct inert variables H^z by dividing by an appropriate power of Λ_z^Z :

$$H_a^\theta \equiv \frac{1}{\sqrt{\Lambda_z^Z}} [E_a^\Theta - H_a^z E_z^\Theta] \quad \text{for} \quad a = z, \theta, \bar{\theta} . \quad (3.12)$$

If e^Θ is expressed in terms of these variables, it takes the form

$$e^\Theta = [e^z + e^z H_z^z + e^\theta H_\theta^z + e^{\bar{z}} H_{\bar{z}}^z] \tau_z^\Theta + [e^\theta H_\theta^\theta + e^z H_z^\theta + e^{\bar{\theta}} H_{\bar{\theta}}^\theta] \sqrt{\Lambda_z^Z} , \quad (3.13)$$

where $\tau_z^\Theta \equiv E_z^\Theta$.

To summarize,

$$(e^Z, e^z, e^\Theta, e^\theta, e^{\bar{z}}, e^{\bar{\theta}}) = (e^z, e^z, e^\theta, e^\theta) \cdot M \cdot Q \quad (3.14)$$

with

$$M = \begin{pmatrix} 1 & H_z^z & 0 & H_z^\theta \\ H_z^z & 1 & H_z^\theta & 0 \\ H_z^\theta & H_z^\theta & H_\theta^\theta & H_\theta^\theta \\ H_\theta^\theta & H_\theta^\theta & H_\theta^\theta & H_\theta^\theta \end{pmatrix}, \quad Q = \begin{pmatrix} \Lambda & 0 & \tau & 0 \\ 0 & \bar{\Lambda} & 0 & \bar{\tau} \\ 0 & 0 & \sqrt{\Lambda} & 0 \\ 0 & 0 & 0 & \sqrt{\Lambda} \end{pmatrix}. \quad (3.15)$$

By construction, the 'H' are inert under the superconformal transformations (3.1)(3.2) and Λ, τ transform as

$$\begin{aligned} \Lambda^{z'} &= e^{-2W} \Lambda^z \\ \tau^{\theta'} &= e^{-W} [\tau^\theta - \Lambda^z (D_\theta W)] \end{aligned} \quad (3.16)$$

The decomposition (3.14) generalizes the parametrization (2.5) of the bosonic theory to superspace. It coincides with the results found by Baulieu, Bellon and Grimm who performed a pregauging-fixing in the vielbein approach, see [8] eqs.(3.22-23), (3.32-37) and (3.40-42). The parametrization (3.11) for H_θ^z has also been considered for supermetrics on SRS's [24].

Let us now investigate the superspace generalization of the transformation laws (2.4). Under a superconformal change of the reference coordinate system,

$$\begin{aligned} (z, \bar{z}, \theta, \bar{\theta}) &\longrightarrow (z'(z, \theta), \bar{z}'(\bar{z}, \bar{\theta}), \theta'(z, \theta), \bar{\theta}'(\bar{z}, \bar{\theta})) \\ D_\theta z' &= \theta' D_\theta \theta' \quad \text{and c.c.} \end{aligned} \quad (3.17)$$

the basis 1-forms transform as

$$\begin{aligned} e^{z'} &= e^{-2w} e^z & \text{and c.c.} \\ e^{\theta'} &= e^{-w} [e^\theta - e^z (D_\theta w)] & \text{and c.c.} \end{aligned}$$

with $e^w \equiv (D_\theta \theta')^{-1}$. Substitution of these expressions into eq.(3.14) yields the transformations of Λ, τ and of the 'H' under these reparametrizations. Using the shorthand notation

$$Y \equiv 1 + (D_\theta w) H_\theta^z,$$

the results may be summarized as follows :

$$\begin{aligned} \Lambda_{z'} &= e^{2w} Y \Lambda_z \\ \tau_{\theta'} &= e^{2w} Y \left[\tau_z + (D_\theta w) H_\theta^z \sqrt{\Lambda_z} \right] \\ H_{\theta'}^{z'} &= e^{-w} H_\theta^z \\ H_{\theta'}^{z'} &= e^w e^{-2w} Y^{-1} H_\theta^z \\ H_{\theta'}^{z'} &= e^{2w} e^{-2w} Y^{-1} [H_\theta^z + (D_\theta \bar{w}) H_\theta^{z'}] \\ H_{\theta'}^{\theta'} &= Y^{1/2} H_\theta^\theta \\ H_{\theta'}^{\theta'} &= e^w e^{-w} Y^{-1/2} [H_\theta^\theta - H_\theta^{z'} (D_\theta w) H_\theta^\theta] \\ H_{\theta'}^{\theta'} &= e^{2w} e^{-w} Y^{-1/2} \left[[H_\theta^\theta + (D_\theta \bar{w}) H_\theta^\theta] - [H_\theta^z + (D_\theta \bar{w}) H_\theta^{z'}] (D_\theta w) H_\theta^\theta \right] \end{aligned} \quad (3.18)$$

The systematics underlying this set of transformations becomes clearer, if they are viewed locally as a special case of superdiffeomorphisms (eqs.(3.55) below). We note that the equation $H_\theta^z = 0$ is invariant under the previous transformations and that it simplifies all other expressions. In particular, it renders H_θ^θ invariant and implies a homogeneous transformation law for H_θ^z . Yet, in the remainder of the text, we will stick to the general scheme and only discuss the restricted geometry $H_\theta^z = 0$ in the concluding chapter 4.

The 'H' can be viewed as *super Beltrami coefficients*. However, due to the relations $(D_\theta)^2 = \partial_z$ and $(D_\theta)^2 = \partial_{\bar{z}}$, not all of the 'H', Λ and τ are independent of each other. In fact, the Beltrami coefficients are related to each other by virtue of the structure relations (3.6) and similarly the *integrating factors* Λ, τ . Given the expressions (3.10) and (3.13), we can unravel the set of equations (3.6) by expanding with respect to the basis $(e^z, e^{\bar{z}}, e^\theta, e^{\bar{\theta}})$ and using $de^z = -e^\theta e^{\bar{\theta}}, de^{\bar{\theta}} = 0$ and

$$d\Lambda = (e^z \partial + e^{\bar{z}} \bar{\partial} + e^\theta D_\theta + e^{\bar{\theta}} D_{\bar{\theta}}) \Lambda.$$

In this way, one obtains two times eight equations, most of which are related to some others by application of D_θ and $D_{\bar{\theta}}$. The only independent equations read

$$\begin{aligned} 0 &= D_\theta (H_\theta^z \Lambda) + 2 H_\theta^z \tau H_\theta^\theta \sqrt{\Lambda} - \Lambda + (H_\theta^\theta)^2 \Lambda \\ 0 &= D_{\bar{\theta}} (H_\theta^z \Lambda) + D_\theta (H_\theta^z \Lambda) + 2 H_\theta^z \tau H_\theta^\theta \sqrt{\Lambda} + 2 H_\theta^\theta H_\theta^z \Lambda + 2 H_\theta^z \tau H_\theta^\theta \sqrt{\Lambda} \\ 0 &= D_{\bar{\theta}} (H_\theta^z \Lambda) + 2 H_\theta^z \tau H_\theta^\theta \sqrt{\Lambda} + (H_\theta^\theta)^2 \Lambda - H_\theta^z \Lambda \\ 0 &= D_{\bar{\theta}} (H_\theta^z \tau) + D_{\bar{\theta}} (H_\theta^\theta \sqrt{\Lambda}) - H_\theta^z \tau - H_\theta^\theta \sqrt{\Lambda} \\ 0 &= D_\theta (H_\theta^\theta \sqrt{\Lambda}) + D_\theta (H_\theta^z \tau) - \tau \\ 0 &= D_{\bar{\theta}} (H_\theta^z \tau) + D_\theta (H_\theta^z \tau) + D_{\bar{\theta}} (H_\theta^\theta \sqrt{\Lambda}) + D_\theta (H_\theta^\theta \sqrt{\Lambda}). \end{aligned} \quad (3.19)$$

These are six equations for eight variables. The first five can be viewed as algebraic equations and used to express $H_\theta^\theta, H_\theta^z, H_\theta^z, H_\theta^z$ and τ in terms of H_θ^z, H_θ^z and Λ (and their derivatives). This is best done by substituting the dependent relations

$$\begin{aligned} 0 &= D_\theta \Lambda - \partial (H_\theta^z \Lambda) - 2 H_\theta^\theta \tau \sqrt{\Lambda} \\ 0 &= D_{\bar{\theta}} \Lambda - \partial (H_\theta^z \Lambda) - 2 H_\theta^\theta \tau \sqrt{\Lambda} \\ 0 &= \bar{\partial} \Lambda - \partial (H_\theta^z \Lambda) - 2 H_\theta^\theta \tau \sqrt{\Lambda} \\ 0 &= D_\theta \tau - \partial (H_\theta^z \tau) - \partial (H_\theta^\theta \sqrt{\Lambda}) \\ 0 &= D_{\bar{\theta}} \tau - \partial (H_\theta^z \tau) - \partial (H_\theta^\theta \sqrt{\Lambda}) \\ 0 &= \bar{\partial} \tau - \partial (H_\theta^z \tau) - \partial (H_\theta^\theta \sqrt{\Lambda}) \end{aligned} \quad (3.20)$$

and it leads to

$$\begin{aligned} (H_\theta^\theta)^2 &= 1 - (D_\theta - H_\theta^z \partial) H_\theta^z \\ H_\theta^\theta &= -\frac{1}{2 H_\theta^\theta} [(D_\theta - H_\theta^z \partial) H_\theta^z + (D_\theta - H_\theta^z \partial) H_\theta^z] \\ H_\theta^z &= (D_\theta - H_\theta^z \partial) H_\theta^z + (H_\theta^\theta)^2 \\ H_\theta^z &= (D_\theta - H_\theta^z \partial) H_\theta^z + \frac{1}{2} H_\theta^\theta (\partial H_\theta^z) \end{aligned} \quad (3.21)$$

$$\begin{aligned}
&= \frac{1}{2H_\theta^\theta} \left[(D_\theta - H_\theta^\theta \partial) H_z^\theta - (\bar{\partial} - H_z^\theta \partial) H_\theta^\theta \right] \\
\tau &= \frac{1}{(H_\theta^\theta)^2} (D_\theta - H_\theta^\theta \partial) (H_\theta^\theta \sqrt{\Lambda}) .
\end{aligned}$$

Thus, there are only two independent Beltrami differentials, H_θ^θ , H_z^θ , and one independent integrating factor, Λ (as well as the complex conjugate variables which occur in the expansion of e^z and e^θ).

Our conclusion that the superconformal structures of (1.1) supersymmetry are parametrized by two odd Beltrami superfields (and the complex conjugate variables), coincides with the results of other authors who used different approaches [13] [8] [26]. (The γ -traceless prepotentials H_α^b of reference [13] correspond to our variables H_θ^θ , H_z^θ and the supersymmetry compensators H^a of these authors are the analogues of our fields H_θ^θ , H_z^θ .) Clearly, the superspace formulation and quantization of superconformally invariant models should be based on the independent Beltrami coefficients H_θ^θ and H_z^θ . As discussed in section 3.7, the superfield H_θ^θ contains the ordinary Beltrami differential amongst its component fields.

So far, we have not yet discussed the last equation in (3.19). Substitution of the previous results in this relation yields a differential equation for the integrating factor Λ . This equation and another dependent relation can also be derived directly from eqs.(3.20):

$$\begin{aligned}
0 &= D_\theta \Lambda - \left[\frac{H_\theta^\theta}{H_\theta^\theta} \right] D_\theta \Lambda - \left[H_z^\theta - \frac{H_\theta^\theta}{H_\theta^\theta} H_\theta^\theta \right] \partial \Lambda - \left[\partial H_\theta^\theta - \frac{H_\theta^\theta}{H_\theta^\theta} \partial H_\theta^\theta \right] \Lambda \\
0 &= \bar{\partial} \Lambda - \left[\frac{H_z^\theta}{H_\theta^\theta} \right] D_\theta \Lambda - \left[H_z^\theta - \frac{H_z^\theta}{H_\theta^\theta} H_\theta^\theta \right] \partial \Lambda - \left[\partial H_z^\theta - \frac{H_z^\theta}{H_\theta^\theta} \partial H_\theta^\theta \right] \Lambda \quad (3.22)
\end{aligned}$$

As we will show in section 3.5, the solutions of the previous equations are only determined up to superconformal transformations. Projection of the second equation to its lowest component, $\Lambda| \equiv \lambda$, $H_z^\theta| \equiv \mu, \dots$, reproduces the bosonic theory equation (2.2) together with additional contributions which are due to supersymmetry.

If we set $H_\theta^\theta = 0 = H_z^\theta$ in the structure relations (3.19), then equations (3.10) (3.13) take the form

$$\begin{aligned}
e^z &= e^{-2v} e^z \\
e^\theta &= e^{-v} [e^\theta - e^z (D_\theta v)]
\end{aligned}$$

with $e^{-2v} \equiv \Lambda$ and $D_\theta v = 0$. Taking into account (3.21), (3.12) and (3.9), we have $e^v = (D_\theta \Theta)^{-1}$ which means that the mapping (3.8) now represents a superconformal transformation. This shows the consistency of the results, since our requirements were limited to (3.8) preserving the structure relations.

Let us come back to the general case. To deduce the infinitesimal transformation laws of the basic fields, we follow the strategy which already proved useful in the bosonic theory. Thus, we consider a ghost vector field

$$\Xi \cdot \partial \equiv \Xi^z(z, \bar{z}, \theta, \bar{\theta}) \partial_z + \Xi^{\bar{z}}(z, \bar{z}, \theta, \bar{\theta}) \partial_{\bar{z}} + \Xi^\theta(z, \bar{z}, \theta, \bar{\theta}) D_\theta + \Xi^{\bar{\theta}}(z, \bar{z}, \theta, \bar{\theta}) D_{\bar{\theta}} ,$$

which generates an infinitesimal change of the coordinates $(z, \bar{z}, \theta, \bar{\theta})$. As before, we introduce C -parameters by

$$(C^z, C^{\bar{z}}, C^\theta, C^{\bar{\theta}}) = (\Xi^z, \Xi^{\bar{z}}, \Xi^\theta, \Xi^{\bar{\theta}}) \cdot M , \quad (3.23)$$

where M is the matrix of Beltrami superfields. Explicitly,

$$\begin{aligned}
C^z &= \Xi^z + \Xi^{\bar{z}} H_z^\theta + \Xi^\theta H_\theta^\theta + \Xi^{\bar{\theta}} H_{\bar{\theta}}^\theta & \text{and c.c.} \\
C^\theta &= \Xi^\theta H_\theta^\theta + \Xi^{\bar{\theta}} H_{\bar{\theta}}^\theta + \Xi^z H_z^\theta + \Xi^{\bar{z}} H_{\bar{z}}^\theta & \text{and c.c.}
\end{aligned} \quad (3.24)$$

Contraction of the 1-forms (3.10) and (3.13) along the vector field $\Xi \cdot \partial$ gives

$$\begin{aligned}
i_{\Xi \cdot \partial} (e^z) &= [\Xi^z + \Xi^{\bar{z}} H_z^\theta + \Xi^\theta H_\theta^\theta + \Xi^{\bar{\theta}} H_{\bar{\theta}}^\theta] \Lambda^z \\
&= C^z \Lambda^z \equiv C^z \\
i_{\Xi \cdot \partial} (e^\theta) &= [\Xi^z + \Xi^{\bar{z}} H_z^\theta + \Xi^\theta H_\theta^\theta + \Xi^{\bar{\theta}} H_{\bar{\theta}}^\theta] \tau_z^\theta + [\Xi^\theta H_\theta^\theta + \Xi^z H_z^\theta + \Xi^{\bar{\theta}} H_{\bar{\theta}}^\theta] \sqrt{\Lambda^z} \\
&= C^z \tau_z^\theta + C^\theta \sqrt{\Lambda^z} \equiv C^\theta .
\end{aligned} \quad (3.25)$$

Thereby,

$$\begin{aligned}
s\Theta &= i_{\Xi \cdot \partial} d\Theta = i_{\Xi \cdot \partial} e^\theta = C^z \tau + C^\theta \sqrt{\Lambda} \\
sZ &= i_{\Xi \cdot \partial} dZ = i_{\Xi \cdot \partial} [e^z - \Theta e^\theta] = C^z \Lambda - \Theta(s\Theta) .
\end{aligned}$$

(In superspace, the s -operator is supposed to act as an antiderivation from the right and the ghost-number is added to the form degree, the Grassmann parity being s-inert.) From $s^2\Theta = 0 = s^2Z$, we now deduce

$$\begin{aligned}
sC^z &= -C^z \Lambda^{-1} [s\Lambda + 2C^\theta \tau \sqrt{\Lambda}] + C^\theta C^\theta \\
sC^\theta &= -\frac{1}{\sqrt{\Lambda}} [(sC^z) \tau + C^z(s\tau)] - \frac{1}{2\Lambda} C^\theta(s\Lambda) .
\end{aligned} \quad (3.26)$$

The transformation laws of the integrating factors and Beltrami coefficients follow by evaluating in two different ways the variations of the differentials $d\Theta$ and dZ ; for instance,

$$s(d\Theta) = -d(s\Theta) = -[e^z \partial + e^{\bar{z}} \bar{\partial} + e^\theta D_\theta + e^{\bar{\theta}} D_{\bar{\theta}}] [C^z \tau + C^\theta \sqrt{\Lambda}]$$

and

$$\begin{aligned}
s(d\Theta) = s e^\theta &= [e^z + e^{\bar{z}} H_z^\theta + e^\theta H_\theta^\theta + e^{\bar{\theta}} H_{\bar{\theta}}^\theta] s\tau + [e^z sH_z^\theta + e^\theta sH_\theta^\theta + e^{\bar{z}} sH_{\bar{z}}^\theta + e^{\bar{\theta}} sH_{\bar{\theta}}^\theta] \tau \\
&\quad + [e^\theta H_\theta^\theta + e^{\bar{\theta}} H_{\bar{\theta}}^\theta + e^z H_z^\theta + e^{\bar{z}} H_{\bar{z}}^\theta] s\sqrt{\Lambda} + [e^\theta sH_\theta^\theta + e^{\bar{\theta}} sH_{\bar{\theta}}^\theta + e^z sH_z^\theta + e^{\bar{z}} sH_{\bar{z}}^\theta] \sqrt{\Lambda}
\end{aligned}$$

lead to the variations of τ and H_θ^θ , H_z^θ , $H_{\bar{\theta}}^\theta$.

From $s(dZ)$ and $s(d\Theta)$, we get

$$\begin{aligned}
s\Lambda &= -\partial(C^z \Lambda) - 2C^\theta \tau \sqrt{\Lambda} \\
s\tau &= -\partial(C^z \tau + C^\theta \sqrt{\Lambda})
\end{aligned} \quad (3.27)$$

and substitution of these expressions into (3.26) yields

$$\begin{aligned} sC^z &= C^z \partial C^z + C^\theta C^\theta \\ sC^\theta &= C^z \partial C^\theta + \frac{1}{2} C^\theta (\partial C^z) \end{aligned} \quad (3.28)$$

The variations of the Beltrami differentials take the form

$$\begin{aligned} sH_\theta^z &= - \left[(D_\theta - H_\theta^z \partial) C^z + (\partial H_\theta^z) C^z - 2 H_\theta^\theta C^\theta \right] \\ sH_\theta^\theta &= - \left[(D_\theta - H_\theta^\theta \partial) C^\theta + (\partial H_\theta^\theta) C^\theta - 2 H_\theta^z C^z \right] \\ sH_z^z &= - \left[(\bar{\partial} - H_z^z \partial) C^z + (\partial H_z^z) C^z - 2 H_z^\theta C^\theta \right] \\ sH_\theta^\theta &= - \left[(D_\theta - H_\theta^\theta \partial) C^\theta + \frac{1}{2} (\partial H_\theta^\theta) C^\theta + C^z (\partial H_\theta^\theta) - \frac{1}{2} H_\theta^\theta (\partial C^z) \right] \\ sH_\theta^z &= - \left[(D_\theta - H_\theta^z \partial) C^z + \frac{1}{2} (\partial H_\theta^z) C^\theta + C^z (\partial H_\theta^z) - \frac{1}{2} H_\theta^\theta (\partial C^z) \right] \\ sH_z^\theta &= - \left[(\bar{\partial} - H_z^\theta \partial) C^\theta + \frac{1}{2} (\partial H_z^\theta) C^\theta + C^z (\partial H_z^\theta) - \frac{1}{2} H_z^z (\partial C^\theta) \right] \end{aligned} \quad (3.29)$$

Thus, the holomorphic factorization is manifestly realized for the s -variations (3.27)-(3.29) which have explicitly been verified to be nilpotent.

We conclude our discussion of the geometry of SRS's with a few comments on the symmetry algebra (3.27)-(3.29).

(i) The expressions C^z and C^θ introduced in eqs.(3.25),

$$\begin{aligned} C^z &= C^z \Lambda_z^z \\ C^\theta &= C^\theta \sqrt{\Lambda_z^z} + C^z \tau_z^\theta \end{aligned} \quad (3.30)$$

are invariant under superconformal changes of the small coordinates (eq.(3.17)). Thereby, we can determine the corresponding transformations of C^z and C^θ from those of Λ_z and τ_z (eqs.(3.18)) :

$$\begin{aligned} C^z &= e^{-2w} Y^{-1} C^z \\ C^\theta &= e^{-w} Y^{-1/2} [C^\theta - C^z (D_\theta w) H_\theta^\theta] \end{aligned} \quad (3.31)$$

The s -variations (3.27)-(3.29) commute with the superconformal transformations (3.18)/(3.31) and therefore provide globally defined expressions on the SRS.

(ii) By construction, eqs.(3.28) are the structure relations associated to the Lie algebra of superdiffeomorphisms. If expressed in terms of the generators $\bar{\Xi}^z$ and $\bar{\Xi}^\theta$, they take the form

$$\begin{aligned} s\bar{\Xi}^z &= (\bar{\Xi} \cdot \partial) \bar{\Xi}^z - \bar{\Xi}^\theta \bar{\Xi}^\theta \\ s\bar{\Xi}^\theta &= (\bar{\Xi} \cdot \partial) \bar{\Xi}^\theta \end{aligned} \quad (3.32)$$

The origin of the quadratic term in $s\bar{\Xi}^z$ can be drawn back to the fact that we expressed the vector field $\bar{\Xi} \cdot \partial$ in terms of the canonical basis of the tangent space :

$$\begin{aligned} \bar{\Xi} \cdot \partial &\equiv \bar{\Xi}^z \partial + \bar{\Xi}^\theta \bar{\partial} + \bar{\Xi}^\theta D_\theta + \bar{\Xi}^\theta D_\theta \\ &= [\bar{\Xi}^z - \bar{\theta} \bar{\Xi}^\theta] \partial + [\bar{\Xi}^z - \bar{\theta} \bar{\Xi}^\theta] \bar{\partial} + \bar{\Xi}^\theta \partial_\theta + \bar{\Xi}^\theta \partial_{\bar{\theta}} \end{aligned}$$

This term disappears if (3.32) is rewritten in terms of the 'natural basis' vector components occurring in the last line.

(iii) The variations of H_θ^z and H_θ^θ close among themselves; similarly those of H_z^z and H_z^θ and those of H_z^z and H_z^θ . Actually, the variation of H_θ^z closes upon itself and its derivatives, because H_θ^θ only depends on these fields. In the terminology of references [12]-[14], the field H_θ^z is a *compensator* for the C^θ -symmetry, i.e. it can be gauged away algebraically by a C^θ -transformation : if $H_\theta^z = 0$, then $sH_\theta^z = 0$ implies

$$C^\theta = \frac{1}{2} D_\theta C^z \quad (3.33)$$

In this case, the structure relations of the superdiffeomorphism algebra reduce to

$$sC^z = C^z \partial C^z + \frac{1}{4} (D_\theta C^z) (D_\theta C^z) \quad (3.34)$$

These structure relations have the same form as those of the superconformal algebra,

$$s\bar{\Xi}^z = \bar{\Xi}^z \partial \bar{\Xi}^z + \frac{1}{4} (D_\theta \bar{\Xi}^z) (D_\theta \bar{\Xi}^z) \quad ,$$

in which $\bar{\Xi}^z$ only depends on z and θ . As emphasized above, the restriction $H_\theta^z = 0$ will not be considered in the main body of the text.

(iv) Equations (3.28) and some of the variations (3.29) involve only space-time derivatives and can be projected to component field expressions in a straightforward way [8]. From the definitions

$$H_z^z \equiv \mu_z^z \quad , \quad H_z^\theta \equiv \frac{i}{2} \alpha_z^\theta \quad (3.35)$$

and

$$C^z \equiv c^z \equiv \xi^z + \xi^z \mu_z^z \quad , \quad C^\theta \equiv \frac{i}{2} \epsilon^\theta \quad , \quad (3.36)$$

(where the bar denotes the projection to the lowest component), we obtain the symmetry algebra of the ordinary Beltrami differentials $\mu, \bar{\mu}$ and of their fermionic partners, the Beltraminos $\alpha, \bar{\alpha}$:

$$\begin{aligned} s\mu &= (\bar{\partial} - \mu \partial + \partial \mu) c + \frac{1}{2} \alpha \epsilon \\ s\alpha &= (\bar{\partial} - \mu \partial + \frac{1}{2} \partial \mu) \epsilon + c \partial \alpha + \frac{1}{2} \alpha \partial c \\ sc &= c \partial c - \frac{1}{4} \epsilon \epsilon \\ s\epsilon &= c \partial \epsilon - \frac{1}{2} \epsilon \partial c \end{aligned} \quad (3.37)$$

Thus, the space-time projection of the superspace algebra (3.29) (3.28) contains the symmetry algebra of the bosonic theory and the holomorphic factorization remains manifestly realized at the component field level². However, as we will see in section 3.7, the geometry was destroyed by the projection procedure and the space-time differential algebra (3.37) is no longer associated with an ordinary Lie algebra, but only with a field-dependent one.

²In the last set of equations, s is supposed to act from the left as usual in component field language and the graduation is given by the sum of the ghost-number and the Grassmann parity; the signs following from the superspace algebra have been modified so as to ensure nilpotency with these conventions.

3.2 Scalar superfields

The coupling of a scalar superfield $\mathcal{X}$ to a superconformal class of metrics is described by the action

$$2i S_{\text{inv}} = \int dZ d\bar{Z} d\Theta d\bar{\Theta} (D_{\Theta} \mathcal{X}) (D_{\bar{\Theta}} \mathcal{X}) , \quad (3.38)$$

where $d\Theta$ and $d\bar{\Theta}$ denote the algebraic duals of the Grassmann variables Θ and $\bar{\Theta}$. This functional is invariant under superconformal changes of coordinates for which the measure transforms with $(D_{\Theta} \Theta')^{-1} (D_{\bar{\Theta}} \bar{\Theta}')^{-1}$, i.e. the Berezinian associated to the superconformal transformation (3.4).

Since this has not been done before, we now derive an explicit expression for (3.38) in terms of Beltrami superfields. For this purpose, we consider again the passage from the small to the capital coordinates for which we have

$$\begin{pmatrix} \partial_Z \\ \partial_{\Theta} \end{pmatrix} = Q^{-1} \cdot M^{-1} \cdot \begin{pmatrix} \partial \\ D_{\bar{\Theta}} \end{pmatrix} . \quad (3.39)$$

The Berezinian of this change of variables reads

$$\left| \frac{\partial(Z, \bar{Z}, \Theta, \bar{\Theta})}{\partial(z, \bar{z}, \theta, \bar{\theta})} \right| = \text{sdet}(M \cdot Q) = (\text{sdet } M) \cdot \sqrt{\Lambda \bar{\Lambda}} \quad (3.40)$$

and the inverse of Q is easily determined :

$$Q^{-1} = \begin{pmatrix} \Lambda^{-1} & 0 & -\Lambda^{-3/2} \tau & 0 \\ 0 & \bar{\Lambda}^{-1} & 0 & -\bar{\Lambda}^{-3/2} \bar{\tau} \\ 0 & 0 & \Lambda^{-1/2} & 0 \\ 0 & 0 & 0 & \bar{\Lambda}^{-1/2} \end{pmatrix} . \quad (3.41)$$

An explicit expression for M^{-1} is obtained by decomposing M into a product of matrices having at least one odd subsector which vanishes :

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ h_{\theta}^z & h_{\theta}^{\bar{z}} & 1 & 0 \\ h_{\bar{\theta}}^z & h_{\bar{\theta}}^{\bar{z}} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ H_z^z & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & h_{\bar{\theta}}^{\theta} & 1 \end{pmatrix} \begin{pmatrix} 1 & H_z^z & 0 & H_z^{\bar{\theta}} \\ 0 & \Delta & H_z^{\theta} & h_{\bar{\theta}}^{\bar{z}} \\ 0 & 0 & I & h_{\bar{\theta}}^{\theta} \\ 0 & 0 & 0 & J \end{pmatrix} . \quad (3.42)$$

Here, Δ, I, J and the 'h' are unknown coefficients which are determined by comparison with the known expression of M :

$$\begin{aligned} \Delta &= 1 - H_z^z H_z^{\bar{z}} , & h_z^{\bar{\theta}} &= -H_z^z H_z^{\bar{\theta}} \\ h_{\theta}^z &= \frac{1}{\Delta} [H_z^z - H_{\theta}^z H_z^z] , & h_{\theta}^{\bar{z}} &= \frac{1}{\Delta} [H_z^z - H_{\theta}^z H_z^z] \\ h_{\bar{\theta}}^z &= \frac{1}{\Delta} [H_z^z - H_{\theta}^z H_z^z] , & h_{\bar{\theta}}^{\bar{z}} &= \frac{1}{\Delta} [H_z^z - H_{\theta}^z H_z^z] \\ h_{\theta}^{\bar{\theta}} &= H_{\theta}^{\bar{\theta}} - \frac{1}{\Delta} [H_{\theta}^z - H_{\theta}^z H_z^z] H_z^{\bar{\theta}} , & I &= H_{\theta}^{\theta} - \frac{1}{\Delta} [H_{\theta}^z - H_{\theta}^z H_z^z] H_z^{\theta} \\ h_{\bar{\theta}}^{\theta} &= \frac{1}{I} \left(H_{\bar{\theta}}^{\theta} - \frac{1}{\Delta} [H_{\bar{\theta}}^z - H_{\bar{\theta}}^z H_z^z] H_z^{\theta} \right) , & J &= H_{\bar{\theta}}^{\bar{\theta}} - h_{\bar{\theta}}^{\theta} h_{\theta}^{\bar{\theta}} - h_{\bar{\theta}}^z H_z^{\bar{\theta}} . \end{aligned}$$

Note that $h_{\theta}^{\bar{\theta}}$ and $h_{\bar{\theta}}^{\theta}$ are not complex conjugate variables. It immediately follows that $\text{sdet } M = \Delta / IJ$ and that

$$M^{-1} = \begin{pmatrix} 1 & * & * & * \\ 0 & \Delta^{-1} & * & * \\ 0 & 0 & I^{-1} & -(IJ)^{-1} h_{\theta}^{\bar{\theta}} \\ 0 & 0 & 0 & J^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -H_z^z & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -h_{\theta}^z & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -h_{\theta}^z & -h_{\theta}^{\bar{z}} & 1 & 0 \\ -h_{\bar{\theta}}^z & -h_{\bar{\theta}}^{\bar{z}} & 0 & 1 \end{pmatrix} . \quad (3.43)$$

Here, the * denote factors whose explicit form may be determined, but is not required for the determination of $D_{\Theta}, D_{\bar{\Theta}}$. Thus,

$$D_{\Theta} = \frac{1}{\sqrt{\Lambda I}} \left[\left(1 + \frac{1}{J} h_{\theta}^{\bar{\theta}} h_{\bar{\theta}}^{\theta} \right) (D_{\theta} - h_{\theta}^z \partial - h_{\theta}^{\bar{z}} \bar{\partial}) - \frac{1}{J} h_{\theta}^{\bar{\theta}} (D_{\bar{\theta}} - h_{\bar{\theta}}^z \bar{\partial} - h_{\bar{\theta}}^{\theta} \partial) \right]$$

$$D_{\bar{\Theta}} = \frac{1}{\sqrt{\Lambda J}} \left[(D_{\bar{\theta}} - h_{\bar{\theta}}^z \bar{\partial} - h_{\bar{\theta}}^{\theta} \partial) - h_{\bar{\theta}}^{\theta} (D_{\theta} - h_{\theta}^z \partial - h_{\theta}^{\bar{z}} \bar{\partial}) \right] , \quad (3.44)$$

which expressions look asymmetrical, but can easily be shown to represent complex conjugate quantities. Finally,

$$\begin{aligned} 2i S_{\text{inv}} &= \int dz d\bar{z} d\theta d\bar{\theta} \left| \frac{\partial(Z, \bar{Z}, \Theta, \bar{\Theta})}{\partial(z, \bar{z}, \theta, \bar{\theta})} \right| (D_{\Theta} \mathcal{X}) (D_{\bar{\Theta}} \mathcal{X}) \\ &= \int dz d\bar{z} d\theta d\bar{\theta} \frac{\Delta}{(IJ)^2} (D_{\theta} - h_{\theta}^z \partial - h_{\theta}^{\bar{z}} \bar{\partial}) \mathcal{X} (D_{\bar{\theta}} - h_{\bar{\theta}}^z \bar{\partial} - h_{\bar{\theta}}^{\theta} \partial) \mathcal{X} . \end{aligned} \quad (3.45)$$

Since the last expression only depends on the Beltrami coefficients and not on the integrating factors, it represents a local functional when projected to the $(z, \bar{z})$ -space. In section 3.7, we will see that it generalizes the action for the bosonic field $X \equiv \mathcal{X}$ in a supersymmetric way. It is invariant under the s -variations (3.29) and

$$s\mathcal{X} = \left(\Xi^z, \Xi^{\bar{z}}, \Xi^{\theta}, \Xi^{\bar{\theta}} \right) \cdot \begin{pmatrix} \partial \\ D_{\theta} \end{pmatrix} \mathcal{X} = \left(C^z, C^{\bar{z}}, C^{\theta}, C^{\bar{\theta}} \right) \cdot M^{-1} \cdot \begin{pmatrix} \partial \\ D_{\bar{\theta}} \end{pmatrix} \mathcal{X} . \quad (3.46)$$

3.3 Superconformal fields and super bc-systems

Extending the definition (2.15), (2.16) of conformal tensors, we define superconformal tensors (of lowest rank) to be given by pairs $(C^z, C^{\bar{z}})$ and $(B_z, B_{\bar{z}})$, respectively, such that the expressions $C^z \partial_z + C^{\bar{z}} \partial_{\bar{z}}$ and $e^z B_z + e^{\bar{z}} B_{\bar{z}}$ are superconformally invariant. From (3.4) and (3.7), one obtains the following transformation properties under superconformal changes of coordinates:

$$\begin{aligned} C^{z'}(Z') &= e^{-2W} C^z(Z) \\ C^{\theta'}(Z') &= e^{-W} [C^{\theta}(Z) - C^z(Z) (D_{\Theta} W)] \\ B_{\theta'}(Z') &= e^W B_{\theta}(Z) \\ B_{z'}(Z') &= e^{2W} [B_z(Z) + (D_{\Theta} W) B_{\theta}(Z)] . \end{aligned} \quad (3.47)$$

Here, $Z = (Z, \bar{Z}, \Theta, \bar{\Theta})$ and $e^{-W} = D_{\Theta}\Theta'$. Analogously, tensors of higher order are defined as pairs $(B_{Z\dots Z}, B_{\Theta Z\dots Z})$ and $(C^{Z\dots Z}, C^{Z\dots Z\Theta})$ such that $(e^Z)^q B_{Z\dots Z} + (e^Z)^{q-1} e^{\Theta} B_{\Theta Z\dots Z}$ and $C^{Z\dots Z} B_{Z\dots Z} + C^{Z\dots Z\Theta} B_{\Theta Z\dots Z}$ are superconformally invariant :

$$\begin{aligned} C^{Z\dots Z'}(\mathcal{Z}') &= e^{-2qW} C^{Z\dots Z'}(\mathcal{Z}) \\ C^{Z\dots Z'\Theta'}(\mathcal{Z}') &= e^{-(2q-1)W} [C^{Z\dots Z'\Theta}(\mathcal{Z}) - C^{Z\dots Z'}(\mathcal{Z}) (D_{\Theta}W)] \\ B_{\Theta Z\dots Z'}(\mathcal{Z}') &= e^{(2q-1)W} B_{\Theta Z\dots Z'}(\mathcal{Z}) \\ B_{Z\dots Z'\Theta}(\mathcal{Z}') &= e^{2qW} [B_{Z\dots Z'}(\mathcal{Z}) + (D_{\Theta}W) B_{\Theta Z\dots Z'}(\mathcal{Z})] \end{aligned} \quad (3.48)$$

From these transformation laws, it follows that the functional

$$\int dZ d\bar{Z} d\Theta d\bar{\Theta} [B_{\Theta Z\dots Z} D_{\Theta} C^{Z\dots Z} + \text{c.c.}] \quad (3.49)$$

is superconformally invariant, but not so the action

$$\int dZ d\bar{Z} d\Theta d\bar{\Theta} [B_{Z\dots Z} D_{\Theta} C^{Z\dots Z\Theta} + \text{c.c.}] \quad (3.50)$$

The remedy to this problem consists of replacing B and C in the last expression by combinations of fields which transform homogeneously, e.g. in the simplest case

$$\begin{aligned} \hat{B}_Z &\equiv B_Z - D_{\Theta}B_{\Theta} \\ \hat{C}^{\Theta} &\equiv C^{\Theta} - \frac{1}{2} D_{\Theta}C^Z \end{aligned}$$

Such combinations also allow for the construction of other superconformal invariants (see, for instance, eq.(3.72) below).

In the literature [25], a ' Z ' index is usually identified with two ' Θ ' indices and all fields are assumed to transform homogeneously. Although the last set of redefinitions amounts to a reduction to such a situation, the basic geometric objects do not behave in this way.

Let us now consider the relation between the conformal variables $(C^Z, C^{\Theta}), \dots, (B_Z, B_{\Theta}), \dots$ and the local fields $(C^Z, C^{\Theta}), \dots, (B_Z, B_{\Theta}), \dots$. From (3.25), we have the relation (3.30) and from (3.10), (3.13), we obtain

$$e^Z B_Z + e^{\Theta} B_{\Theta} = [e^Z + e^Z H_{\bar{Z}}^Z + e^{\Theta} H_{\Theta}^Z + e^{\bar{\Theta}} H_{\bar{\Theta}}^Z] B_Z + [e^{\bar{\Theta}} H_{\bar{\Theta}}^{\Theta} + e^Z H_{\bar{Z}}^{\Theta} + e^{\bar{\Theta}} H_{\bar{\Theta}}^{\Theta}] B_{\Theta}$$

with

$$\begin{aligned} B_{\Theta} &= \sqrt{\Lambda_Z^Z} B_{\Theta} \\ B_Z &= \Lambda_Z^Z B_Z + \tau_Z^{\Theta} B_{\Theta} \end{aligned} \quad (3.50)$$

In the previous equations, the factors Λ, τ and the B, C fields depend on $(z, \bar{z}, \theta, \bar{\theta})$ while B and C depend on $(Z, \bar{Z}, \Theta, \bar{\Theta})$.

The generalization of (3.30) to higher order tensors is given by

$$\begin{aligned} C^{Z\dots Z} &= C^{Z\dots Z} (\Lambda_Z^Z)^q \\ C^{Z\dots Z\Theta} &= C^{Z\dots Z\Theta} (\Lambda_Z^Z)^{\frac{2q-1}{2}} + C^{Z\dots Z} \tau_Z^{\Theta} (\Lambda_Z^Z)^{q-1} \\ B_{\Theta Z\dots Z} &= (\Lambda_Z^Z)^{\frac{2q-1}{2}} B_{\Theta Z\dots Z} \\ B_{Z\dots Z} &= (\Lambda_Z^Z)^q B_{Z\dots Z} + (\Lambda_Z^Z)^{q-1} \tau_Z^{\Theta} B_{\Theta Z\dots Z} \end{aligned} \quad (3.51)$$

These relations are consistent with the superconformal transformations of the capital coordinates, eqs.(3.16), (3.48), and yield those of the small coordinates by comparison with (3.18):

$$\begin{aligned} C^{z'\dots z'} &= e^{-2q_w} Y^{-q} C^{z\dots z} \\ C^{z'\dots z'\theta'} &= e^{-(2q-1)w} Y^{-(2q-1)/2} [C^{z\dots z\theta} - C^{z\dots z} (D_{\theta}w) H_{\theta}^{\theta}] \\ B_{\theta' z' \dots z'} &= e^{(2q-1)w} Y^{(2q-1)/2} B_{\theta z\dots z} \\ B_{z' \dots z'} &= e^{2q_w} Y^q [B_{z\dots z} + (D_{\theta}w) H_{\theta}^{\theta} B_{\theta z\dots z}] \end{aligned} \quad (3.52)$$

Here, we recall $e^w \equiv (D_{\theta}\theta')^{-1}$ and $Y \equiv 1 + (D_{\theta}w)H_{\theta}^z$. An explicit expression for (3.49) in terms of the small coordinates will be derived in section 3.6.

The chiral currents $\mathcal{J}_{\mathcal{B}}(X)$ of the bosonic theory, eq.(2.21), generalize to pairs of superfields $(\mathcal{J}_{\mathcal{B}}, \mathcal{J}_{\Theta})$ transforming as the $\mathcal{B}$ -fields. A superconformally invariant action is given by substituting D_{Θ} by $\mathcal{J}_{\Theta}$ in eq.(3.49). From

$$J_{\mathcal{B}} = (\text{sdet } M) \sqrt{\Lambda_Z^Z} \mathcal{J}_{\Theta} \quad (3.53)$$

and the relations (3.51), we obtain

$$\int dZ d\bar{Z} d\Theta d\bar{\Theta} [B_{\Theta Z\dots Z} \mathcal{J}_{\Theta}(\mathcal{X}) C^{Z\dots Z} + \text{c.c.}] = \int dz d\bar{z} d\theta d\bar{\theta} [B_{\theta z\dots z} J_{\mathcal{B}}(\mathcal{X}) C^{z\dots z} + \text{c.c.}] \quad (3.54)$$

3.4 Superdiffeomorphisms

The action of a superdiffeomorphism

$$(z, \bar{z}, \theta, \bar{\theta}) \longrightarrow (z'(z, \bar{z}, \theta, \bar{\theta}), \bar{z}'(z, \bar{z}, \theta, \bar{\theta}), \theta'(z, \bar{z}, \theta, \bar{\theta}), \bar{\theta}'(z, \bar{z}, \theta, \bar{\theta}))$$

on the Beltrami coefficients, integrating factors and B, C fields can be derived along the lines of the bosonic theory. Using the shorthand notation

$$\begin{aligned} U_z^{z'} &\equiv [\partial z' + \theta' \partial \theta'] + [\partial \bar{z}' + \bar{\theta}' \partial \bar{\theta}'] H_{z'}^{z'} + [\partial \theta'] H_{\theta'}^{z'} + [\partial \bar{\theta}'] H_{\bar{\theta}'}^{z'} \\ V_z^{\theta'} &\equiv [\partial \theta'] H_{\theta'}^{\theta'} + [\partial \bar{\theta}'] H_{\bar{\theta}'}^{\theta'} + [\partial z' + \theta' \partial \theta'] H_{z'}^{\theta'} + [\partial \bar{z}' + \bar{\theta}' \partial \bar{\theta}'] H_{\bar{z}'}^{\theta'} + [\partial \theta'] H_{\theta'}^{\theta'} \end{aligned}$$

the results are summarized as follows :

$$\begin{aligned} \Lambda_z &= U_z^{z'} \Lambda_{z'} \\ \tau_z &= U_z^{z'} \tau_{z'} + V_z^{\theta'} \sqrt{\Lambda_{z'}} \\ H_z^z &= (U_z^{z'})^{-1} \{ [\partial z' + \theta' \partial \theta'] + [\partial \bar{z}' + \bar{\theta}' \partial \bar{\theta}'] H_{z'}^{z'} + [\partial \theta'] H_{\theta'}^{z'} + [\partial \bar{\theta}'] H_{\bar{\theta}'}^{z'} \} \\ H_z^{\theta'} &= (U_z^{z'})^{-1} \{ [D_{\theta} z' - \theta' D_{\theta} \theta'] + [D_{\theta} \bar{z}' - \bar{\theta}' D_{\theta} \bar{\theta}'] H_{z'}^{z'} + [D_{\theta} \theta'] H_{\theta'}^{z'} + [D_{\theta} \bar{\theta}'] H_{\bar{\theta}'}^{z'} \} \\ H_{\theta}^z &= (U_z^{z'})^{-1} \{ [D_{\theta} z' - \theta' D_{\theta} \theta'] + [D_{\theta} \bar{z}' - \bar{\theta}' D_{\theta} \bar{\theta}'] H_{z'}^{z'} + [D_{\theta} \theta'] H_{\theta'}^{z'} + [D_{\theta} \bar{\theta}'] H_{\bar{\theta}'}^{z'} \} \\ H_{\theta}^{\theta'} &= (U_z^{z'})^{-1/2} \{ [D_{\theta} \theta'] H_{\theta'}^{\theta'} + [D_{\theta} \bar{\theta}'] H_{\bar{\theta}'}^{\theta'} + [D_{\theta} z' - \bar{\theta}' D_{\theta} \bar{\theta}'] H_{z'}^{\theta'} + [D_{\theta} \bar{z}' - \bar{\theta}' D_{\theta} \bar{\theta}'] H_{\bar{z}'}^{\theta'} - H_{\theta}^z V_z^{\theta'} \} \end{aligned} \quad (3.55)$$

or explicitly

$$\begin{aligned}\delta_{\Xi} \Gamma^z &= (\Xi \cdot \partial) \Gamma^z - (\Gamma \cdot \partial) \Xi^z - 2 \Xi^\theta \Gamma^\theta & \text{and c.c.} \\ \delta_{\Xi} \Gamma^\theta &= (\Xi \cdot \partial) \Gamma^\theta - (\Gamma \cdot \partial) \Gamma^\theta & \text{and c.c.}\end{aligned}$$

3.5 Intermediate coordinate system

As illustrated by eq.(3.44), the derivatives $\partial_Z, \partial_{\bar{Z}}, \partial_\theta, \partial_{\bar{\theta}}$ represent complicated linear combinations of the reference derivatives $\partial, \bar{\partial}, D_\theta, D_{\bar{\theta}}$. The operators D_θ and $D_{\bar{\theta}}$ occur in the invariant action for $\mathcal{X}$, but none of the derivatives $\partial_Z, \dots, D_\theta$ naturally appear in the symmetry transformations or in the integrating factor equations (IFEQ's). In this section, we will introduce a kind of "intermediate coordinate system" which reproduces the derivatives occurring in the IFEQ's. As an immediate application, we can determine the ambiguity on the solutions of the IFEQ's and carry out a further discussion of super bc-systems (subsequent section).

For the introduction of the intermediate or 'tilde' coordinates,

$$(z, \bar{z}, \theta, \bar{\theta}) \longrightarrow (\tilde{z}, \tilde{\bar{z}}, \tilde{\theta}, \tilde{\bar{\theta}}) \equiv (Z, \bar{Z}, \Theta, \bar{\Theta}) \longrightarrow (Z, \bar{Z}, \Theta, \bar{\Theta}) \quad , \quad (3.58)$$

we disregard complex conjugation and rely on the fact that the structure equations for e^z, e^θ and $e^{\bar{z}}, e^{\bar{\theta}}$ then become totally independent of each other. The matrices describing the passage from $(z, \bar{z}, \theta, \bar{\theta})$ to $(\tilde{z}, \tilde{\bar{z}}, \tilde{\theta}, \tilde{\bar{\theta}})$ are given by

$$M_1 Q_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ H_z^z & 1 & H_z^\theta & 0 \\ H_\theta^z & 0 & H_\theta^\theta & 0 \\ H_{\bar{\theta}}^z & 0 & H_{\bar{\theta}}^\theta & 1 \end{pmatrix} \begin{pmatrix} \Lambda & 0 & \tau & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{\Lambda} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (3.59)$$

Here, Λ, τ and the 'H' are the same fields as before and they satisfy the same equations (eqs.(3.19)-(3.22)). By construction, the intermediate coordinates are *not* related by complex conjugation :

$$\tilde{\bar{z}} \neq \bar{\tilde{z}} \quad , \quad \tilde{\bar{\theta}} \neq \bar{\tilde{\theta}} \quad .$$

We immediately have $\text{sdet}(M_1 Q_1) = \sqrt{\Lambda} / H_\theta^\theta$ and

$$Q_1^{-1} M_1^{-1} = \begin{pmatrix} \Lambda^{-1} & 0 & -\Lambda^{-3/2} \tau & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \Lambda^{-1/2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -H_z^z + (H_\theta^\theta)^{-1} H_z^\theta H_\theta^z & 1 & -(H_\theta^\theta)^{-1} H_z^\theta & 0 \\ -(H_\theta^\theta)^{-1} H_\theta^z & 0 & (H_\theta^\theta)^{-1} & 0 \\ -H_\theta^z + (H_\theta^\theta)^{-1} H_\theta^\theta H_\theta^z & 0 & -(H_\theta^\theta)^{-1} H_\theta^\theta & 1 \end{pmatrix} \quad (3.60)$$

Henceforth,

$$\begin{aligned}\tilde{\partial} &= \frac{1}{\Lambda} \left[\partial - \frac{\tau}{\sqrt{\Lambda} H_\theta^\theta} (D - H_\theta^z \partial) \right] \\ \tilde{\bar{\partial}} &= \bar{\partial} - \frac{H_z^\theta}{H_\theta^\theta} D - (H_z^z - \frac{H_z^\theta}{H_\theta^\theta} H_\theta^z) \partial\end{aligned}$$

$$\begin{aligned}H_\theta^\theta &= (U_z^z)^{-1/2} \{ [\tilde{\partial} \theta] H_\theta^\theta + [\tilde{\partial} \bar{z} + \bar{\theta} \tilde{\partial} \bar{\theta}] H_\theta^\theta + [\tilde{\partial} \bar{\theta}] H_\theta^\theta \} - H_z^z V_z^\theta \\ H_\theta^\theta &= (U_z^z)^{-1/2} \{ [D_\theta \theta] H_\theta^\theta + [D_\theta \bar{z} - \bar{\theta} D_\theta \bar{\theta}] H_\theta^\theta + [D_\theta \bar{\theta}] H_\theta^\theta \} - H_\theta^z V_z^\theta \\ C^{z\dots z} &= (U_z^z)^{-q} C^{z\dots z'} \\ C^{z\dots z\theta} &= (U_z^z)^{-\frac{2q-1}{2}} C^{z\dots z'\theta} - (U_z^z)^{-\frac{2q+1}{2}} V_z^\theta C^{z\dots z'} \\ B_{\theta z\dots z} &= (U_z^z)^{-\frac{2q-1}{2}} B_{\theta z'\dots z'} \\ B_{z\dots z} &= (U_z^z)^{-q} B_{z'\dots z'} + (U_z^z)^{q-1} V_z^\theta B_{\theta z'\dots z'}\end{aligned}$$

The action of an infinitesimal superdiffeomorphism generated by the vector field $\Xi(z, \bar{z}, \theta, \bar{\theta}) \cdot \partial$ follows from the previous set of equations by expanding to first order :

$$\begin{aligned}z' &\simeq z + (\Xi \cdot \partial) z = z + \Xi^z - \theta \Xi^\theta & \text{and c.c.} \\ \theta' &\simeq \theta + (\Xi \cdot \partial) \theta = \theta + \Xi^\theta & \text{and c.c.}\end{aligned}$$

The induced transformation laws of the Beltrami coefficients coincide with the s -variations (3.29), but with the (negative of the) ghost fields C replaced by the symmetry parameters Ξ according to eq.(3.24). The other variations depend on the expressions

$$\begin{aligned}R &\equiv \partial \Xi^z + (\partial \Xi^z) H_z^z + (\partial \Xi^\theta) H_\theta^z + (\partial \Xi^\theta) H_\theta^\theta \\ S_z^\theta &\equiv (\partial \Xi^\theta) H_\theta^\theta + (\partial \Xi^z) H_z^\theta + (\partial \Xi^\theta) H_\theta^\theta\end{aligned}$$

and have the explicit form

$$\begin{aligned}\delta_{\Xi} \Lambda_z &= (\Xi \cdot \partial) \Lambda_z + R \Lambda_z \\ \delta_{\Xi} \tau_z &= (\Xi \cdot \partial) \tau_z + R \tau_z + S_z^\theta \sqrt{\Lambda_z} \\ \delta_{\Xi} C^{z\dots z} &= (\Xi \cdot \partial) C^{z\dots z} - q R C^{z\dots z} \\ \delta_{\Xi} C^{z\dots z\theta} &= (\Xi \cdot \partial) C^{z\dots z\theta} - \frac{2q-1}{2} R C^{z\dots z\theta} - S_z^\theta C^{z\dots z} \\ \delta_{\Xi} B_{\theta z\dots z} &= (\Xi \cdot \partial) B_{\theta z\dots z} + \frac{2q-1}{2} R B_{\theta z\dots z} \\ \delta_{\Xi} B_{z\dots z} &= (\Xi \cdot \partial) B_{z\dots z} + q R B_{z\dots z} + S_z^\theta B_{\theta z\dots z}\end{aligned} \quad (3.56)$$

As a consequence of the structure relations, the variations of Λ and τ coincide with (the negative of) their s -operations, eqs.(3.27).

By construction, the algebra of the Ξ -transformations is given by the Lie bracket :

$$[\delta_{\Xi_1}, \delta_{\Xi_2}] = -\delta_{\Xi_{12}} \quad \text{with} \quad \Xi_{12} \cdot \partial \equiv [\Xi_1 \cdot \partial, \Xi_2 \cdot \partial] \quad . \quad (3.57)$$

Moreover, the Ξ -variations of C^z, C^θ and of the 'H' commute with the corresponding s -variations, $[s, \delta_{\Xi}] = 0$.

If one decomposes the ghost vector field C according to $(C^z, C^z, C^\theta, C^\theta) = (\Gamma^z, \Gamma^z, \Gamma^\theta, \Gamma^\theta) \cdot M$, then the variations (3.56) of C^z and C^θ are equivalent to $\Gamma \cdot \partial$ transforming with the Lie bracket,

$$\delta_{\Xi}(\Gamma \cdot \partial) = [\Xi \cdot \partial, \Gamma \cdot \partial] \quad ,$$

$$\tilde{D} = \frac{1}{\sqrt{\Lambda H_\theta}} (D - H_\theta^z \partial) \quad (3.61)$$

$$\tilde{\bar{D}} = \bar{D} - \frac{H_\theta^\theta}{H_\theta^z} D - (H_\theta^z - \frac{H_\theta^\theta}{H_\theta^z} H_\theta^z) \partial,$$

where $(\tilde{D})^2 = \tilde{\partial}$ and $(\tilde{\bar{D}})^2 = \tilde{\bar{\partial}}$ as a consequence of the construction.

In terms of the tilde derivatives, the IFEQ's (3.22) read

$$\tilde{D}\Lambda = \left(\partial H_\theta^z - \frac{H_\theta^\theta}{H_\theta^z} \partial H_\theta^z \right) \Lambda \quad (3.62)$$

$$\tilde{\bar{D}}\Lambda = \left(\partial H_\theta^z - \frac{H_\theta^\theta}{H_\theta^z} \partial H_\theta^z \right) \Lambda.$$

The passage from the intermediate to the capital coordinates is described by matrices whose form is complementary to (3.59) :

$$M_2 Q_2 = \begin{pmatrix} 1 & k_z^z & 0 & k_z^\theta \\ 0 & 1 & 0 & 0 \\ 0 & k_\theta^z & 1 & k_\theta^\theta \\ 0 & k_\theta^z & 0 & k_\theta^\theta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \bar{L} & 0 & \bar{T} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{\bar{L}} \end{pmatrix}. \quad (3.63)$$

Here, $\bar{L}$, $\bar{T}$ and the 'k' are determined by the condition $MQ = (M_1 Q_1) \cdot (M_2 Q_2)$, e.g.

$$\bar{L} = \left[1 - H_\theta^z H_\theta^z - \frac{H_\theta^\theta}{H_\theta^z} (H_\theta^z - H_\theta^z H_\theta^z) \right] \bar{\Lambda}$$

$$k_\theta^z = \left[H_\theta^z - H_\theta^z H_\theta^z - \frac{H_\theta^\theta}{H_\theta^z} (H_\theta^z - H_\theta^z H_\theta^z) \right] \frac{\bar{\Lambda}}{\bar{L}}.$$

Again, $\bar{L}$, $\bar{T}$ and the 'k' are related to each other by (the complex conjugate of) eqs.(3.19)-(3.22). The form of $(M_2 Q_2)^{-1}$ is similar to $(M_1 Q_1)^{-1}$ and leads to the explicit expressions

$$\partial_z = \tilde{\partial} - \frac{k_z^\theta}{k_\theta^\theta} \tilde{D} - (k_z^z - \frac{k_z^\theta}{k_\theta^\theta} k_\theta^z) \tilde{\bar{\partial}}$$

$$\partial_2 = \frac{1}{\bar{L}} \left[\tilde{\partial} - \frac{\bar{T}}{\sqrt{\bar{L}}} (\tilde{D} - k_\theta^z \tilde{\bar{\partial}}) \right] \quad (3.64)$$

$$D_\theta = \tilde{D} - \frac{k_\theta^\theta}{k_\theta^z} \tilde{D} - (k_\theta^z - \frac{k_\theta^\theta}{k_\theta^z} k_\theta^z) \tilde{\bar{\partial}}$$

$$D_\theta = \frac{1}{\sqrt{\bar{L} k_\theta^\theta}} [\tilde{D} - k_\theta^z \tilde{\bar{\partial}}].$$

With this machinery at hand, it is extremely easy to discuss the ambiguity of the solutions of the basic IFEQ, i.e. the first of eqs.(3.62). The homogeneous equation reads $0 = \tilde{D} \ln \Lambda$ and implies $0 = \tilde{\partial} \ln \Lambda$. Combining these relations, we get

$$0 = (\tilde{D} - k_\theta^z \tilde{\bar{\partial}}) \ln \Lambda \propto D_\theta \ln \Lambda.$$

Henceforth, we can always multiply an arbitrary solution of (3.62) by a function of Z and Θ :

$$\Lambda' = e^{f(Z, \Theta)} \Lambda. \quad (3.65)$$

Comparison with (3.16) shows that this freedom just corresponds to a superconformal transformation. (The induced change of τ can also be seen to come out correctly by substituting (3.65) in the expression (3.21) for τ .)

3.6 Super bc-systems continued

Using the change of variables (3.51) for $C^{z,z}$ and $B_{\theta,z,z}$, the associated superconformal invariant (3.49) becomes

$$\int d^4 Z B_{\theta,z,z} D_\theta C^{z,z} = \int d^4 z \text{sdet} (M_1 Q_1 M_2 Q_2) \left[\Lambda^{-\frac{2q+1}{2}} B_{\theta,\dots} \right] \frac{1}{\sqrt{\bar{L} k_\theta^\theta}} (\tilde{D} - k_\theta^z \tilde{\bar{\partial}}) [\Lambda^q C^{z,z}]$$

and substitution of the IFEQ's (3.62) yields

$$\int d^4 z \frac{B_{\theta,z,z}}{H_\theta^\theta (k_\theta^\theta)^2} \left\{ (\tilde{D} - k_\theta^z \tilde{\bar{\partial}}) C^{z,z} + q C^{z,z} \left[(\partial H_\theta^z - \frac{H_\theta^\theta}{H_\theta^z} \partial H_\theta^z) - k_\theta^z (\partial H_\theta^z - \frac{H_\theta^\theta}{H_\theta^z} \partial H_\theta^z) \right] \right\}.$$

Since $\tilde{D}$, $\tilde{\bar{\partial}}$ and k_θ^z , k_θ^θ only depend on the Beltrami coefficients, the last expression represents a well-defined local action which is superdiffeomorphism-invariant by construction. For $q = 1$, this action for B_θ and C^z does not have much resemblance to a BRS invariant gauge-fixing functional associated to the action for the scalar superfield $\mathcal{X}$ (even when restricted to $H_\theta^z = 0$, in which case it takes the form

$$- \int d^4 z \frac{B_{\theta,z}}{(k_\theta^\theta)^2} \{ s H_\theta^z - k_\theta^z s H_\theta^z \}.$$

Other local functionals can be constructed by considering a more complicated change of variables for B . However, in this case, the superdiffeomorphism invariance of the final action is no longer automatically guaranteed, and it seems better to proceed the other way round, i.e. to start from a local and manifestly diffeomorphism invariant functional and work one's way up to an expression involving only capital coordinates. We will now present such a construction for a local action which is a linear combination of s -variations of Beltrami superfields. If restricted to the bosonic theory, our procedure provides an alternative derivation of the corresponding results for the gauge-fixing action and therefore it should also prove to be useful in this respect for the supersymmetric case.

Under a superdiffeomorphism $(z, \bar{z}, \theta, \bar{\theta}) \rightarrow (z', \bar{z}', \theta', \bar{\theta}')$, the basis 1-forms change as $(e^z, e^{\bar{z}}, e^\theta, e^{\bar{\theta}}) = (e^{z'}, e^{\bar{z}'}, e^{\theta'}, e^{\bar{\theta}'}) \cdot m$ where

$$m \equiv \begin{pmatrix} \partial z' + \theta' \partial \theta' & \partial \bar{z}' + \bar{\theta}' \partial \bar{\theta}' & \partial \theta' & \partial \bar{\theta}' \\ \bar{\partial} z' + \theta' \bar{\partial} \theta' & \bar{\partial} \bar{z}' + \bar{\theta}' \bar{\partial} \bar{\theta}' & \bar{\partial} \theta' & \bar{\partial} \bar{\theta}' \\ D z' - \theta' D \theta' & D \bar{z}' - \bar{\theta}' D \bar{\theta}' & D \theta' & D \bar{\theta}' \\ \bar{D} z' - \theta' \bar{D} \theta' & \bar{D} \bar{z}' - \bar{\theta}' \bar{D} \bar{\theta}' & \bar{D} \theta' & \bar{D} \bar{\theta}' \end{pmatrix}.$$

The columns of the matrix MQ transform like vectors under superdiffeomorphisms, e.g. for the first column,

$$V = m \cdot V' \quad \text{with} \quad V \equiv \begin{pmatrix} \Lambda H_z^z \\ \Lambda H_\theta^z \\ \Lambda H_\theta^z \end{pmatrix}.$$

Since the s -operator commutes with superdiffeomorphisms, sV also transforms like a vector:

$$sV = m \cdot (sV)'.$$

In order to construct an invariant action for sV , we introduce a density covector $B = (B_1, B_2, B_3, B_4)$ transforming as

$$B' = \frac{1}{\text{sdet } m} B \cdot m.$$

Then, it is clear that the functional

$$S \equiv \int d^4z B \cdot (sV) \quad (3.66)$$

is invariant under superdiffeomorphisms. It has the explicit form

$$S = \int d^4z \left\{ \Lambda [B_2(sH_z^z) + B_3(sH_\theta^z) + B_4(sH_\theta^z)] + \Lambda^{-1}(s\Lambda) [B \cdot V] \right\}. \quad (3.67)$$

Here, $B \cdot V$ is a scalar density and therefore the choice $B \cdot V = 0$ is consistent with supercoordinate invariance. This condition allows one to express B_1 in terms of the other 'B',

$$B_1 = - (B_2 H_z^z + B_3 H_\theta^z + B_4 H_\theta^z), \quad (3.68)$$

and to drop the last term in the action (3.67). Absorbing Λ in the definition of B , $\hat{B} \equiv \Lambda B$, the functional (3.67) becomes

$$S = \int d^4z \left\{ \hat{B}_2(sH_z^z) + \hat{B}_3(sH_\theta^z) + \hat{B}_4(sH_\theta^z) \right\}, \quad (3.69)$$

with ' $\hat{B}$ ' transforming as

$$\hat{B}' = \frac{\Lambda'}{\Lambda} \frac{1}{\text{sdet } m} \hat{B} \cdot m.$$

In the bosonic theory, B has only two components and (3.69) reads $S = \int d^2z \hat{B}_2(s\mu_z^z)$, the transformation law of $\hat{B}_2$ coinciding with the one of b_z , eq.(2.25).

Let us now express (3.69) in terms of the intermediate coordinates introduced in the last section and in terms of the capital coordinates. For this purpose, we consider the standard change of variables for C^z and C^θ , eq.(3.30), and redefine B by

$$\begin{aligned} \hat{B} &\equiv \frac{1}{\text{sdet}(M_1 Q_1)} B \cdot (M_1 Q_1) \\ &= (0, \hat{B}_2, \hat{B}_3, \hat{B}_4), \end{aligned}$$

where we used (3.68) to pass to the last line. Substituting the explicit form of the ' sH ' and the IFEQ's (3.62), the action (3.69) becomes

$$S = - \int d^4z \left\{ \hat{B}_2(\hat{\partial} C^z) + \hat{B}_3(\hat{D} C^z - 2C^\theta) + \hat{B}_4(\hat{D} C^z) \right\}. \quad (3.70)$$

The passage to the coordinates $(Z, \bar{Z}, \Theta, \bar{\Theta})$ is realized by a similar redefinition,

$$\begin{aligned} B &\equiv \frac{1}{\text{sdet}(MQ)} B \cdot (MQ) \\ &= (0, B_2, B_3, B_4), \end{aligned} \quad (3.71)$$

which implies

$$S = - \int d^4Z \left\{ B_2(\partial_Z C^z) + B_3(D_\Theta C^z - 2C^\theta) + B_4(D_\Theta C^z) \right\}. \quad (3.72)$$

In the bosonic case, eqs.(3.71) and (3.72) read $B_2 = \lambda^{-2} \hat{B}_2$ and $S = \int d^2Z B_2 \partial_Z C^z$, which coincides with our previous results.

Under a superconformal change of the capital coordinates, the B -fields transform as

$$\begin{aligned} B'_2 &= e^{3W} e^{-\bar{W}} B_2 \\ B'_3 &= e^{2W} e^{\bar{W}} B_3 \\ B'_4 &= e^{3W} [B_4 - B_2(D_\Theta \bar{W})] \end{aligned} \quad (3.73)$$

The actions (3.69)(3.70) and (3.72) admit an extra gauge invariance which is obvious in the latter two cases: by virtue of $(\bar{D})^2 = \bar{\partial}$ and $(D_\Theta)^2 = \partial_Z$, (3.70) and (3.72) are invariant under

$$\begin{aligned} \delta \bar{B}_2 &= \alpha, & \delta \bar{B}_3 &= 0, & \delta \bar{B}_4 &= -\bar{D}\alpha \\ \delta B_2 &= A, & \delta B_3 &= 0, & \delta B_4 &= -D_\Theta A. \end{aligned} \quad (3.74)$$

Here, α and A are odd superfields, since the ' B '-fields are paired with s -variations in the initial action. This "shift symmetry" allows the elimination of the B_2 -field from the theory.

Let us first consider the gauge where $B_2 = 0$. This condition is invariant under superdiffeomorphisms $(z, \bar{z}, \theta, \bar{\theta}) \rightarrow (z', \bar{z}', \theta', \bar{\theta}')$ and under the transformations (3.73). In this case, the remaining B -fields $B_3 \equiv B_{\theta z}$ and $B_4 \equiv B_{\theta \bar{z}}$ behave in the standard way under superconformal transformations. By virtue of (3.71), the condition $B_2 = 0$ is equivalent to

$$0 = B_2 + B_1 H_z^z + B_3 H_\theta^z + B_4 H_\theta^z$$

and substitution of (3.68) in this relation yields

$$B_2 = - \frac{1}{1 - H_z^z H_z^z} [B_3 (H_\theta^z - H_\theta^z H_z^z) + B_4 (H_\theta^z - H_\theta^z H_z^z)] \quad (3.75)$$

With this value of B_2 , the action (3.69) is still superdiffeomorphism invariant, but not BRS invariant. To summarize, the action (3.72) with $B_2 = 0$ and B_3, B_4 transforming in the standard way is equivalent to the local action (3.69) with B_2 depending on B_3, B_4 and the ' H ' according to eq.(3.75).

Alternatively, the B_2 -field in (3.69) may be eliminated by using its shift symmetry induced by (3.74). Although the vanishing of B_2 is invariant under superconformal changes of the small coordinates, it is not under superdiffeomorphisms and therefore any such transformation must be accompanied by a compensating shift transformation.

3.7 Component field results

Let us have a closer look at the field content of the basic superfields:

$$\begin{aligned} H_\theta^{\dot{\theta}} &= \sigma_\theta^{\dot{\theta}} + \theta v^{\dot{\theta}} + \bar{\theta} \mu_{\dot{\theta}} + \theta \bar{\theta} \alpha_{\dot{\theta}}^{\theta} \quad \text{and c.c.} \\ H_\theta^{\theta} &= \tau_\theta^{\theta} + \theta \varphi + \bar{\theta} w^{\theta} + \theta \bar{\theta} \psi_{\theta} \quad \text{and c.c.} \end{aligned} \quad (3.76)$$

The only physical fields are the Beltrami differentials $\mu_{\dot{\theta}}^{\dot{\theta}}$, $\mu_{\dot{\theta}}^{\theta}$ and their fermionic partners, the Beltraminos $\alpha_{\dot{\theta}}^{\theta}$, $\alpha_{\dot{\theta}}^{\dot{\theta}}$. Since the superfield $H_\theta^{\dot{\theta}}$ involves only non-physical fields, it is natural to restrict the geometry to the case where $H_\theta^{\dot{\theta}} = 0$ (and c.c.). As pointed out in section 3.1, this condition is superconformally invariant and so is the condition $C^\theta = 1/2 D_\theta C^z$ following from $sH_\theta^{\dot{\theta}} = 0$. In the following, we will consider a further restriction of the geometry to the so-called *Wess-Zumino (WZ) supergauge*,

$$H_\theta^{\dot{\theta}} = 0, \quad \sigma_\theta^{\dot{\theta}} = 0 = v^{\dot{\theta}} \quad \text{and c.c.} \quad (3.77)$$

and derive component field results for this case. The calculations are somewhat tedious, albeit completely straightforward and the results have a simple and physically transparent form.

A component field expression for the superspace actions can be deduced from the Berezinian integrals $\int d\theta d\bar{\theta} \mathcal{L}(z, \bar{z}, \theta, \bar{\theta}) = (D_\theta D_{\bar{\theta}} \mathcal{L})|$ by using the θ -expansions of the superfields. In the WZ-gauge and suppressing all indices, the invariant action (3.45) for

$$\mathcal{X} = X + \theta[i\lambda_\theta] + \bar{\theta}[-i\bar{\lambda}_{\bar{\theta}}] + \theta\bar{\theta}[iF]$$

becomes

$$\begin{aligned} -2i S_{\text{inv}}^{(WZ)} &= \int dz d\bar{z} \left\{ \frac{1}{1-\mu\bar{\mu}} [(\partial - \bar{\mu}\bar{\partial})X(\bar{\partial} - \mu\partial)X \right. \\ &\quad \left. + \alpha\lambda(\partial - \bar{\mu}\bar{\partial})X + \bar{\alpha}\bar{\lambda}(\bar{\partial} - \mu\partial)X + \frac{1}{2}(\alpha\lambda)(\bar{\alpha}\bar{\lambda}) \right] \\ &\quad \left. + [\lambda(\bar{\partial} - \mu\partial)\lambda + \bar{\lambda}(\partial - \bar{\mu}\bar{\partial})\bar{\lambda} + (1-\mu\bar{\mu})F^2] \right\} \quad (3.78) \end{aligned}$$

and the functional (B.1) takes the form³

$$-2i S_{\text{eff}}^{(WZ)} = \int dz d\bar{z} \left\{ b_{xx}(s\mu_x^z) + b_{\theta x}(s\alpha_\theta^{\dot{\theta}}) + \text{c.c.} \right\} \quad (3.79)$$

For the ghost superfields C^z and C^θ , the restriction to the WZ-gauge implies

$$\begin{aligned} 0 &= sH_\theta^{\dot{\theta}} = -(D_\theta C^z - 2C^\theta) \\ 0 &= sH_\theta^{\theta} = -(D_{\bar{\theta}} C^z) \\ 0 &= s(D_\theta H_\theta^{\dot{\theta}}) = -(D_\theta D_{\bar{\theta}} C^z) \end{aligned}$$

(In the last two lines, we used the rules

$$\begin{aligned} s(\mathcal{F}) &\equiv (s\mathcal{F}) \\ D(s\mathcal{F}) &\equiv s(D\mathcal{F}) \quad , \quad \bar{D}(s\mathcal{F}) \equiv s(\bar{D}\mathcal{F}) \end{aligned} \quad (3.80)$$

³Use the definitions (3.80) below.

which are assumed to hold for any superfield $\mathcal{F}$.) Therefore, the only independent symmetry parameters to be left in the WZ-gauge are those of ordinary diffeomorphisms and those of local supersymmetry transformations: from

$$|\Xi^z| \equiv \xi^z, \quad |\Xi^{\dot{z}}| \equiv \xi^{\dot{z}}, \quad |\Xi^\theta| \equiv \frac{i}{2}\xi^\theta, \quad |\Xi^{\bar{\theta}}| \equiv -\frac{i}{2}\xi^{\bar{\theta}}$$

and eq.(3.24), we conclude that

$$\begin{aligned} c^z &\equiv C^z = \xi^z + \xi^{\dot{z}}\mu_{\dot{z}}^z \\ \frac{i}{2}c^\theta &\equiv C^\theta = \frac{i}{2}(\xi^\theta + \xi^{\bar{\theta}}\alpha_{\bar{\theta}}^\theta) \end{aligned} \quad (3.81)$$

The variations of the matter fields now follow from (3.46) by applying the projection rules (3.80) to $\mathcal{F} = \mathcal{X}, D\mathcal{X}, \bar{D}\mathcal{X}, D\bar{D}\mathcal{X}$. The final result reads

$$\begin{aligned} sX &= (\xi \cdot \partial)X + \frac{1}{2}(\xi^\theta\lambda + \xi^{\bar{\theta}}\bar{\lambda}) \\ s\lambda &= (\xi \cdot \partial)\lambda + \frac{1}{2}(\partial\xi + \mu\bar{\partial}\bar{\xi})\lambda + \frac{1}{2}\xi^\theta D_z X - \frac{i}{2}\xi^{\bar{\theta}} F \\ s\bar{\lambda} &= (\xi \cdot \partial)\bar{\lambda} + \frac{1}{2}(\bar{\partial}\bar{\xi} + \bar{\mu}\partial\xi)\bar{\lambda} + \frac{1}{2}\xi^{\bar{\theta}} D_{\bar{z}} X + \frac{i}{2}\xi^\theta F \\ sF &= (\xi \cdot \partial)F + \frac{1}{2}(\partial\xi + \mu\bar{\partial}\bar{\xi})F + \frac{1}{2}(\bar{\partial}\bar{\xi} + \bar{\mu}\partial\xi)F - \frac{i}{2}[\xi^\theta D_z \bar{\lambda} - \xi^{\bar{\theta}} D_{\bar{z}} \lambda] \end{aligned} \quad (3.82)$$

where the ' D ' correspond to the so-called supercovariant derivatives of component field supergravity [11]:

$$\begin{aligned} D_z X &\equiv \frac{1}{1-\mu\bar{\mu}} \left[(\bar{\partial} - \mu\partial)X + \frac{1}{2}\mu\bar{\alpha}\bar{\lambda} - \frac{1}{2}\alpha\lambda \right] \\ D_z \lambda &\equiv \frac{1}{1-\mu\bar{\mu}} \left[(\bar{\partial} - \mu\partial - \frac{1}{2}\partial\mu)\lambda - \frac{1}{2}\alpha(D_z X) - \frac{i}{2}\mu\bar{\alpha}F \right] \end{aligned} \quad (3.83)$$

($D_z X$ and $D_z \bar{\lambda}$ are obtained by putting bars everywhere and replacing ' i ' by ' $-i$ '). The derivatives (3.83) behave covariantly under the s -operation,

$$\begin{aligned} s(D_z X) &= (\xi \cdot \partial)D_z X + (\bar{\partial}\bar{\xi} + \bar{\mu}\partial\xi)D_z X + \frac{1}{2}[\bar{\partial}\xi^\theta + (\bar{\partial}\bar{\xi})\bar{\alpha}]\bar{\lambda} + \frac{1}{2}[\xi^\theta D_z \bar{\lambda} + \xi^{\bar{\theta}} D_{\bar{z}} \bar{\lambda}] \\ s(D_z \lambda) &= (\xi \cdot \partial)D_z \lambda + (\bar{\partial}\bar{\xi} + \bar{\mu}\partial\xi)D_z \lambda + \frac{1}{2}(\partial\xi + \mu\bar{\partial}\bar{\xi})D_z \lambda \\ &\quad - \frac{i}{2}[\bar{\partial}\xi^\theta + (\bar{\partial}\bar{\xi})\bar{\alpha}]F - \frac{i}{2}\xi^\theta D_z F + \frac{1}{2}\xi^{\bar{\theta}} D_{\bar{z}} D_z X \end{aligned}$$

where

$$\begin{aligned} D_z \bar{\lambda} &\equiv \frac{1}{1-\mu\bar{\mu}} \left[(\bar{\partial} - \mu\partial - \frac{1}{2}\mu\bar{\partial}\bar{\mu})\bar{\lambda} + \frac{1}{2}\mu\bar{\alpha}(D_z X) - \frac{i}{2}\alpha F \right] \\ D_z F &\equiv \frac{1}{1-\mu\bar{\mu}} \left[(\bar{\partial} - \mu\partial - \frac{1}{2}\partial\mu + \frac{1}{2}\mu\bar{\partial}\bar{\mu})F + \frac{i}{2}\alpha(D_z \bar{\lambda}) + \frac{i}{2}\mu\bar{\alpha}(D_{\bar{z}} \lambda) \right] \\ D_z D_z X &\equiv \frac{1}{1-\mu\bar{\mu}} \left[(\bar{\partial} - \mu\partial - \partial\mu)D_z X - \frac{1}{2}\alpha(D_z \lambda) + \frac{1}{2}\mu\bar{\alpha}(D_{\bar{z}} \bar{\lambda}) - \frac{1}{2}(\partial\alpha)\lambda \right] \end{aligned} \quad (3.84)$$

and $[D_z, D_z]X = 0$. In eqs.(3.82), the signs in front of the supersymmetry transformations have been chosen in such a way that the s -operator is nilpotent with the usual component field conventions (see footnote to eq.(3.37)).

The bosonic contributions to the variations (3.82) and the derivatives (3.83) (3.84) identify $X, \lambda, \bar{\lambda}$ and F as space-time fields with weights $(0,0), (-\frac{1}{2}, 0), (0, -\frac{1}{2})$ and $(-\frac{1}{2}, -\frac{1}{2})$, respectively. From these equations, it should be clear how the derivative of a field of weight (r, l) looks like (see also reference [8]).

Application of (3.80) to $\mathcal{F} = \bar{D}H_\theta^z, D\bar{D}H_\theta^z, C^z, DC^z$ reproduces the s -variations (3.37) of μ, α and c, ϵ . The variations of c and ϵ are equivalent to

$$\begin{aligned} s\xi^z &= (\xi \cdot \partial)\xi^z - \frac{1}{4} \frac{1}{1 - \mu\bar{\mu}} (\xi^\theta \xi^\theta - \mu \xi^\theta \xi^\theta) \\ s\xi^\theta &= (\xi \cdot \partial)\xi^\theta - \frac{1}{2} \xi^\theta (\partial\xi + \mu \partial\bar{\xi}) + \frac{1}{4} \frac{1}{1 - \mu\bar{\mu}} (\xi^\theta \xi^\theta - \mu \xi^\theta \xi^\theta) \alpha, \end{aligned} \quad (3.85)$$

which appear to be the structure relations of a field-dependent Lie algebra. This field-dependence is a generic phenomenon for WZ-type gauges and cannot be removed by a change of generators (as explicitly shown for the case of super YM-theories in reference [27]).

A superspace solution of the consistency condition for the integrated anomaly, $s\mathcal{A} = 0$, is readily determined in the case where $H_\theta^z = 0$:

$$\mathcal{A}(H_\theta^z = 0) + \text{c.c.} = \int dz d\bar{z} d\theta d\bar{\theta} \left\{ C^z \partial_z^2 D_\theta H_\theta^z \right\} + \text{c.c.} \quad (3.86)$$

In the WZ-gauge, this becomes

$$\mathcal{A}^{(WZ)} + \text{c.c.} = \int dz d\bar{z} \left\{ c \partial^2 \mu - \epsilon \partial^2 \alpha \right\} + \text{c.c.}, \quad (3.87)$$

which shows that (3.86) represents the extension of the bosonic theory anomaly (A.15) to the (1,1) supersymmetric case with $H_\theta^z = 0$. The anomaly (3.86) reflects the non-conservation of the stress-energy supertensor

$$T_{z\theta} \equiv \frac{\delta S_q}{\delta \alpha_z^\theta} + \theta \frac{\delta S_q}{\delta \mu_{z^2}} + \bar{\theta} \frac{\delta S_q}{\delta v^z} + \theta \bar{\theta} \frac{\delta S_q}{\delta \sigma_\theta^z} \quad \text{and c.c.} \quad (3.88)$$

at the quantum level. In the WZ-gauge, we have the explicit expressions

$$\begin{aligned} \frac{\delta S_q^{(WZ)}}{\delta \mu} &= -(\mathcal{D}_z X)(\mathcal{D}_z X) - \lambda \partial \lambda + \bar{\mu} F^2 + [c \partial + 2 \partial c] b_{zz} - \frac{1}{2} [\epsilon \partial + 3 \partial \epsilon] b_{\theta z} \\ \frac{\delta S_q^{(WZ)}}{\delta \alpha} &= +\lambda (\mathcal{D}_z X) + [c \partial + \frac{3}{2} \partial c] b_{\theta z} - \frac{1}{2} b_{zz} \epsilon. \end{aligned} \quad (3.89)$$

Eqs.(3.78), (3.82) confirm the results previously obtained for the Beltrami parametrization of 2-dimensional supergravity by squeezing out all super-Weyl degrees of freedom, see references [7] and [8]. (The only discrepancy consists of the additional F -dependent term in our supercovariant derivative of $\lambda, \bar{\lambda}$, see eqs.(3.83); the presence of these terms ensures the covariance of the derivatives and the closure of the algebra.) It is quite remarkable

and satisfactory that the two rather different procedures lead to the same component field results⁴.

To summarize our space-time results, we have determined the complete quantum action, all transformation laws and an explicit expression for the anomaly. The expressions derived here can be used to discuss further aspects of the quantum theory like Ward identities and WZ-actions on the space-time manifold. However, in doing so, one should keep in mind that the projection to a minimal number of component fields has destroyed the general geometric framework and led to a field-dependent symmetry algebra.

3.8 Geometric interpretation

Before presenting a geometric interpretation of super Beltrami differentials along the lines of the bosonic theory, we would like to stress a striking mathematical novelty: we now have two independent Beltrami coefficients, $H_\theta^z, H_\theta^{\bar{z}}$ (and their c.c.), and the superconformal transformation law of one of them depends on the other, see eq.(3.18).

The space $\mathcal{SM}(M)$ of superconformal structures on the $(2+2)$ -dimensional supermanifold M is parametrized by the Beltrami coefficients $H_\theta^z, H_\theta^{\bar{z}}$ (and their c.c.). The quotient of this space by the group of those superdiffeomorphisms which are homotopic to the identity defines super-Teichmüller space $ST(M)$. For compact SRS's of genus $g > 1$, the space $ST(M)$ is a supermanifold which may be locally parametrized by $6g - 6$ even coordinates m^k and $4g - 4$ odd coordinates ζ^a [18]. Accordingly, the tangent vectors $\{\mu^{(k)z}, H_\theta^{(k)z}, H_\theta^{(k)\bar{z}}\}$ of the bosonic theory become super-tangent vectors with $6g - 6$ odd components $H_\theta^{(k)z}, H_\theta^{(k)\bar{z}}$ and $4g - 4$ even components $H_\theta^{(a)z}, H_\theta^{(a)\bar{z}}$. These vectors are given at the points $(H_\theta^z, H_\theta^{\bar{z}}, H_\theta^z, H_\theta^{\bar{z}})$ of $\mathcal{SM}(M)$ and locally depend on coordinates $(z, \bar{z}, \theta, \bar{\theta})$ of M . Their infinitesimal transformation laws are obtained by dropping the H -independent terms in the s -variations (3.29). The resulting s -variations are still nilpotent.

As a direct application of our discussions, we can define *quasi-superconformal* (q.s.c.) mappings by weakening the superconformal condition [28]. A smooth mapping from the SRS to itself,

$$\begin{aligned} f : M &\longrightarrow M \\ p &\longmapsto \hat{p}, \end{aligned}$$

is said to be q.s.c. on M if it is q.s.c. with respect to any local coordinate charts $(z, \bar{z}, \theta, \bar{\theta})$ and $(\hat{z}, \hat{\bar{z}}, \hat{\theta}, \hat{\bar{\theta}})$ around p and $\hat{p}$, respectively. This last condition means that the *super Beltrami equations*

$$\begin{aligned} [D_\theta \hat{z} - \hat{\theta}(\mathcal{D}_\theta \hat{\theta})] - H_\theta^z [\partial_z \hat{z} + \hat{\theta}(\partial_z \hat{\theta})] &= 0 \quad \text{and c.c.} \\ [D_\theta \hat{\bar{z}} - \hat{\bar{\theta}}(\mathcal{D}_\theta \hat{\bar{\theta}})] - H_\theta^{\bar{z}} [\partial_{\bar{z}} \hat{\bar{z}} + \hat{\bar{\theta}}(\partial_{\bar{z}} \hat{\bar{\theta}})] &= 0 \quad \text{and c.c.} \end{aligned} \quad (3.90)$$

are satisfied for any given smooth functions H_θ^z and $H_\theta^{\bar{z}}$ transforming as in (3.18). If we apply D_θ to the second equation and project it down to its lowest component, we see that it contains the ordinary Beltrami equation $(\bar{\partial} - \mu \partial)f = 0$ which defines a quasi-conformal

⁴For a projection method similar to ours in supergravity, see reference [14].

Chapter 4

Concluding Remarks

In the present work, we have studied two-dimensional superconformal structures without any reference to metrics or vielbeins. These structures are parametrized by super Beltrami differentials whose transformation laws exhibit holomorphic factorization in terms of a natural set of variables. To the extent that these questions have been addressed before¹, the results obtained here confirm those of the supergravity (metric) approach of references [8] [7]. Coincidence of our parametrization (3.39) with the supergravity expressions for the frame fields obtained by solving the torsion constraints on the covariant derivatives [13] [29] is expected to hold after appropriate changes of variables. In comparison to these supergravity approaches, our procedure appears to be more economical (and possibly more satisfactory as far as global aspects are concerned).

Further advantages with respect to other approaches considered in the literature include the following. In contrast to minimal component field approaches, the superspace formalism does not involve field-dependent symmetry algebras which are obscure from the mathematical point of view. (This problem of component field approaches is avoided by taking into account all components of the superfields and not just those remaining in a WZ-type gauge.) In our purely algebraic treatment of quantization and anomalies, no use was made of path integrals whose rigorous definition generally poses severe problems. The fact of eliminating the Weyl degrees of freedom from the beginning on and achieving the holomorphic factorization completely avoids the occurrence of the so-called holomorphic anomaly [18] which is *not* an intrinsic phenomenon of the theory (see section VII.A of ref. [18]).

In conclusion, we would like to discuss briefly the restricted geometry where $H_\theta^z = 0$. This condition is compatible with superconformal invariance, it considerably simplifies many equations and it seems natural from the point of view of physical applications (e.g. the condition $sH_\theta^z = 0$ allows one to eliminate the ghost field C^θ for which there does not appear to be a dynamical term in the gauge-fixing action, see Appendix B). Yet the crucial question is whether or not the induced restriction of the superdiffeomorphism group still represents an honest, i.e. field-independent group. If we look at the infinitesimal transformations, then the condition $\delta_\xi H_\theta^z = 0$ and its complex conjugate imply the following

¹I.e. the parametrization of $(e^z, \dots, e^\theta)$, the superspace symmetry algebra and most component field expressions

relations between the symmetry parameters:

$$\begin{aligned}\Xi^\theta &= \frac{1}{2} \left[D\Xi^\dagger + (D\Xi^\dagger)H_\theta^\dagger + (D\Xi^\theta)H_\theta^\dagger \right] \\ \bar{\Xi}^\theta &= \frac{1}{2} \left[\bar{D}\Xi^\dagger + (\bar{D}\Xi^\dagger)H_\theta^\dagger + (\bar{D}\Xi^\theta)H_\theta^\dagger \right] .\end{aligned}$$

Application of $1/2 H_\theta^\dagger D$ to the second equation and substitution of the result into the first one, yields a differential equation for Ξ^θ :

$$\left[1 - \frac{1}{4} H_\theta^\dagger D(H_\theta^\dagger \bar{D}) \right] \Xi^\theta = \frac{1}{2} [D\Xi^\dagger + (D\Xi^\dagger)H_\theta^\dagger] - \frac{1}{4} D \left[\bar{D}\Xi^\dagger + (\bar{D}\Xi^\dagger)H_\theta^\dagger \right] H_\theta^\dagger.$$

The differential operator on the l.h.s. is of second order and its inversion is non-trivial which makes it difficult to draw further conclusions at this point. However, a further study should be worthwhile and might provide some new insights into the global structure of SRS's and their moduli.

Although most of our discussions have been rather complete and general, we would like to postpone a further elaboration on the following issues that were only mentioned or touched upon: the general form of the gauge-fixing functional and of the anomaly of superdiffeomorphisms, the construction of the associated WZ-action and the study of Ward identities. (Some of these topics, like the higher genus WZ-action, still represent a subject of current investigation in the purely bosonic case [6].)

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Appendix A

Bosonic theory

This Appendix represents a complement to chapter 2. The main goal is to derive the quantum action for the bosonic string. This action is identified as a particular conformal model in the main text.

A.1 General framework of the metric approach

Consider a Riemannian 2-manifold (M, g) , i.e. a real, smooth 2-manifold M which is equipped with a positive-definite metric g . Using the complex notation $dz = dx + i dy$, $d\bar{z} = dx - i dy$, the line element associated to the metric can always be written as [1]

$$ds^2 = |\rho|^2 |dz + d\bar{z} \mu_z|^2 . \quad (\text{A.1})$$

Here, ρ and $\mu \equiv \mu_z$ are smooth complex-valued functions of $z, \bar{z}$ and the positive-definiteness of the metric is expressed by the condition $|\mu| < 1$. The function ρ is usually called the *conformal factor* and μ the *Beltrami coefficient* [1]. Eq.(A.1) is often referred to as the *Beltrami parametrization* of the metric [4].

If we decompose the metric with respect to orthonormal frame fields, $ds^2 = e^z e^{\bar{z}}$, we obtain a similar parametrization for the zweibeins:

$$\begin{aligned}e^z &= \rho (dz + d\bar{z} \mu) \\ e^{\bar{z}} &= \bar{\rho} (d\bar{z} + dz \bar{\mu}) .\end{aligned} \quad (\text{A.2})$$

Weyl transformations of the metric are simply implemented by a rescaling of ρ , i.e. infinitesimally

$$\delta_S \rho = -k \rho ,$$

where k is a complex-valued function of z and $\bar{z}$. (Properly speaking, $k = k_W + i k_L$, where the real-valued functions k_W and k_L parametrize Weyl rescalings and rotations, respectively.) Since the function μ is Weyl-inert, it parametrizes conformal classes of metrics.

To determine the transformation laws of ρ and μ under infinitesimal reparametrizations of the surface, we consider a small change of coordinates

$$\begin{aligned}z &\longrightarrow z + \xi(z, \bar{z}) \\ \bar{z} &\longrightarrow \bar{z} + \bar{\xi}(z, \bar{z}) .\end{aligned}$$

From $ds(z, \bar{z}) = \rho(z, \bar{z})[dz + d\bar{z}\mu(z, \bar{z})]$ and

$$\begin{aligned} ds(z + \xi, \bar{z} + \bar{\xi}) - ds(z, \bar{z}) &= \delta_\xi(ds) + \mathcal{O}(\xi^2) \\ &= (\delta_\xi \rho)[dz + d\bar{z}\mu] + \rho d\bar{z}(\delta_\xi \mu) + \mathcal{O}(\xi^2), \end{aligned}$$

one deduces that

$$\begin{aligned} \delta_\xi \mu &= (\bar{\partial} - \mu \partial)c + (\partial \mu)c \\ \delta_\xi \rho &= (\xi \partial + \bar{\xi} \bar{\partial})\rho + \rho(\partial \xi + \mu \bar{\partial} \bar{\xi}) \end{aligned} \quad (\text{A.3})$$

where $c \equiv \xi + \bar{\xi}\mu$ and $\partial \equiv \partial/\partial z$, $\bar{\partial} \equiv \partial/\partial \bar{z}$. Thus, the holomorphic factorization (see section 2.1) is realized for μ , but not for the conformal factor ρ . Altogether, this variable transforms as

$$\delta \rho = (\delta_\xi + \delta_S)\rho = (\xi \partial + \bar{\xi} \bar{\partial})\rho + \rho[\partial \xi + \mu \bar{\partial} \bar{\xi} - k].$$

A.2 BRS quantization of the bosonic string

The coupling of d real-valued scalar fields

$$\begin{aligned} \vec{X} : M &\longrightarrow \mathbf{R}^d \\ (z, \bar{z}) &\longmapsto \vec{X}(z, \bar{z}) = (X^1(z, \bar{z}), \dots, X^d(z, \bar{z})) \end{aligned}$$

to the metric (g_{mn}) is described by the functional

$$S_{\text{inv}}[\vec{X}, g] = \frac{1}{2} \sum_{m,n=1}^2 \int d^2 z \sqrt{g} g^{mn} (\partial_m \vec{X}) \cdot (\partial_n \vec{X}) \quad (\text{A.4})$$

$$= \frac{1}{2i} \int dz \wedge d\bar{z} \frac{1}{1 - \mu \bar{\mu}} (\bar{\partial} - \mu \partial) \vec{X} \cdot (\partial - \bar{\mu} \bar{\partial}) \vec{X}, \quad (\text{A.5})$$

where the dot denotes the (pseudo-) Euclidean scalar product in $\mathbf{R}^d$. Since the action does not depend on the conformal factor ρ , it is manifestly Weyl invariant. It is also invariant under diffeomorphisms (under which $\vec{X}$ transforms as $\delta_\xi \vec{X} = (\xi \partial + \bar{\xi} \bar{\partial}) \vec{X}$).

The gauge-fixing of the functional (A.4) can be realized by choosing a background metric

$$ds_0^2 = |\rho_0|^2 |dz + d\bar{z}\mu_0|^2.$$

and a Landau-type gauge [30]. The Weyl ghost k occurring in the BRS variation of the conformal factor ρ may be eliminated both from the quantum action and the BRS transformations without destroying the nilpotency of the latter [30]. In the following, we will only consider general coordinate transformations and the form (A.5) of the classical action.

To implement the gauge conditions $\mu = \mu_0$ and $\bar{\mu} = \bar{\mu}_0$ in our Lagrangian framework, we introduce auxiliary fields $d_{z\bar{z}}, d_{z\bar{z}}$ and the associated anti-ghosts $b_{z\bar{z}}, b_{z\bar{z}}$. The BRS

operation is then defined by turning the gauge parameters $\xi, \bar{\xi}$ of diffeomorphisms into ghost fields [2]–[4]:

$$\begin{aligned} s\mu &= (\bar{\partial} - \mu \partial)c + (\partial \mu)c \\ s\mu_0 &= 0 \\ sc &= c\partial c \\ sb &= -d \\ sd &= 0 \end{aligned} \quad (\text{A.6})$$

(analogously for the complex conjugate expressions) and

$$s\vec{X} = (\xi \cdot \partial) \vec{X} = \frac{1}{1 - \mu \bar{\mu}} [c(\partial - \bar{\mu} \bar{\partial}) + \bar{c}(\bar{\partial} - \mu \partial)] \vec{X}. \quad (\text{A.7})$$

In a Landau-type gauge, the gauge-fixing action reads

$$\begin{aligned} 2i S_{gf} &= -s \int dz \wedge d\bar{z} \{ b(\mu - \mu_0) + \text{c.c.} \} \\ &= \int dz \wedge d\bar{z} \{ d(\mu - \mu_0) + b(s\mu) + \text{c.c.} \}. \end{aligned} \quad (\text{A.8})$$

After eliminating the auxiliary field from the previous expression by its algebraic equation of motion, $\mu = \mu_0$, the complete quantum action takes the form

$$S_q|_{\mu=\mu_0} = S_{\text{inv}}[\vec{X}, \mu_0] + \frac{1}{2i} \int dz \wedge d\bar{z} \{ b[(\bar{\partial} - \mu_0 \partial) + (\partial \mu_0)]c + \text{c.c.} \}. \quad (\text{A.9})$$

Obviously, this elimination is not consistent with the s -variations (A.6). However, there is a natural modification of (A.6) which represents an invariance of (A.9):

$$\begin{aligned} \bar{s}\mu_0 &= (\bar{\partial} - \mu_0 \partial)c + (\partial \mu_0)c \\ \bar{s}c &= c\partial c \\ \bar{s}b &= 0 \end{aligned} \quad (\text{A.10})$$

and

$$\bar{s}\vec{X} = \frac{1}{1 - \mu_0 \bar{\mu}_0} [c(\partial - \bar{\mu}_0 \bar{\partial}) + \bar{c}(\bar{\partial} - \mu_0 \partial)] \vec{X}.$$

It should be noted that the BRS variations s and $\bar{s}$ only differ by equations of motion:

$$\begin{aligned} \bar{s}\mu_0 &= s\mu_0 + \left(\frac{\delta S_q}{\delta b} \right) \Big|_{\mu=\mu_0} \\ \bar{s}b &= sb - \frac{\delta S_q}{\delta \mu_0} \\ \bar{s}c &= sc \\ \bar{s}\vec{X} &= (s\vec{X})|_{\mu=\mu_0}. \end{aligned}$$

To simplify the notation in the remainder of this section, we simply write s for $\bar{s}$ and μ for μ_0 , while keeping in mind that μ now represents a fixed external field. Let us summarize the equations thus obtained for reference:

$$S_q = S_{\text{inv}}[\vec{X}, \mu] + S_{gf}[\mu, b, c] \quad (\text{A.11})$$

with

$$S_{gf}[\mu, b, c] = \frac{1}{2i} \int dz \wedge d\bar{z} \{ b(s\mu) + \text{c.c.} \} \quad (\text{A.12})$$

and

$$\begin{aligned} s\mu &= (\bar{\partial} - \mu\partial + \partial\mu)c \\ sc &= c\partial c \\ sb &= 0 \\ s\bar{X} &= \frac{1}{1-\mu\bar{\mu}} [c(\partial - \bar{\mu}\bar{\partial}) + \bar{c}(\bar{\partial} - \mu\partial)] \bar{X} \end{aligned} \quad (\text{A.13})$$

The classical stress-energy tensor associated to the functional (A.11) is obtained by differentiation with respect to the external field μ :

$$T_{zz} \equiv \frac{\delta S_q}{\delta \mu_z} = \frac{-1}{(1-\mu\bar{\mu})^2} (\partial - \bar{\mu}\bar{\partial}) \bar{X} \cdot (\partial - \bar{\mu}\bar{\partial}) \bar{X} + [c\partial + 2\partial c]b \quad (\text{A.14})$$

The quantum theory associated to (A.11)-(A.13) has been discussed in references [2]-[4] and here we only remark that the BRS-invariance of the functional S_q is spoiled at the quantum level by a diffeomorphism anomaly :

$$A + \bar{A} = \int dz \wedge d\bar{z} \{ c\partial^3\mu + \bar{c}\bar{\partial}^3\bar{\mu} \} \quad (\text{A.15})$$

Both of its contributions separately satisfy the WZ consistency condition: $sA = 0 = s\bar{A}$.

In conclusion, we would like to mention some problems arising in the global formulation of the quantum theory. First, the expression (A.15) does not have a well-defined global meaning as a solution of $sA = 0 = s\bar{A}$. This problem can be solved by the introduction of a projective connection [5]. Another issue is that of global zero modes which manifests itself as follows. By partial integration, the gauge-fixing functional (A.12) becomes

$$S_{gf} = \frac{1}{2i} \int dz \wedge d\bar{z} \{ c[(\bar{\partial} - \mu\partial) - 2(\partial\mu)]b + \text{c.c.} \} \quad (\text{A.16})$$

In general, the kernels of the differential operators acting on c and b in (A.12) and (A.16) are non-trivial and require a further gauge-fixing. These so-called global zero modes can be taken care of in a geometric way [21] [17], but, so far, a complete and satisfactory solution for the zero modes associated to c^* is still lacking.

Appendix B

(1,1) Supersymmetric Theory

B.1 Metric approach

In references [13] [29] [8], a superspace generalization for the Beltrami parametrization of the zweibein has been found for (1,1) supersymmetry. This generalization amounts to expressing the supervielbein forms in terms of superconformal factors and fields which parametrize superconformal classes of frames. The choice of variables in reference [8] corresponds to the parametrization of superconformal structures which we found by different considerations in section 3.1. For any choice of variables, the superconformal invariance of the matter field action in supergravity is reflected by the fact that it does not depend on the conformal factors. In analogy to section A.2, we will now study the quantization of this action which is explicitly given by eq.(3.45).

B.2 Gauge-fixing for the superstring

To perform the gauge-fixing for the action of $\mathcal{X}$, we consider background fields $H^{(0)}$ and a Landau-type gauge as in the bosonic theory. To implement the gauge conditions

$$H_\theta^z = H_\theta^{(0)z} \quad , \quad H_\theta^{\bar{z}} = H_\theta^{(0)\bar{z}} \quad \text{and c.c.} \quad ,$$

we introduce auxiliary fields D_θ, D_{θ^*} and the associated anti-ghosts B_θ, B_{θ^*} and proceed as in eqs.(A.6) to (A.13). After eliminating the auxiliary fields and dropping the 'zeros', one is left with the action

$$-2i S_{gf} = \int dz d\bar{z} d\theta d\bar{\theta} \{ B_\theta(sH_\theta^z) + B_{\theta^*}(sH_\theta^{\bar{z}}) + \text{c.c.} \} \quad , \quad (\text{B.1})$$

where $sB_\theta = 0 = sB_{\theta^*}$. To verify that this expression represents an admissible gauge-fixing (i.e. defines a non-degenerate action in space-time), we set $H_\theta^z = 0 = H_\theta^{\bar{z}}$ and expand the remaining superfields in terms of components, e.g.

$$\begin{aligned} B_\theta &= b_0 + \theta b_1 + \bar{\theta} b_2 + \theta\bar{\theta} b_3 \\ C^z &= c_0 + \theta c_1 + \bar{\theta} c_2 + \theta\bar{\theta} c_3 \end{aligned}$$

and similarly for B_{θ} and C^{θ} . Then it is not difficult to check that the resulting space-time functional is non-degenerate and that it reduces to

$$\int dz d\bar{z} \left\{ b_1 \bar{\partial} c_0 - b_0 \bar{\partial} c_1 + \text{c.c.} \right\}$$

by the use of algebraic equations of motion. This last expression coincides with the conformal gauge action which was previously studied in the literature (see [25] and the first of references [9]). This shows that our expression (B.1) represents a local version of this type of action. Moreover, by projecting down to space-time fields (see section 3.7), we see that (B.1) contains the gauge-fixing functional (A.12) of the bosonic theory.

Despite all these nice features, the action (B.1) has the unpleasant property that its BRS invariance is not compatible with superconformal invariance: if we require (B.1) to be superconformally invariant, then the transformation laws of the ' H ' and ' C ', eqs.(3.18)(3.31), imply those of the ' B ', i.e. explicitly

$$\begin{aligned} B_{\theta'} &= e^{zw} Y B_{\theta} \\ B_{\bar{\theta}'} &= e^{\bar{z}\bar{w}} \left[B_{\bar{\theta}} + B_{\theta} Y^{-1} (D_{\theta} w) \right] \end{aligned} \quad (\text{B.2})$$

Here, the factor $e^w \equiv (D_{\theta} \theta')^{-1}$ is s -inert, but $Y \equiv 1 + (D_{\theta} w) H_{\theta}$ is not and therefore the transformed B -fields are no longer BRS invariant: s commutes with the superconformal transformations of B only up to the equation of motion of B_{θ} , i.e. $s H_{\theta}^z = 0$. (In the bosonic theory, we encountered a similar phenomenon, but only at the level of diffeomorphisms. Here, the situation appears to be worse, since the superconformal transformations are needed for patching together the local expressions on the SRS.) The obvious remedy consists of restricting the geometry by $H_{\theta}^z = 0$ and $s H_{\theta}^z = 0$. This last condition eliminates the ghost field C^{θ} for which there is no dynamical term in the action (B.1). The admissibility of this procedure is discussed in chapter 4.

Appendix C

Super Riemann surface approach to (1,0) supersymmetry

(1,0) superspace is locally parametrized by $(z, \bar{z}, \theta)$ where θ denotes an anticommuting variable. (1,0) supergravity and its quantization have been studied in the last two references of [13] and in [14] [15] [18] [31], but so far a formulation with manifest holomorphic factorization has not been given. Therefore, we now summarize such a formulation for (1,0) SRS's along the lines of chapter 3.

C.1 General framework

In the (1,0) case, a change of coordinates $(Z, \bar{Z}, \Theta) \rightarrow (Z', \bar{Z}', \Theta')$ is superconformal, if it satisfies (i) $Z' = Z'(Z, \Theta)$, $\bar{Z}' = \bar{Z}'(\bar{Z})$, $\Theta' = \Theta'(Z, \Theta)$ and (ii) $D_{\Theta} Z' = \Theta'(D_{\Theta} \Theta')$. Under such a coordinate transformation, the canonical tangent space vectors $\partial_Z, \partial_{\bar{Z}}, D_{\Theta}$ change according to (3.4) and $\partial'_Z = (\partial_Z \bar{Z}) \partial_{\bar{Z}}$.

The structure relations for the cotangent space basis

$$\begin{aligned} e^Z &= dZ + \Theta d\Theta \\ e^{\bar{Z}} &= d\bar{Z} \\ e^{\Theta} &= d\Theta \end{aligned} \quad (\text{C.1})$$

now take the form

$$\begin{aligned} 0 &= de^Z + e^{\Theta} e^{\Theta} \\ 0 &= de^{\bar{Z}} \\ 0 &= de^{\Theta} \end{aligned} \quad (\text{C.2})$$

Proceeding as in section 3.1, we find

$$(e^Z, e^{\bar{Z}}, e^{\Theta}) = (e^z, e^{\bar{z}}, e^{\theta}) \cdot M \cdot Q \quad (\text{C.3})$$

with

$$M = \begin{pmatrix} 1 & H_z & 0 \\ H_z^z & 1 & H_z^{\theta} \\ H_{\theta}^z & H_{\theta}^{\bar{z}} & H_{\theta}^{\theta} \end{pmatrix}, \quad Q = \begin{pmatrix} \Lambda & 0 & \tau \\ 0 & \Omega & 0 \\ 0 & 0 & \sqrt{\Lambda} \end{pmatrix} \quad (\text{C.4})$$

Here, Λ, τ and the 'H' locally depend on the derivatives of $Z, \dots, \bar{\Theta}$ as before while $\Omega, \bar{Z} = \bar{\partial}Z$.

In the present case, the structure relations (C.2) contain as independent equations

$$\begin{aligned}
0 &= \bar{\partial}\Lambda - \partial(H_z^\tau \Lambda) - 2H_z^\theta \tau \sqrt{\Lambda} \\
0 &= D_\theta \Omega - \bar{\partial}(H_\theta^\tau \Omega) \\
0 &= D_\theta(H_\theta^\tau \Omega) - H_z^\tau \Omega \\
0 &= D(H_\theta^\tau \Lambda) - \Lambda + 2H_\theta^\tau \tau H_\theta^\theta \sqrt{\Lambda} + (H_\theta^\theta)^2 \Lambda \\
0 &= D(H_z^\tau \Lambda) - \bar{\partial}(H_\theta^\theta \Lambda) - 2H_z^\tau \tau H_\theta^\theta \sqrt{\Lambda} - 2H_\theta^\tau H_z^\theta \sqrt{\Lambda} - 2H_\theta^\theta H_z^\theta \Lambda \\
0 &= D_\theta(H_\theta^\tau \tau) + D_\theta(H_\theta^\theta \sqrt{\Lambda}) - \tau.
\end{aligned} \tag{C.5}$$

The last four equations can be used to express $H_z^\tau, H_\theta^\theta, H_z^\theta$ and τ in terms of the Beltrami differentials $H_z^\tau, H_\theta^\tau, H_\theta^\tau$ and the integrating factor Λ , the latter occurring only in τ . Explicitly, one finds

$$\begin{aligned}
H_z^\tau &= (D_\theta - H_\theta^\tau \bar{\partial}) H_\theta^\tau \\
(H_\theta^\theta)^2 &= 1 - (D_\theta - H_\theta^\tau \bar{\partial}) H_\theta^\tau \\
H_z^\theta &= \frac{1}{2H_\theta^\theta} \left[(D_\theta - H_\theta^\tau \bar{\partial}) H_z^\tau - (\bar{\partial} - H_z^\tau \partial) H_\theta^\tau \right] \\
\tau &= \frac{1}{(H_\theta^\theta)^2} (D_\theta - H_\theta^\tau \bar{\partial})(H_\theta^\theta \sqrt{\Lambda}).
\end{aligned} \tag{C.6}$$

Thus, the (1,0) superconformal structures are parametrized by one even and two odd Beltrami superfields. This is in agreement with the results of the metric approach [14] where H_z^τ and H_θ^θ correspond to the prepotentials H_{--++} and H_{+-} while H_θ^τ corresponds to the supersymmetry compensator H^+ .

The differential equations for the independent integrating factors Λ and Ω read

$$\begin{aligned}
0 &= \bar{\partial}\Lambda - \left[\frac{H_z^\tau}{H_\theta^\theta} \right] D\Lambda - \left[H_z^\tau - \frac{H_z^\theta H_\theta^\tau}{H_\theta^\theta} \right] \partial\Lambda - \left[\partial H_z^\tau - \frac{H_z^\theta \partial H_\theta^\tau}{H_\theta^\theta} \right] \Lambda \\
0 &= \left[D_\theta - H_\theta^\tau \bar{\partial} - \bar{\partial} H_\theta^\tau \right] \Omega.
\end{aligned} \tag{C.7}$$

They contain the bosonic theory equations for $\lambda \equiv \Lambda$ and $\bar{\lambda} \equiv \Omega$.

The transformation laws of all fields under an infinitesimal change of coordinates generated by the vector field $\Xi(z, \bar{z}, \theta) \cdot \partial = \Xi^z \partial_z + \Xi^{\bar{z}} \partial_{\bar{z}} + \Xi^\theta D_\theta$ are determined as in the (1,1) theory and are best expressed in terms of the C -variables defined by

$$\begin{aligned}
i_{\Xi \partial} (e^z) &= \left[\Xi^z + \Xi^{\bar{z}} H_z^\tau + \Xi^\theta H_\theta^\tau \right] \Lambda_z^z \equiv C^z \Lambda_z^z \\
i_{\Xi \partial} (e^{\bar{z}}) &= \left[\Xi^{\bar{z}} + \Xi^z H_z^\tau + \Xi^\theta H_\theta^\tau \right] \Omega_z^{\bar{z}} \equiv C^{\bar{z}} \Omega_z^{\bar{z}} \\
i_{\Xi \partial} (e^\theta) &= \left[\Xi^\theta H_\theta^\theta + \Xi^z H_z^\theta \right] \sqrt{\Lambda_z^z} + \left[\Xi^{\bar{z}} + \Xi^z H_z^\tau + \Xi^\theta H_\theta^\tau \right] \tau_z^\theta \\
&\equiv C^\theta \sqrt{\Lambda_z^z} + C^{\bar{z}} \tau_z^\theta.
\end{aligned} \tag{C.8}$$

In summary,

$$\begin{aligned}
sC^z &= C^z \partial C^z + C^\theta C^\theta \\
sC^{\bar{z}} &= C^{\bar{z}} \bar{\partial} C^{\bar{z}} \\
sC^\theta &= C^z \partial C^\theta + \frac{1}{2} C^\theta (\partial C^z)
\end{aligned} \tag{C.9}$$

and

$$\begin{aligned}
sH_\theta^\tau &= - \left[(D_\theta - H_\theta^\tau \bar{\partial}) C^z + (\partial H_\theta^\tau) C^z - 2H_\theta^\theta C^\theta \right] \\
sH_z^\tau &= - \left[(\bar{\partial} - H_z^\tau \partial) C^z + (\partial H_z^\tau) C^z - 2H_z^\theta C^\theta \right] \\
sH_\theta^\tau &= - \left[(D_\theta - H_\theta^\tau \bar{\partial}) C^z + (\bar{\partial} H_\theta^\tau) C^z \right] \\
sH_z^\tau &= - \left[(\partial - H_z^\tau \bar{\partial}) C^z + (\partial H_z^\tau) C^z \right] \\
sH_\theta^\theta &= - \left[(D_\theta - H_\theta^\tau \bar{\partial}) C^\theta + \frac{1}{2} (\partial H_\theta^\tau) C^\theta + C^z (\partial H_\theta^\theta) - \frac{1}{2} H_\theta^\theta (\partial C^z) \right] \\
sH_z^\theta &= - \left[(\bar{\partial} - H_z^\tau \partial) C^\theta + \frac{1}{2} (\partial H_z^\tau) C^\theta + C^z (\partial H_z^\theta) - \frac{1}{2} H_z^\theta (\partial C^z) \right].
\end{aligned} \tag{C.10}$$

The geometric interpretation of super Beltrami differentials given for the (1,1) theory also applies to the present case. The only difference consists of the fact that the odd dimension of the genus $g > 1$ super-Teichmüller space is now equal to $2g - 2$.

C.2 Scalar superfields

From the previous expressions, it follows that

$$\begin{pmatrix} \partial_z \\ \partial_{\bar{z}} \\ D_\theta \end{pmatrix} = Q^{-1} \cdot M^{-1} \cdot \begin{pmatrix} \partial \\ \bar{\partial} \\ D_\theta \end{pmatrix}$$

with

$$Q^{-1} = \begin{pmatrix} \Lambda^{-1} & 0 & -(\tau \Lambda^{-3/2}) \\ 0 & \Omega^{-1} & 0 \\ 0 & 0 & \Lambda^{-1/2} \end{pmatrix}.$$

To determine M^{-1} , we decompose M in analogy to (3.42),

$$M = \begin{pmatrix} 1 & 0 & 0 \\ H_z^\tau & 1 & 0 \\ H_\theta^\tau & h_\theta^\tau & 1 \end{pmatrix} \begin{pmatrix} 1 & H_z^\tau & 0 \\ 0 & \Delta & H_z^\theta \\ 0 & 0 & G \end{pmatrix},$$

the form of h_θ^τ, Δ and G following by comparison with (C.4) :

$$\begin{aligned}
h_\theta^\tau &= \frac{1}{\Delta} [H_\theta^\tau - H_\theta^\tau H_z^\tau] \\
\Delta &= 1 - H_z^\tau H_z^\tau, \quad G = H_\theta^\theta - h_\theta^\tau H_z^\theta.
\end{aligned}$$

Now, one easily finds $\text{sdet } M = \Delta/G$ and

$$M^{-1} = \begin{pmatrix} 1 & -\Delta H_z^\sharp & -(\Delta G)^{-1} H_z^\sharp H_z^\theta \\ 0 & \Delta^{-1} & -(\Delta G)^{-1} H_z^\theta \\ 0 & 0 & G^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -H_z^\sharp & 1 & 0 \\ -h_\theta^\sharp & -h_\theta^\sharp & 1 \end{pmatrix}, \quad (C.11)$$

with $h_\theta^\sharp = H_\theta^\sharp - h_\theta^\sharp H_z^\sharp = 1/\Delta [H_\theta^\sharp - H_\theta^\sharp H_z^\sharp]$. Henceforth,

$$\begin{aligned} D_\Theta &= \frac{1}{\sqrt{\Lambda G}} [D - h_\theta^\sharp \partial - h_\theta^\sharp \bar{\partial}] \\ \partial_{\bar{z}} &= \frac{1}{\Omega \Delta} \left[(\bar{\partial} - H_z^\sharp \partial) - \frac{1}{G} H_z^\theta (D - h_\theta^\sharp \partial - h_\theta^\sharp \bar{\partial}) \right] \end{aligned} \quad (C.12)$$

and substitution of these expressions into the action for the scalar superfield $\mathcal{X}$ yields

$$\begin{aligned} 2i S_{\text{inv}} &= \int dZ d\bar{Z} d\Theta (D_\Theta \mathcal{X}) (\partial_{\bar{z}} \mathcal{X}) \\ &= \int dz d\bar{z} d\theta \left[\frac{\partial(Z, \bar{Z}, \Theta)}{\partial(z, \bar{z}, \theta)} \right] (D_\Theta \mathcal{X}) (\partial_{\bar{z}} \mathcal{X}) \\ &= \int dz d\bar{z} d\theta \frac{1}{G^2} (D - h_\theta^\sharp \partial - h_\theta^\sharp \bar{\partial}) \mathcal{X} \cdot (\bar{\partial} - H_z^\sharp \partial) \mathcal{X}. \end{aligned} \quad (C.13)$$

This expression does not depend on the integrating factors, nor does the variation of the matter superfield $\mathcal{X}$:

$$s\mathcal{X} = \begin{pmatrix} \Xi^\sharp & \Xi^\sharp, \Xi^\theta \end{pmatrix} \cdot \begin{pmatrix} \partial \\ \bar{\partial} \\ D_\theta \end{pmatrix} \mathcal{X} = (C^\sharp, C^\sharp, C^\theta) \cdot M^{-1} \cdot \begin{pmatrix} \partial \\ \bar{\partial} \\ D_\theta \end{pmatrix} \mathcal{X}. \quad (C.14)$$

The gauge-fixing of S_{inv} proceeds along the lines of the (1,1)-theory and leads to the action

$$-2i S_{gf} = \int dz d\bar{z} d\theta \{ B_{zz} (sH_\theta^\sharp) + B_{z\bar{z}} (sH_\theta^\theta) + B_{\theta z} (sH_z^\sharp) \}, \quad (C.15)$$

the problems with superconformal invariance being the same as in the (1,1) case.

C.3 Superconformal fields and super bc-systems

In the (Z, Θ) -sector, superconformal tensors are defined as in the (1,1) theory, the associated invariant action being given by

$$\int dZ d\bar{Z} d\Theta B_{\Theta Z} \partial_{\bar{z}} C^{z, \bar{z}}. \quad (C.16)$$

In the $\bar{Z}$ -sector, superconformal fields do not come in pairs: the tensors $B_{\bar{z} \dots \bar{z}}$ and $C^{\bar{z} \dots \bar{z}}$ with q indices transform as

$$B_{\bar{z} \dots \bar{z}}(\mathcal{Z}') = B_{\bar{z} \dots \bar{z}}(\mathcal{Z}) (\partial_{\bar{z}} \bar{Z}')^{-q}, \quad C^{\bar{z} \dots \bar{z}}(\mathcal{Z}') = C^{\bar{z} \dots \bar{z}}(\mathcal{Z}) (\partial_{\bar{z}} \bar{Z}')^q \quad (C.17)$$

and admit the invariant action

$$\int dZ d\bar{Z} d\Theta B_{\bar{z} \dots \bar{z}} D_\Theta C^{z, \bar{z}}. \quad (C.18)$$

In the (Z, Θ) -sector, the passage to the small coordinates is realized as before and in the other sector, we simply have the rescalings

$$C^{z, \dots, \bar{z}} = C^{z, \dots, \bar{z}} (\Omega_z^2)^q, \quad B_{z, \dots, \bar{z}} = (\Omega_z^2)^q B_{z, \dots, \bar{z}}.$$

C.4 Component field results

The basic superfields have the form

$$\begin{aligned} H_z^\sharp &= \mu_z^\sharp + \theta [i \alpha_z^\theta] \\ H_\theta^\sharp &= \rho_\theta^\sharp + \theta \mu_z^\sharp \\ H_\theta^{\bar{z}} &= \tau_\theta^{\bar{z}} + \theta A \end{aligned} \quad (C.19)$$

and, in the WZ-gauge, the only variables to survive are the physical fields $\mu_z^\sharp, \mu_z^{\bar{z}}$ and α_z^θ . The restriction to the WZ-gauge implies $C^\theta = 1/2 D_\theta C^\sharp$ and $0 = sH_\theta^{\bar{z}} = -(D_\theta C^{\bar{z}})$. The matter multiplet is given by $\mathcal{X} = X + \theta [i \lambda_\theta]$ and the component field expressions for the quantum action and the symmetry transformations are derived as for the (1,1) theory. The results for the (1,0) theory coincide with the truncation $0 = \bar{\alpha} = \bar{\lambda} = \bar{F} = \xi^\theta$ of eqs.(3.37), (3.78)-(3.85) and (3.89).

For $H_\theta^{\bar{z}} = 0$, a superspace expression for the anomaly is given by

$$\mathcal{A}(H_\theta^{\bar{z}} = 0) + \bar{\mathcal{A}} = \int dz d\bar{z} d\theta \{ C^\sharp \partial_{\bar{z}}^2 D_\theta H_z^\sharp \} + \int dz d\bar{z} d\theta \{ C^{\bar{z}} \partial_{\bar{z}}^2 H_\theta^{\bar{z}} \}. \quad (C.20)$$

The second term is s -invariant in the general case, but the first only is if $H_\theta^{\bar{z}} = 0$. The anomaly (C.19) reflects the quantum non-conservation of the stress-energy superoperators

$$T_{z\theta} \equiv \frac{\delta S_q}{\delta \alpha_z^\theta} + \theta \frac{\delta S_q}{\delta \mu_z^\sharp}, \quad T_{z\bar{z}} \equiv \frac{\delta S_q}{\delta \mu_z^{\bar{z}}} + \theta \frac{\delta S_q}{\delta \rho_\theta^{\bar{z}}}, \quad (C.21)$$

and, in the WZ-gauge, it reduces to

$$\mathcal{A}^{(WZ)} + \bar{\mathcal{A}}^{(WZ)} = \int dz d\bar{z} \{ c \partial^3 \mu - \epsilon \partial^2 \alpha \} + \int dz d\bar{z} \{ \bar{c} \partial^3 \bar{\mu} \}.$$

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