CERN-TH.5743/90
CTP TAMU-48/90
IC/90/100
USC-90/HEP09

W_∞ gravity and super- W_∞ gravity

E. Bergshoeff,[†] C.N. Pope,^{*§} L.J. Romans,[‡] E. Sezgin^{*} and X. Shen^{*}

ABSTRACT

We construct gauge theories of the W_∞ algebra and its super-extension, and discuss their relation to earlier results for the gauging of the “classical” contraction to the w_∞ algebra.

[†] Theory Division, CERN, CH-1211, Geneva 23, Switzerland.

^{*} Center for Theoretical Physics, Texas A&M University, College Station, TX 77843-4242.

[§] Supported in part by the U.S. Department of Energy, under grant DE-AS05-81ER40039.

[‡] Department of Physics, University of Southern California, Los Angeles, CA 90089-0484. Supported in part by the U.S. Department of Energy, under grant DE-FG03-84ER40168.

CERN-TH.5743/90
CTP TAMU-48/90
IC/90/100
USC-90/HEP09
June 1990

1. Introduction

W_N algebras are extensions of Virasoro algebras which contain generators with all integer conformal spins $2 \leq s \leq N$ [1]. Lagrangian field theories possessing W_N symmetry may be relevant to multimatrix models [2] which, in turn, are relevant for the non-perturbative analysis of string theory [3]. Furthermore, W_N -symmetric field theories may provide a stringy description of consistent interacting target-space higher-spin field theories [4].

A particular $N \rightarrow \infty$ limit of W_N , denoted by w_∞ , was described in [5], and its supersymmetric extensions were obtained in [6, 7]. These algebras are all linear. However, only the (super-) Virasoro subsector of the (super-) w_∞ algebra admits a central extension. More recently, two new linear algebras, called W_{∞} [8, 9] and $W_{1+\infty}$ [10], have been constructed with generators having all integer conformal spins $s \geq 2$ and $s \geq 1$, respectively. Both of these algebras do admit central terms for generators of all conformal spins. A super- W_∞ algebra whose bosonic sector is $W_\infty \oplus W_{1+\infty}$ has been given in [11]. This algebra has $N = 2$ supersymmetry, and again has central terms for generators of all conformal spins.

A field-theoretic realisation of the W_3 algebra was recently given in [12], where a free action realising a chiral W_3 symmetry was Noether-coupled to spin-2 and spin-3 background gauge fields. This work was extended in [13] to a non-chiral gauged theory of W_3 , in which both the left-handed and right-handed W_3 symmetries of the original action are made local. A crucial ingredient in this construction was the introduction of a “nested” covariant derivative, defined via auxiliary fields in the first-order formalism. An extension of this idea played a central rôle in the construction of w_∞ gravity in [14], where the w_∞ symmetry was realised non-linearly on a set of scalar fields. This realisation was given a geometrical interpretation in terms of the coset space $w_{1+\infty}/w_\infty$. By making use of compensating Stueckelberg symmetries, the w_∞ gauging could be truncated to W_N for any finite N .

By means of a coset construction, the W_∞ or $W_{1+\infty}$ symmetry can also be realised non-linearly on scalars [14]; however, no Lagrangian has been found that is invariant under these symmetries. In the case of w_∞ , it seems that the non-linearity of the realisation is essential in order to obtain a non-trivial Lagrangian. For W_∞ , on the other hand, it turns out that one can obtain a non-trivial theory with a linear realisation, although at the expense of having higher-derivative terms. The transformation rules in fact comprise precisely the linear terms in the non-linear realisation given in [14].

In this paper we first construct a Lagrangian field theory with a local chiral W_∞ symmetry. As a first step towards a covariant formulation of the theory we then give the non-chiral version, in which both the left-handed and right-handed W_∞ symmetries are gauged. Our method is rather universal, and should be applicable to any (infinite-dimensional) Lie algebra. We then extend our results to the $N = 2$ super- W_∞ algebra, and construct a non-chiral supersymmetric Lagrangian with heterotic fermions. One can truncate this theory to $N = 1$ by taking the bosonic and fermionic fields to be real. We describe also a differential operator realisation of the truncated super-algebra.

2. Chiral and non-chiral W_∞ gravity

The W_∞ algebra is given by [8, 9]

$$[V_m^i, V_n^j] = \sum_{\ell=0}^{\infty} q^\ell g_{2\ell}^{ij}(m, n) V_{m+n}^{i+j-2\ell} + \text{central terms.} \quad (1)$$

Here V_m^i are the Fourier modes of currents V^i which have (quasi-) conformal spin $s = i + 2$, with $i = 0, 1, 2, \dots$, and the structure constants $g_\ell^{ij}(m, n)$ are given by

$$g_\ell^{ij}(m, n) = \frac{1}{2(\ell+1)!} \phi_\ell^{ij} N_\ell^{ij}(m, n), \quad (2)$$

where ϕ_ℓ^{ij} are functions given in terms of a Saalschützian ${}_4F_3(1)$ generalised hypergeometric function

$$\phi_\ell^{ij} = {}_4F_3 \left[\begin{matrix} -\frac{1}{2}, \frac{3}{2}, -\frac{\ell}{2} - \frac{1}{2}, -\frac{\ell}{2} \\ -i - \frac{1}{2}, -j - \frac{1}{2}, i+j-\ell+\frac{5}{2} \end{matrix}; 1 \right], \quad (3)$$

and $N_\ell^{ij}(m, n)$ is the “Clebsch-Gordan” part, which is a polynomial in m and n :

$$N_\ell^{ij}(m, n) = \sum_{k=0}^{\ell+1} (-1)^k \binom{\ell+1}{k} (2i+2-\ell)_k [2j+2-k]_{\ell+1-k} [i+1+m]_{\ell+1-k} [j+1+n]_k. \quad (4)$$

The definitions of the generalised hypergeometric functions and Pochhammer symbols are standard; they may be found for example in ref. [9]. We shall, without loss of generality, take the parameter q to be equal to unity in the remainder of this paper. Note that although the sum on the right-hand side of (1) runs to infinity, the occurrence of zeros in the functions ϕ_ℓ^{ij} and $N_\ell^{ij}(m, n)$ ensures that the sequence of generators with descending conformal spins terminates with spin 2 or 3.

The chiral W_∞ algebra can be realised in terms of the bilinear currents [15]

$$V^i(z) = \sum_{k=0}^i (-1)^{k+1} \alpha_k^i \partial^{k+1} \varphi(z) \partial^{i-k+1} \varphi^*(z), \quad (5)$$

where φ is a free complex scalar, with 2-point function

$$\varphi(z) \varphi^*(w) \sim -\log(z-w), \quad (6)$$

and the coefficients appearing in (5) are determined to be

$$\alpha_k^i = \frac{2^{-i-1}(i+2)!}{(2i+1)!(i+1)!} \binom{i+1}{k} \binom{i+1}{k+1}. \quad (7)$$

Note that for a given spin $s = i + 2$ we can write down $(i + 1)$ different bilinear currents. However, all but one of these can be expressed in terms of derivatives of lower-spin currents. Therefore, there is just one independent current at each spin. The consequence of this is that for any non-degenerate choice of the coefficients α_i^* the algebra closes. Of course, the actual calculation of the structure constants for a given choice of α_i^* is a non-trivial combinatorial problem. The choice (7) is made so that the operator products of the $V^i(z)$ satisfy the W_∞ algebra as given in [8,9]; in particular, the requirement that central terms should arise only between generators of the same conformal spin fixes the α_i^* up to $(i$ -dependent) overall factors. One can check that the operator products of the currents $V^i(z)$ take the form

$$V^i(z)V^j(w) \sim -\sum_{\ell=0}^{\infty} g_{2\ell}^{ij}(\partial_z, \partial_w)^{(0)} \left(\frac{V^{i+j-2\ell}(w)}{z-w} \right) + \text{central terms.} \quad (8)$$

This is equivalent to the Fourier-mode form of the W_∞ algebra (1), with the parameter q taken to have the value 1. The functions $g_\ell^{ij}(m, n)^{(0)}$ are given by replacing the polynomials $N_\ell^{ij}(m, n)$ in (2) by

$$M_\ell^{ij}(m, n) \equiv \sum_{k=0}^{\ell+1} (-1)^k \binom{\ell+1}{k} (2i - \ell + 2)k[2j + 2 - k]_{\ell+1-k} m^{\ell+1-k} n^k. \quad (9)$$

In other words, one picks out the leading component, with total degree $(\ell + 1)$ in m and n .

Under the symmetry generated by the current V^i , the transformation of any field f is given by the standard OPE expression [16]

$$\delta_{k_i} f = \oint \frac{dz}{2\pi i} k_i(z) V^i(z) f. \quad (10)$$

In particular, from (5) and (6), we see that under a k_i transformation the field φ transforms as

$$\delta_{k_i} \varphi = \sum_{k=0}^i \alpha_k^* \partial^k (k_i \partial^{i-k+1} \varphi). \quad (11)$$

To find a field-theoretic realisation of the W_∞ algebra described above, we start with a free Lagrangian for a complex scalar φ :

$$\mathcal{L} = \partial_+ \varphi^* \partial_- \varphi. \quad (12)$$

where $\partial_\pm$ denotes $\partial_t \pm \partial_\sigma$, with τ and σ being coordinates on the world sheet. This Lagrangian has a global chiral W_∞ symmetry under which φ transforms in a manner analogous to the OPE result (11):

$$\delta_{k_i} \varphi = \sum_{k=0}^i \alpha_k^* \partial^k (k_i \partial^{i-k+1} \varphi), \quad (13)$$

where the parameters k_i are functions of x^+ only.

To gauge the chiral W_∞ symmetry, we now allow the parameters k_i to depend on x^- as well as x^+ . We must also introduce gauge fields A_i and add the (gauge field)-(current) terms to the Lagrangian:

$$\mathcal{L} = \partial_+ \varphi^* \partial_- \varphi + \sum_{i \geq 0} A_i V^i. \quad (14)$$

The Noether procedure now goes as follows. The variation of φ in the kinetic term is cancelled by the leading-order transformation of the gauge field which is of the form $\delta A_i = \partial_- k_i + \dots$. We next vary the currents by using the formula (10) with f taken to be V^j . The result can conveniently be expressed in the form

$$\delta_{k_i} V^i(x) = \sum_{\ell \geq 0} \int dy k_j(y) g_{2\ell}^{ij}(\partial_x, \partial_y)^{(0)} (\delta(x-y) V^{i+j-2\ell}(x)). \quad (15)$$

By integrating by parts, one can see that this variation is cancelled by adding terms to the transformation rule of the gauge field A_i so that its total variation $\delta A_i = \partial_- k_i + \hat{\delta} A_i$ is given by

$$\delta A_i = \partial_- k_i + \sum_{\ell \geq 0} g_{2\ell}^{i, i-j+2\ell} (\partial_A, \partial_k)^{(0)} A_j k_{i-j+2\ell}, \quad (16)$$

where ∂_A and ∂_k are the ∂_+ derivatives acting on A and k , respectively. The term $\hat{\delta} A_i$ is a co-adjoint transformation of A_i , while V^i transforms in the adjoint representation of W_∞ ; i.e. $f(\hat{\delta} A_i V^i + A_i \delta V^i) = 0$. Unlike the case of finite-dimensional semi-simple Lie algebras, one must distinguish between the adjoint and co-adjoint representations of W_∞ since there is no Cartan-Killing metric available to raise or lower indices. One consequence of this is that whereas the transformation (15) of V^i under k_j for given i and j involves a finite number of terms, the transformation (16) of A_i under k_j will involve an infinite number.

We next consider the simultaneous gauging of both the left-handed and right-handed W_∞ symmetries. Following the construction of refs [13,14], we introduce gauge fields $A_{\pm i}$ corresponding to the left- and right-handed W_∞ symmetries, and auxiliary fields $J_{\pm}$. We then find that the Lagrangian for non-chiral W_∞ is given by

$$\begin{aligned} \mathcal{L}_{\text{bosonic}} = & -\partial_+ \varphi^* \partial_- \varphi - J_+^* J_- + \partial_- \varphi J_+^* + \partial_+ \varphi^* J_- - J_+ J_-^* + \partial_- \varphi^* J_+ + \partial_+ \varphi J_-^* \\ & + \sum_{i \geq 0} (A_{+i} V_-^i(J_-) + A_{-i} V_+^i(J_+)), \end{aligned} \quad (17)$$

where $V_\pm^i(J_\pm) = \sum_k (-1)^{k+1} \alpha_k^* \partial_\pm^k J_\pm \partial_\pm^{i-k} J_\pm^*$. This Lagrangian has the following non-chiral W_∞ symmetry

$$\delta \varphi = \sum_{i \geq 0} \sum_{k=0}^i \alpha_k^* \partial_+^k (k_{-i} \partial_+^{i-k} J_+) + \alpha_k^* \partial_-^k (k_{+i} \partial_-^{i-k} J_-),$$

$$\begin{aligned}
\delta A_{\pm i} &= \partial_{\pm} k_{\pm i} + \sum_{\ell \geq 0} \sum_{j=0}^{i+2\ell} g_{2\ell}^{j, i-j+2\ell} (\partial_A, \partial_k)^{(0)} A_{\pm j} k_{\pm i-j+2\ell}, \\
\delta J_{\pm} &= \partial_{\pm} (\delta_{k_{\mp}} \varphi).
\end{aligned} \tag{18}$$

In the transformation rule for the gauge fields, it is to be understood that ∂_A and ∂_k denote ∂_- derivatives when acting on A_+ and k_+ , and ∂_+ derivatives when acting on A_- and k_- .

The field equations of the complex auxiliary fields $J_{\pm}$ give

$$\begin{aligned}
J_{\pm} &= \partial_{\pm} \varphi - \delta \varphi|_{k_{\pm} = A_{\pm}} \\
&= \partial_{\pm} \varphi - \sum_{i \geq 0} \sum_{k=0}^i \left[\alpha_i^k \partial_{\pm}^k (A_{-i} \partial_{\pm}^{i-k} J_+) + \alpha_i^k \partial_{\pm}^k (A_{+i} \partial_{\pm}^{i-k} J_-) \right],
\end{aligned} \tag{19}$$

and define a set of nested covariant derivatives generalising those of ref. [13].

Note that the method of gauging we have used above is rather universal and does not depend on the specific form of the structure constants of the algebra. The method equally applies to any set of currents satisfying an (infinite-dimensional) Lie algebra. These currents will generate symmetries whose action on fields is given by (10), which can be gauged in a Lagrangian of the form (14) or (17). Note also that the nested covariant derivative in the non-chiral case, defined by the first line in equation (19), reduces to the normal covariant derivative for a finite-dimensional gauge group where φ transforms in the usual way.

3. Super- W_{∞} Gravity

Using the ideas described in the previous section, we now proceed to construct a gauged theory of super- W_{∞} gravity. To realise the super- W_{∞} algebra, in addition to the currents V^i defined in (5), we also introduce the following currents

$$\tilde{V}^i = \sum_{k=0}^{i+1} (-1)^k \beta_i^k \partial_{\pm}^k \psi^* \partial_{\pm}^{i-k+1} \psi, \tag{20}$$

$$G^{\alpha} = \sum_{k=0}^{\alpha} (-1)^k \gamma_k^{\alpha} \partial_{\pm}^{\alpha-k+1} \varphi^* \partial_{\pm}^k \varphi, \tag{21}$$

where ψ is a complex fermion with two-point function

$$\psi^*(z) \psi(w) \sim \frac{1}{z-w}, \tag{22}$$

and

$$\beta_k^i = \frac{(i+1)(k+1)}{(i+2)(i-k+1)} \alpha_k^i, \quad \gamma_k^i = \frac{2(k+1)}{(i+2)} \alpha_k^i. \tag{23}$$

The currents $\tilde{V}^i$ have (quasi-) conformal spins $s = i+2$, $i \geq -1$, whilst the G^{α} have (quasi-) conformal spins $s = \alpha + \frac{3}{2}$, $\alpha \geq 0$.

One finds that the currents (5), (20) and (21) obey the super- W_{∞} algebra, which, in terms of the Fourier modes of the generators, is given by (1) together with [11]

$$[\tilde{V}_m^i, \tilde{V}_n^j] = \sum_{\ell=0}^{\infty} \hat{g}_{2\ell}^{ij}(m, n) \tilde{V}_{m+n}^{i+j-2\ell} + \text{central terms}, \tag{24}$$

$$\{\tilde{G}_r^{\alpha}, \tilde{G}_s^{\beta}\} = \sum_{\ell=0}^{\infty} \left(b_{\ell}^{\alpha\beta}(r, s) V_{r+s}^{\alpha+\beta-\ell} + \tilde{b}_{\ell}^{\alpha\beta}(r, s) \tilde{V}_{r+s}^{\alpha+\beta-\ell} \right) + \text{central terms}, \tag{25}$$

$$[V_m^i, G_r^{\alpha}] = \sum_{\ell=0}^{\infty} a_{\ell}^{i\alpha}(m, r) G_{m+r}^{\alpha+i-\ell+1}, \tag{26}$$

$$[\tilde{V}_m^i, G_r^{\alpha}] = \sum_{\ell=0}^{\infty} \tilde{a}_{\ell}^{i\alpha}(m, r) G_{m+r}^{\alpha+i-\ell+1}, \tag{27}$$

$$[V_m^i, \tilde{G}_r^{\alpha}] = \sum_{\ell=0}^{\infty} (-)^{\ell+1} a_{\ell}^{i\alpha}(m, r) \tilde{G}_{m+r}^{\alpha+i-\ell+1}, \tag{28}$$

$$[\tilde{V}_m^i, \tilde{G}_r^{\alpha}] = \sum_{\ell=0}^{\infty} (-)^{\ell+1} \tilde{a}_{\ell}^{i\alpha}(m, r) \tilde{G}_{m+r}^{\alpha+i-\ell+1}. \tag{29}$$

Explicit expressions for the structure constants $\hat{g}, \dots, \tilde{a}$ can be found in ref. [11].

At this point it would be straightforward to write down the supersymmetric analogue of the chiral gauging discussed in section 2. Instead, we shall proceed directly to a non-chiral supersymmetric gauging, from which the chiral theory may be easily obtained by truncation. In this context, the term “non-chiral” requires some clarification. For the bosonic case, the motivation for considering the non-chiral theory is that it would presumably arise naturally from a fully-covariant formulation through gauge fixing. When one includes fermions, however, similar reasoning would not require the fermionic sector to be non-chiral. Thus the simplest “covariantisable” supersymmetric extension should describe left- (or right-) moving fermions coupled to non-chiral bosons. We shall describe such a heterotic theory below.

Following the steps described in section 2 for constructing the bosonic theory, we introduce, in addition to the fields $A_{\pm}^i$ for the generators $V_{\pm}^i$ of (non-chiral) W_{∞} , gauge fields $\tilde{A}^i$, χ^{α} and $\tilde{\chi}^{\alpha}$ corresponding to the additional generators $\tilde{V}^i$, G^{α} and $\tilde{G}^{\alpha}$ of super W_{∞} . We find that the heterotic super- W_{∞} -symmetric Lagrangian is

$$\mathcal{L}_{\text{super}} = \mathcal{L}_{\text{bosonic}} - \psi^* \partial_- \psi + \sum_{i \geq -1} \tilde{A}_i \tilde{V}^i + \sum_{\alpha \geq 0} \left(\chi_{\alpha} G^{\alpha}(J_+) + \tilde{\chi}_{\alpha} \tilde{G}^{\alpha}(J_+) \right), \tag{30}$$

where $\mathcal{L}_{\text{bosonic}}$ is given by (17), and the currents $\tilde{V}^i$ and $G^{\alpha}(J_+)$ are those defined in (20) and (21) with $\partial \rightarrow \partial_+$, and ∂_{φ} replaced by the auxiliary field J_+ . The above Lagrangian

has the following non-chiral super- W_∞ symmetry:

$$\delta\varphi = \sum_{i \geq 0} \sum_{k=0}^i \left[\alpha_k^i \partial_+^k (k_- \partial_+^{i-k} J_+) + \alpha_k^i \partial_-^k (k_+ \partial_-^{i-k} J_-) \right] - \sum_{\alpha \geq 0} \sum_{k=0}^\alpha (-)^{\alpha-k} \gamma_k^\alpha \partial_+^{\alpha-k} (c_\alpha \partial_+^k \psi),$$

$$\delta\psi = \sum_{i \geq -1} \sum_{k=0}^{i+1} \beta_k^i \partial_+^k (\tilde{k}_i \partial_+^{i-k+1} \psi) + \sum_{\alpha \geq 0} \sum_{k=0}^\alpha \gamma_k^\alpha \partial_+^k (c_\alpha \partial_+^{\alpha-k} J_+^*),$$

$$\delta A_{+,i} = \partial_+ k_{+,i} + \sum_{\ell \geq 0} \sum_{j=0}^{i+2\ell} g_{\ell}^{j,i-j+2\ell} (\partial_A, \partial_k)^{(0)} A_{+,j} k_{+,i-j+2\ell}$$

$$\begin{aligned} \delta A_{-,i} = \partial_- k_{-,i} + \sum_{\ell \geq 0} \sum_{j=0}^{i+2\ell} g_{\ell}^{j,i-j+2\ell} (\partial_A, \partial_k)^{(0)} A_{-,j} k_{-,i-j+2\ell}, \\ - \sum_{\ell \geq 0} \sum_{\alpha=0}^{i+\ell} (-)^{\ell} b_{\ell}^{\alpha,i-\alpha-\ell} (\partial_\chi, \partial_\epsilon)^{(0)} (\tilde{\chi}_\alpha \epsilon_{i-\alpha-\ell} + \chi_\alpha \tilde{\epsilon}_{i-\alpha-\ell}), \end{aligned}$$

$$\begin{aligned} \delta \tilde{A}_i = \partial_- \tilde{k}_i + \sum_{\ell \geq 0} \sum_{j=-1}^{i+2\ell+1} \tilde{g}_{\ell}^{j,i-j+2\ell} (\partial_A, \partial_k)^{(0)} \tilde{A}_j \tilde{k}_{i-j+2\ell} \\ - \sum_{\ell \geq 0} \sum_{\alpha=0}^{i+\ell} (-)^{\ell} \tilde{b}_{\ell}^{\alpha,i-\alpha-\ell} (\partial_\chi, \partial_\epsilon)^{(0)} (\tilde{\chi}_\alpha \epsilon_{i-\alpha-\ell} + \chi_\alpha \tilde{\epsilon}_{i-\alpha-\ell}), \end{aligned}$$

$$\begin{aligned} \delta \chi_\alpha = \partial_- \epsilon_\alpha + \sum_{\ell \geq 0} \sum_{j=0}^{i+2\ell} (-)^{\ell} \left[a_{\ell}^{j,\alpha-j+\ell-1} (\partial_A, \partial_k)^{(0)} A_{-,j} \epsilon_{\alpha-j+\ell-1} - a_{\ell}^{j,\alpha-j+\ell-1} (\partial_k, \partial_\chi)^{(0)} k_{-,j} \chi_{\alpha-j+\ell-1} \right] \\ + \sum_{\ell \geq 0} \sum_{j=-1}^{i+2\ell+1} (-)^{\ell} \left[\tilde{a}_{\ell}^{j,\alpha-j+\ell-1} (\partial_A, \partial_k)^{(0)} \tilde{A}_j \epsilon_{\alpha-j+\ell-1} - \tilde{a}_{\ell}^{j,\alpha-j+\ell-1} (\partial_k, \partial_\chi)^{(0)} \tilde{k}_{j} \chi_{\alpha-j+\ell-1} \right], \end{aligned}$$

$$\delta J_+ = \partial_+ (\delta k_- \varphi + \delta \epsilon \varphi), \quad \delta J_- = \partial_- (\delta k_+ \varphi). \quad (31)$$

In the preceding expressions, the quantities with superscript (0) are related to the corresponding quantities of eqns. (24)–(29) in the same manner as that described in section 2 for the W_∞ structure constants. It is to be understood that the derivatives appearing in these

functions are ∂_- when acting on A_+ or k_+ , and ∂_+ when acting on A_- , $\tilde{A}$, χ , k_- , $\tilde{k}$ or ϵ .

The chiral truncation is obtained by simply omitting the fields J_- and A_+ in the action and transformation rules, and substituting $\partial_+ \varphi$ for J_+ everywhere. If one now considers the chiral W_∞ and $W_{1+\infty}$ transformations with parameters k_- and $\tilde{k}_-$, one finds that their action upon the fields φ and ψ is given by differential operators $\delta_{k_-} \varphi = \Lambda(k_-) \varphi$ and $\delta_{\tilde{k}_-} \psi = \Sigma(k_-) \psi$, respectively. These differential operators can be written as

$$\Lambda(k) = \sum_{i \geq 0} \sum_{m=0}^i \lambda_m^i (\partial^m k_i) \partial^{i-m+1}, \quad \Sigma(k) = \sum_{i \geq -1} \sum_{m=0}^{i+1} \sigma_m^i (\partial^m k_i) \partial^{i-m+1}, \quad (32)$$

where

$$\lambda_m^i = \sum_{n=m}^i \binom{n}{m} \alpha_n^i, \quad \sigma_m^i = \sum_{n=m}^{i+1} \binom{n}{m} \beta_n^i. \quad (33)$$

We find that the coefficients λ_m^i and σ_m^i obey the following recursion relations

$$\begin{aligned} \lambda_m^i &= \lambda_m^{i-1} + \frac{1}{2} \lambda_{m-1}^{i-1} + \frac{1-i^2}{4(4i^2-1)} \lambda_{m-2}^{i-2}, \\ \sigma_m^i &= \sigma_m^{i-1} + \frac{1}{2} \sigma_{m-1}^{i-1} - \frac{i^2}{4(4i^2-1)} \sigma_{m-2}^{i-2}. \end{aligned} \quad (34)$$

The operators $\Lambda(k)$ and $\Sigma(k)$ are closely related to the realisations of W_∞ and $W_{1+\infty}$ in terms of differential operators discussed in ref. [17]. (The differences are analogous to the differences between the structure constants $g_{\ell}^{ij}(m,n)^{(0)}$ and $g_{\ell}^{ij}(m,n)$; they are related by redefinitions of generators [9].)

The gauged $N=2$ theory constructed above can be truncated to a theory gauging the $N=1$ super- $W_{\infty/2}$ algebra [11], obtained by replacing the complex scalar φ and complex fermion ψ by real ones. This picks out the even-bosonic-spin currents, which generate $W_{\infty/2} \oplus W_{(1+\infty)/2}$, and fermionic currents corresponding to the real part of G^α [11]. If we consider the chiral truncation of this theory, we can extend the differential-operator realisation of the bosonic transformations to transformations with fermionic parameters. We do this by introducing super-differential operators, $D(\epsilon)$, acting on an $N=1$ superfield $\Phi = \varphi + \theta \psi$ as $\delta_\epsilon \Phi = D(\epsilon) \Phi$. The operator $D(\epsilon)$ can be written as

$$D(\epsilon) = T_m^\alpha \left[(-1)^\alpha \frac{\alpha+1}{\alpha-m+1} (\partial^m \epsilon_\alpha) \partial^{\alpha-m} \partial_\theta + \theta (\partial^m \epsilon_\alpha) \partial^{\alpha-m+1} \right], \quad (35)$$

where

$$T_m^\alpha = \sum_{n=m}^\alpha \binom{n}{m} \gamma_n^\alpha. \quad (36)$$

The extension of this realisation to the $N=2$ algebra is more complicated.

4. Conclusions

In this paper we have constructed gauge theories of the W_∞ and super- W_∞ algebras. Although the methods that we have used closely parallel those used in previous work for the gauging of W_3 [13], and W_N and w_∞ [14], there is one important difference, namely that in this paper the currents that generate the algebras are built from higher-derivative bilinears in fields, whereas those in [13] and [14] are built from higher powers of first derivatives of fields. Because of the different approaches used in the two cases, there is as yet no natural way to recover the previous w_∞ gauge theory as a limit of the W_∞ theory.

The results presented here for non-chiral gauge theories can be expected to be interpretable as light-cone gauge-fixed versions of underlying covariant theories. In the case of W_3 such a covariant gauge theory has been constructed recently [18]. It would be interesting to see whether a similar covariant theory can be constructed for the (super) W_∞ algebras. In addition, it would be interesting to extend this work by adding kinetic terms for the gauge fields. This would presumably give rise to higher-spin analogues of the two-dimensional gravity action. Recently, an action for topological W_N -gravity has been obtained [19], and it would be interesting to investigate its w_∞ and W_∞ limits. There are also some very interesting indications of a possible connection between W_∞ gravity and self-dual four-dimensional gravity [20,21]. Finally, it should be remarked that all the discussion so far has been concerned with the classical theory. Important questions remain about the quantisation of these theories, and the issue of possible anomalies in the higher-spin symmetries.

Acknowledgements

Two of the authors (E.B. and E.S.) would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics. Two of the others (C.N.P. and L.J.R.) are grateful to Keke Li for discussions, and to the California Institute of Technology for hospitality during the completion of this work.

References

- [1] A.B. Zamolodchikov, *Teo. Mat. Fiz.* **65** (1985) 347;
V.A. Fateev and S. Lykhanov, *Int. J. Mod. Phys. A* **3** (1988) 507.
- [2] M. Awada and Z. Qiu, "Non-perturbative higher-spin gravity in two dimensions," and "Non-perturbative 2-d quantum gravity and the generalised KdV hierarchy," preprints, UFIFT-HEP-90-2,4.

- [3] E. Brezin and V.A. Kazakov, *Phys. Lett. B* **236** (1990) 144;
D.J. Gross and A.A. Migdal, *Phys. Rev. Lett.* **64** (1990) 127;
M. Douglas and S. Shenker, "Strings in less than 1 dimension," Rutgers preprint RU-89-34.
- [4] A. Bilal and J.L. Gervais, *Nucl. Phys. B* **326** (1989) 222.
- [5] I. Bakas, "The structure of the W_∞ algebra," Maryland preprint, PP-90-085 (1990);
A. Bilal, *Phys. Lett. B* **227** (1989) 406.
- [6] E. Sezgin and E. Sokatchev, *Phys. Lett. B* **227** (1989) 103.
- [7] C.N. Pope and X. Shen, *Phys. Lett. B* **236** (1990) 21.
- [8] C.N. Pope, L.J. Romans and X. Shen, *Phys. Lett. B* **236** (1990) 173.
- [9] C.N. Pope, L.J. Romans and X. Shen, " W_∞ and the Racah-Wigner algebra," preprint, CTP TAMU-72/89, USC-89/HEP040, to appear in *Nucl. Phys. B*.
- [10] C.N. Pope, L.J. Romans and X. Shen, "A new higher-spin algebra and the Lone-Star product," preprint, CTP TAMU-17/90 (1990), USC-90/HEP01 (1990), to appear in *Phys. Lett. B*.
- [11] E. Bergshoeff, C.N. Pope, L.J. Romans, E. Sezgin and X. Shen, "The Super W_∞ algebra," preprint, CTP TAMU-42/90 (1990), to appear in *Phys. Lett. B*.
- [12] C.M. Hull, "Gauging the Zamolodchikov W -algebra," preprint, QMC/PH/89/18 (1989).
- [13] K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, "A new gauge theory of W -type algebras," Stony Brook preprint, ITP-SB-90-19 (1990); Talk presented by P. van Nieuwenhuizen at Strings 90, Texas A&M University, March 1990.
- [14] E. Bergshoeff, C.N. Pope, L.J. Romans, E. Sezgin, X. Shen and K.S. Stelle, " w_∞ Gravity," preprint, CTP TAMU-31/90, to appear in *Phys. Lett. B*.
- [15] I. Bakas and E. Kiritsis, "Bosonic realisation of a universal W -algebra and Z_∞ parafermions," preprint, UCB-PTH-90/8.
- [16] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, *Nucl. Phys. B* **241** (1984) 333.
- [17] C.N. Pope, L.J. Romans and X. Shen, "Ideals of Kac-Moody algebras and realisations of W_∞ ," preprint, CTP-TAMU-25/90, USC-90/HEP05 (1990), to appear in *Phys. Lett. B*.
- [18] K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, "Covariant formulation of classical W -gravity," Stony Brook preprint, ITP-SB-90-26 (1990).
- [19] K. Li, "Action for topological W gravity," preprint, CALT-68-1635 (1990).
- [20] Q.-H. Park, *Phys. Lett. B* **238** (1990) 287.
- [21] E. Witten, "Surprises with topological field theories," Princeton preprint IASNS-HEP-90/37 (1990).