DFTT-10/90
DFPD/90/TH 12
UCLA 90/TEP/25
March 1990

SPECIAL GEOMETRY WITHOUT SPECIAL COORDINATES

Leonardo Castellani

Istituto Nazionale di Fisica Nucleare, Sezione di Torino
Via P. Giuria 1, I-10125 Turin, Italy

Riccardo D' Auria

Dipartimento di Fisica, Università di Padova and
Istituto Nazionale di Fisica Nucleare, Via Marzolo 8,
I-35131 Padua, Italy

and

Sergio Ferrara

CERN Theoretical Division, 1211 Geneva 23, Switzerland
and
Department of Physics, University of California
Los Angeles, CA 90024-1547, U.S.A.

Abstract

We present a detailed geometrical derivation of $N=2$, 4D supergravity coupled to abelian vector multiplets $(1, 1/2, 0)$. The theory is covariant under reparametrizations of the manifold $\mathcal{M}$ spanned by the scalar fields. By requiring the closure of the supersymmetry transformation laws, we find a coordinate-free characterization of $\mathcal{M}$. The geometry of $\mathcal{M}$, called "special geometry", is relevant to compactified string theories, since it is common to the moduli spaces of Calabi-Yau threefolds and $c=9$ (2,2) superconformal field theories.

* This work is supported in part by the Ministero della Pubblica Istruzione, and by the Dept of Energy of U.S.A. under contract DOE-AT03-88ER40 384, TASK E.

1 - Introduction

In this paper we give a geometric formulation of $N=2$, 4D Supergravity coupled to (abelian) vector matter. An important feature of our framework is the covariance of the theory under reparametrizations of $\mathcal{M}$, the manifold spanned by the scalar fields.

Closure of the $N=2$ supersymmetry transformations restricts the choice of possible metrics for $\mathcal{M}$, and this restriction is given in a $\mathcal{M}$ -covariant form. We find a solution for the metric of $\mathcal{M}$ in terms of $n+1$ complex scalar functions $L^I(z, \bar{z})$ of the coordinates $z, \bar{z}$ of $\mathcal{M}$ ($I=0, 1, \dots, n$). These functions have $U(1)$ Kähler weight = 1, are $U(1)$ -covariantly analytic* in $z, \bar{z}$, and do satisfy one real constraint; they are otherwise arbitrary.

By a particular choice of these functions, we retrieve the coordinate-dependent formulation of refs [1,2a], where the metric of $\mathcal{M}$ depends on one complex analytic function $f(z)$, and defines what is called "special Kähler geometry". Some of these results were already presented by the authors in ref. [2b].

The geometry of $\mathcal{M}$ has recently become of interest because of its relation to the moduli space of Calabi-Yau compactifications of superstring theories [3,4,5], or in more general terms to the so-called Zamolodchikov metric [6] for the moduli of (2,2) superconformal field theories.

A number of authors [3-10] have emphasized, in the recent past, that the metric of the moduli space of Calabi-Yau manifolds should not only be Kähler, as implied by $N=1$ space-time supersymmetry, but also "special Kähler", as implied by $N=2$ space-time supersymmetry when one regards (2,2) theories as possible vacua for type II superstrings.

The constraints on $\mathcal{M}$ due to $N=2$ supersymmetry were found in refs [1,2a], and were formulated in a particular coordinate system of $\mathcal{M}$, the "special gauge". More precisely, the metric of $\mathcal{M}$ is constrained to be generated by a Kähler potential

$$G(z, \bar{z}) = -\ln z^I N_{IJ} \bar{z}^J \quad I = 0, 1, \dots, n \quad (1.1)$$

with

$$N_{IJ}(z, \bar{z}) = \frac{1}{4} \frac{\partial}{\partial X^I} \frac{\partial}{\partial X^J} F(X) + h.c. \quad (1.2)$$

$$z^I = \frac{X^I}{X^0} = (1, z^i) \quad (1.3)$$

and where $F(X^I)$ is a homogeneous holomorphic function of degree 2 in $n+1$ variables X^I . This implies

$$F(X) = (X^0)^2 f(z), \quad f(z) \text{ analytic} \quad (1.4)$$

* In more rigorous terms, these functions are holomorphic sections of a $U(1)$ bundle.

and the only freedom in the geometry of $\mathcal{M}$ resides in the arbitrary analytic function $f(z)$.

The Riemann curvature of $\mathcal{M}$ is computed from (1) to be [2a]:

$$R_{ij^*k^*} = -g_{j^*(i}g_{k^*)^*} + \frac{1}{2}Y^{-2}Q_{ikm}\tilde{Q}_{j^*l^*n^*}g^{mn^*} \quad (1.5)$$

with

$$Y = e^{-G} = z^I N_{IJ} \bar{z}^J = 2f + 2\bar{f} - (f_i - \bar{f}_{i^*})(z^i - \bar{z}^{i^*}) \quad (1.6)$$

$$Q_{ijk} = \frac{1}{4}\partial_i\partial_j\partial_k f(z) \quad (1.7)$$

$$(f_i \equiv \frac{\partial}{\partial z^i}f, \quad \partial_i \equiv \frac{\partial}{\partial z^i})$$

In the case of rigid N=2 supersymmetry, i.e. when one considers a system of n vector multiplets in a flat supergravity background, the Kähler potential of the scalar manifold $\mathcal{M}$ takes the form

$$G = z^i \bar{f}_{i^*} + \bar{z}^{i^*} f_i \quad i = 1, \dots, n \quad (1.8)$$

and the corresponding Riemann curvature is:

$$R_{ij^*kl^*} = +\frac{1}{2}Q_{ikm}(z)\tilde{Q}_{j^*l^*n^*}(\bar{z})g^{mn^*} \quad (1.9)$$

Formulas (1.5-9) hold only in a particular coordinate system of $\mathcal{M}$: the metric and the curvature computed through eq. (1.1) are not covariant under reparametrizations of the Kähler manifold $\mathcal{M}$; they are only covariant under rigid $Sp(2(n+1), R)$ symplectic transformations [4] acting on the $2(n+1)$ vectors $(X^I, \frac{\partial}{\partial X^I}F(X))$. Therefore g_{ij^*} and $R_{ij^*kl^*}$ can be expressed in terms of an analytic function $f(z)$ only in some parametrizations of $\mathcal{M}$.

As a byproduct of our geometric formulation, we derive in Sect. 4 the covariant generalization of the above formulas. By requiring the closure of the N=2 supersymmetry algebra, we arrive at the following expression for the curvature of $\mathcal{M}$:

$$R_{ij^*kl^*} = -g_{j^*(i}g_{k^*)^*} + \frac{1}{2}e^{2G}T_{ikm}\tilde{T}_{j^*l^*n^*}g^{mn^*} \quad (1.10)$$

where T_{ikm} is a "good" covariant tensor on $\mathcal{M}$, totally symmetric and analytic, given in terms of covariantly holomorphic scalar functions $L^I(z)$ appearing in the supersymmetry transformation laws of the spin 1 fields.

In heterotic string theory (or actually in a compactification of 10D anomaly-free supergravity [11]), the tensor T_{ij^*} is related to the Yukawa couplings of the E_6 families

[7,10,12,13,14]. A derivation of formula (8) for the moduli space of an abstract (2,2) superconformal field theory has recently been given in ref. [10]. In the large radius limit of Calabi-Yau manifolds, explicit formulas for the holomorphic function $f(z)$ have been found as integrals over Calabi-Yau spaces in [4,8,9,13,14]. In this case the moduli space is a direct product $\mathcal{M} = \mathcal{M}_A \times \mathcal{M}_B$, where $\mathcal{M}_A$ and $\mathcal{M}_B$ are two special Kähler manifolds as defined by formulas (1.1)-(1.7), with Y functions respectively given by [8,9]

$$Y_A = \int_{C_3} J \wedge J \wedge J, \quad Y_B = i \int_{C_3} \Omega \wedge \bar{\Omega} \quad (1.11)$$

where J and Ω are (1,1) and (3,0) forms parametrized by the real (1,1) and complex (2,1) moduli respectively. The corresponding holomorphic functions $f(z)$ are given by

$$f^A = \int_{C_3} \tilde{J} \wedge \tilde{J} \wedge \tilde{J}, \quad (1.12)$$

$$f^B = -\frac{i}{2} \int_{C_3} \hat{\Omega} \wedge (\alpha_0 + z^i \alpha_i) \quad (1.13)$$

where $\tilde{J}$ is a complexified Kähler form

$$\tilde{J} = B + iJ = \sum_{i=1}^{h(1,1)} z^i V_i \quad (1.14)$$

(V_i is a basis in $H^{(1,1)}$ Dolbeault cohomology), and $\hat{\Omega}$ is a holomorphic three-form in projective coordinates

$$\hat{\Omega}(z) = (X^0)^{-1} \Omega(X^i), \quad z^I = \frac{X^I}{X^0} \quad (1.15)$$

α_0, α_i are (with β^0, β^i) a cohomology basis in H^3 dual to the homology cycles A_a, B^a .

If one introduces homogeneous coordinates for the moduli spaces $\mathcal{M}_A, \mathcal{M}_B$, the Y functions can be rewritten as

$$Y_{A(B)} = \bar{X}^I F_I^{A(B)} + X^I \bar{F}_I^{A(B)} \quad (1.16)$$

where

$$F^{A(B)}(X) = (X^0)^2 f^{A(B)} \quad (1.17)$$

String arguments and space-time supersymmetry imply that eq. (1.12) is only valid in σ -model perturbation theory, while eq. (1.13) is an exact formula. On the other hand all the equations (1.1)-(1.17) are general, since they are merely a consequence of N=2 space-time supersymmetry or alternatively of world-sheet superconformal invariance.

Although we only discuss the local properties of special geometry, global properties can be discussed as well. Actually these properties, studied for particularly simple systems, such as superstrings propagating on orbifolds, indicate that the moduli space is a

2 - The geometry of $N=2$, $D=4$ supergravity coupled to vector matter

We first establish our conventions. The physical supergravity $N=2$ multiplet $(2, 3/2, 1)$ is given by $(V_\mu^a, \Psi_{A\mu}, A_\mu)$ where V_μ^a is the vierbein, $\Psi_{A\mu}$ ($A=1,2$) are the two Majorana gravitini and A_μ is the graviphoton. The n $N=2$ vector multiplets $(1, 1/2, 0^+, 0^-)$ contain the fields $(A_\mu^I, \Lambda_A^I, M^I)$ where A_μ^I ($I=1, \dots, n$) are the n gauge vectors, Λ_A^I are the gauginos and M^I are complex scalar fields. For simplicity we assume an abelian gauge group. This restriction does not influence the geometry of the Kähler manifold $\mathcal{M}$ spanned by the scalars M^I .

The Majorana spinors $\Psi_{A\mu}$ and Λ_A^I are decomposed into their chiral and antichiral components according to the definitions:

$$\psi_A \equiv \frac{1 + \gamma_5}{2} \Psi_A \quad \lambda^I_A \equiv \frac{1 + \gamma_5}{2} \Lambda_A^I \quad (2.1)$$

$$\bar{\psi}^A \equiv \frac{1 - \gamma_5}{2} \bar{\Psi}^A \quad \bar{\lambda}^I_A \equiv \frac{1 - \gamma_5}{2} \bar{\Lambda}_A^I \quad (2.2)$$

Therefore the upper and lower position of the $SO(2)$ index $A=1,2$ is related to the chirality of the spinor.

The manifold $\mathcal{M}$ of the scalar fields $M^I(z)$ is assumed to be a Kähler manifold of complex dimension n and Kähler potential $G(z^i, \bar{z}^{\bar{i}})$, $z^i, \bar{z}^{\bar{i}}$, $i=1, \dots, n$ being complex coordinates on $\mathcal{M}$. M^I and $\bar{M}^I$ are holomorphic and antiholomorphic scalar functions of the coordinates:

$$M^I = M^I(z)$$

$$\bar{M}^I = \bar{M}^I(\bar{z}) \quad (2.2)$$

and are assumed to be functionally independent. They could, in fact, be identified with the coordinates $z^i, \bar{z}^{\bar{i}}$ of $\mathcal{M}$. We prefer to consider the multiplet index I a "flat" index, inert under $\mathcal{M}$ reparametrizations.

We also introduce a $U(1)$ -Kähler connection 1-form Q defined as follows:

$$Q = -\frac{i}{2}(\partial_i G dz^i - \bar{\partial}_{\bar{i}} G d\bar{z}^{\bar{i}}) \quad (2.3)$$

and the Kähler 2-form

$$K = ig_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}} \quad (2.4)$$

Locally we have

$$g_{i\bar{j}} = \partial_i \bar{\partial}_{\bar{j}} G(z^i, \bar{z}^{\bar{i}}) \quad (2.5a)$$

$$K = dQ \quad (2.5b)$$

special Kähler orbifold [15] rather than a manifold, the fixed points corresponding to the enhanced gauge symmetry of heterotic string compactifications. The Kähler potential, because of its relation to the superpotential, is related to the theory of modular forms of the duality group of the background string parameters.

Recently some attempts have been made [10,16] to understand in an intrinsic way the special geometry of Calabi-Yau moduli space, as well as its relation to general properties of superconformal field theories. In all these approaches the special geometry is connected to some underlying properties of the internal conformal field theory (or Calabi-Yau space).

Our point of view is somewhat orthogonal, in the sense that we only require the closure of the supersymmetry transformations of the $N=2$ vector multiplets, in a geometrical framework that ensures from the start the covariance under reparametrizations of $\mathcal{M}$.

Our method allows us to define special geometries both for $N=2$ rigid and local supersymmetry. These geometries are actually very different: for example a flat metric is special only in the rigid case, in the sense that it satisfies eq. (1.9) but not eq. (1.5). It corresponds in a special gauge to a choice of $f(z)$ quadratic in the coordinates.

We recall that also for the $N=1$ chiral multiplets, and $N=2$ hypermultiplets, rigid and local supersymmetry yield different geometries for the scalar manifold. Indeed for $N=1$ chiral multiplets rigid supersymmetry requires a Kähler structure, while local supersymmetry requires $\mathcal{M}$ to be a Hodge manifold. For hypermultiplets the rigid case corresponds to hyper-Kähler geometry, while the local case corresponds to quaternionic manifolds [17,18,19]

The paper is organized as follows. In Sect. 2 we set up the notations and the basic geometric structure of the theory. In Sect. 3 we discuss the curvature parametrizations, i.e. the most general form of the supersymmetry transformation rules. In Sect. 4 we analyse the Bianchi identities, which correspond to the closure of the supersymmetry transformations, and in Sect. 5 we derive the constraint (1.10) on the geometry of $\mathcal{M}$. In Sect. 6 we find a solution of eq. (1.10), a Kähler metric that generalizes the one derived from the potential (1.1). Sect. 7 contains some concluding remarks. Finally, a Table summarizes the main results.

The Kähler transformation

$$G \rightarrow G + f(z) + \bar{f}(\bar{z}) \quad (2.6)$$

acts on the $U(1)$ connection as

$$Q \rightarrow Q + Im \, df \quad (2.7)$$

and leaves invariant the Kähler 2-form (2.4).

$N=2$ Supergravity in superspace is formulated in terms of the superspace 1-forms $(V^a, \psi_A, \bar{\psi}^A, A)$ associated to the supergravity physical fields, and the spin connection ω^{ab} . Their associated curvatures are * :

$$\begin{aligned} R^a &= dV^a - \omega^{ab}V_b - i\psi_A\gamma^a\psi^A \equiv DV^a - i\psi_A\gamma^a\psi^A & (2.8a) \\ \rho_A &= d\psi_A - \frac{1}{4}\omega^{ab}\gamma_{ab}\psi_A + \frac{i}{2}Q\psi_A \equiv \nabla\psi_A & (2.8b) \\ \rho^A &= d\bar{\psi}^A - \frac{1}{4}\omega^{ab}\gamma_{ab}\bar{\psi}^A - \frac{i}{2}Q\bar{\psi}^A \equiv \nabla\bar{\psi}^A & (2.8c) \\ F &= dA + \varepsilon_{AB}L\bar{\psi}^A\psi^B + \varepsilon^{AB}\bar{L}\bar{\psi}_A\psi_B & (2.8d) \\ R^{ab} &= d\omega^{ab} - \omega^a{}_c\omega^{cb} & (2.8e) \end{aligned}$$

where $\mathcal{D}$ and ∇ are the $SO(1,3)$ and $SO(1,3) \times U(1)$ covariant differentials, respectively. The function L and its complex conjugate $\bar{L}$ are assumed to be functions of the scalar fields M^I and therefore of the coordinates $z^i, \bar{z}^i$ on $\mathcal{M}$. In absence of matter multiplets the Kähler connection Q is zero and the functions $L, \bar{L}$ are constants which can be taken equal to 1.

We note that the spinor fields $\psi_A, \bar{\psi}^A$ are charged under the $U(1)$ Kähler transformation (2.6) and we have arbitrarily normalized the Kähler weight p of ψ_A and $\bar{\psi}^A$ to be $\frac{1}{2}$ and $-\frac{1}{2}$ respectively:

$$\psi'_A = e^{xp}(-\frac{i}{2}Im f)\psi_A \quad (2.9a)$$

$$\bar{\psi}'^A = e^{xp}(\frac{i}{2}Im f)\bar{\psi}^A \quad (2.9b)$$

In general the $U(1)$ -Kähler covariant differential of a field Φ of weight p is defined as

$$\nabla^{U(1)}\Phi = (d + ipQ)\Phi \quad (2.10)$$

which implies

* wedge products are omitted in the following.

$$\nabla_i^{U(1)}\Phi = \partial_i\Phi + \frac{p}{2}G_i\Phi \quad (G_i \equiv \partial_i G) \quad (2.11a)$$

$$\nabla_{i'}^{U(1)}\Phi = \partial_{i'}\Phi - \frac{p}{2}G_{i'}\Phi \quad (G_{i'} \equiv \partial_{i'} G) \quad (2.11b)$$

In particular

$$\nabla\nabla\Phi = ipK\Phi \quad (2.12)$$

and therefore, using (2.4),

$$\nabla_{[i}\nabla_{j]}\Phi = \nabla_{[i}\nabla_{j]}\Phi = 0 \quad (2.13a)$$

$$\nabla_i\nabla_{j'}\Phi = -p g_{ij'}\Phi \quad (2.13b)$$

Since ψ_A and $\bar{\psi}^A$ are $U(1)$ -charged, the same must be true for the scalar functions $L, \bar{L}$ appearing on the right-hand side of eq. (2.8d). Indeed A and F are neutral, so that the weights of L and $\bar{L}$ must be +1 and -1 respectively.

We now consider the vector multiplets. We introduce the superspace 1-form A^I for the gauge vector, the superspace fermionic 0-forms $\lambda^{IA}, \lambda_A^I$ for the chiral and antichiral gauginos and the complex bosonic fields M^I and $\bar{M}^I$ for the scalar fields.

Their superspace curvatures or field strengths are defined as:

$$F^I = dA^I + \varepsilon_{AB}L^I\bar{\psi}^A\psi^B + \varepsilon^{AB}\bar{L}^I\bar{\psi}_A\psi_B \quad (2.14a)$$

$$\nabla\lambda^{IA} = d\lambda^{IA} - \frac{1}{4}\omega^{ab}\gamma_{ab}\lambda^{IA} - \frac{i}{2}Q\lambda^{IA} \quad (2.14b)$$

$$\nabla\lambda_A^I = d\lambda_A^I - \frac{1}{4}\omega^{ab}\gamma_{ab}\lambda_A^I + \frac{i}{2}Q\lambda_A^I \quad (2.14c)$$

$$dM^I = \partial_i M^I(z)dz^i; \quad d\bar{M}^I = \partial_{i'}\bar{M}^I(\bar{z})d\bar{z}^{i'} \quad (2.14d)$$

The n complex 0-forms $L^I, \bar{L}^I$ appearing in the definition of F^I are complex scalar fields, functions of the (anti)holomorphic $M^I(\bar{M}^I)$ fields. Their $U(1)$ weight is +1 and -1 respectively.

Since $z^i, \bar{z}^{i'}$ are functions in superspace, we may expand the $\mathcal{M}$ coframe $(dz^i, d\bar{z}^{i'})$ in terms of the superspace coframe $(V^a, \psi_A, \bar{\psi}^A)$:

$$dz^i = Z_a^i V^a + \bar{\lambda}^{IA}\psi_A \quad (2.15a)$$

$$d\bar{z}^{i'} = \bar{Z}_a^{i'} V^a + \bar{\lambda}_A^{i'}\bar{\psi}^A \quad (2.15b)$$

where λ^{IA} and λ_A^i are defined by eq.s (2.15). Note that the chiral and antichiral spacetime spinors $\lambda^{IA}, \lambda_A^i$ are also (1,0) and (0,1) vectors on $\mathcal{M}$, respectively. Substituting eq.(2.15) into (2.14d) we find

$$dM^I = \partial_i M^I Z_a^i V^a + \partial_i M^I \bar{\lambda}^{iA} \psi_A \quad (2.16a)$$

$$d\bar{M}^I = \partial_i \bar{M}^I \bar{Z}_a^i V^a + \partial_i \bar{M}^I \bar{\lambda}^{i*} \bar{\psi}_A \quad (2.16b)$$

Comparing eq.s (2.16) with the general superspace expansion

$$dM^I = M_a^I V^a + \bar{\lambda}^{iA} \psi_A \quad (2.17a)$$

$$d\bar{M} = \bar{M}_a^I V^a + \bar{\lambda}^{i*} \bar{\psi}_A \quad (2.17b)$$

we obtain the identifications:

$$\partial_i M^I Z_a^i = M_a^I; \quad \partial_i \bar{M}^I \bar{Z}_a^i = \bar{M}_a^I \quad (2.18a)$$

$$\lambda^{IA} = \partial_i M^I \lambda^{iA}; \quad \lambda^{IA} = \partial_i \bar{M}^I \bar{\lambda}^{i*} \quad (2.18b)$$

The restrictions on the Kähler geometry of $\mathcal{M}$ are more easily found by analysing the Bianchi identities on the coordinate fields $z^i, \bar{z}^{i*}$ and their superpartners $\lambda^{iA}, \bar{\lambda}^{i*}$ (this is done in Sect. 5.). Thus we define the curvatures $\nabla \lambda^{iA}, \nabla \bar{\lambda}^{i*}$ where the ∇ differential is also covariant with respect to general (anti)holomorphic reparametrizations on $\mathcal{M}$:

$$\nabla \lambda^{iA} \equiv d\lambda^{iA} - \frac{1}{4} \omega^{ab} \gamma_{ab} \lambda^{iA} - \frac{i}{2} Q \lambda^{iA} - \Gamma^i_j \lambda^{jA} \quad (2.19a)$$

$$\nabla \bar{\lambda}^{i*} \equiv d\bar{\lambda}^{i*} - \frac{1}{4} \omega^{ab} \gamma_{ab} \bar{\lambda}^{i*} + \frac{i}{2} Q \bar{\lambda}^{i*} - \Gamma^{i*}_{j*} \bar{\lambda}^{j*} \quad (2.19b)$$

where $\Gamma^i_j \equiv \Gamma^i_{jk} dz^k$ and $\Gamma^{i*}_{j*} \equiv \Gamma^{i*}_{j*k} d\bar{z}^k$ are the Christoffel connection 1-forms on $\mathcal{M}$. In the following we will use the curvatures $(dz^i, \nabla \lambda^{iA}, F^I)$ and their complex conjugates.

Exterior differentiation of eq.s (2.8) yields the Bianchi identities of the gravitational multiplet:

$$DR^a + R^{ab} V_b - i\bar{\psi}^A \gamma^a \rho_A + i\bar{\rho}^A \gamma^a \psi_A = 0 \quad (2.20a)$$

$$\nabla \rho_A + \frac{1}{4} \gamma_{ab} R^{ab} \psi_A + \frac{1}{2} g_{ij} dz^i d\bar{z}^{j*} \psi_A = 0 \quad (2.20b)$$

$$\nabla \rho^A + \frac{1}{4} \gamma_{ab} R^{ab} \bar{\psi}^A + \frac{1}{2} g_{ij} dz^i d\bar{z}^{j*} \bar{\psi}^A = 0 \quad (2.20c)$$

$$dF - \epsilon_{AB} \nabla \bar{L} \bar{\psi}^A \bar{\psi}^B + 2\epsilon_{AB} L \bar{\psi}^A \rho^B - \epsilon^{AB} \nabla \bar{L} \bar{\psi}_A \psi_B + 2\epsilon^{AB} \bar{L} \bar{\psi}_A \rho_B = 0 \quad (2.20d)$$

$$DR^{ab} = 0 \quad (2.20e)$$

Similarly, differentiating $(dz^i, \nabla \lambda^{iA}, F^I)$ we find the Bianchi identities of the matter multiplet:

$$d^2 z^i = 0 \quad (2.21a)$$

$$\nabla^2 \lambda^{iA} + \frac{1}{4} R^{ab} \gamma_{ab} \lambda^{iA} + \frac{i}{2} R[Q] \lambda^{iA} + R[\Gamma]^i_j \lambda^{jA} = 0 \quad (2.21b)$$

$$dF^I - \epsilon_{AB} \nabla L^I \bar{\psi}^A \bar{\psi}^B + 2\epsilon_{AB} L^I \bar{\psi}^A \rho^B - \epsilon^{AB} \nabla \bar{L}^I \bar{\psi}_A \psi_B + 2\epsilon^{AB} \bar{L}^I \bar{\psi}_A \rho_B = 0 \quad (2.21c)$$

where $R[Q] = dQ$ and $R[\Gamma]^i_j$ is the Riemann curvature 2-form on $\mathcal{M}$:

$$R^i_j = 2R^i_{jk*} d\bar{z}^{k*} dz^j \quad (2.22)$$

Analogous equations hold for the conjugate multiplet $(d\bar{z}^{i*}, \nabla \bar{\lambda}^{i*}, F^I)$.

The appearance of the Riemann curvature of $\mathcal{M}$ in eq. (2.21b) makes it plausible that possible restrictions on the geometry of $\mathcal{M}$ should be found in the analysis of this Bianchi identity. As we show in Sect. 4, this is indeed the case, and justifies the use of the $\mathcal{M}$ vectors $\lambda^{iA}, \bar{\lambda}^{i*}$ instead of the $\mathcal{M}$ scalars $\lambda^{IA}, \bar{\lambda}^{I*}$.

3 - The parametrizations of the curvatures, or the most general Ansatz for the transformation rules

In order to solve the coupled system of 3- and 2-form equations (2.20-21) one must parametrize the curvatures (R^a, ρ^A, F, R^{ab}) and $(dz^i, \nabla \lambda^{iA}, F^I)$ in superspace. This is equivalent to write down the most general transformation rules for the fields V^a, ψ, A, ω and z^i, λ^{iA}, A^I . We recall that in our geometric approach the supersymmetry transformations are obtained after replacing ψ^A in the parametrizations by the supersymmetry parameter ϵ^A . For example $\delta_{susy} z^i = \bar{\epsilon}_A \lambda^{iA}$, see eq.(3.4a). The gravitino supersymmetry variation contains the additional derivative term $\nabla \epsilon^A$, since ψ^A is the "connection" of the local $N=2$ supersymmetry *

The parametrizations must satisfy the following requirements:

- i) they must be "rheonomic", i.e. their "outer components" (along one or more fermionic cotangent vectors ψ^A) are expressible in terms of their $V^a \dots V^b$ components.
- ii) they must be covariant under $SO(1,3) \times U(1)$ and under reparametrizations of $\mathcal{M}$.
- iii) they must be invariant under the global rescaling

$$\omega^{ab} \rightarrow \omega^{ab}; \quad V^a \rightarrow w V^a; \quad \psi \rightarrow w^{\frac{1}{2}} \psi$$

* we recall that the invariances of the theory are, in our framework [20], the superspace diffeomorphisms. On a generic field μ^A , these have the form: $\delta \mu^A(y) = \mu^A(y + \delta y) - \mu^A(y) = (\nabla \delta y)^A + i\delta y R^A$, where $i\delta y$ is the contraction along the tangent vector δy .

$$\begin{aligned}
A &\rightarrow wA; & A^I &\rightarrow wA^I \\
\lambda^{iA} &\rightarrow w^{-\frac{1}{2}}\lambda^{iA}; \\
\lambda_A^{i*} &\rightarrow w^{-\frac{1}{2}}\lambda_A^{i*}; \\
z^i &\rightarrow z^i; \bar{z}^{i*} &\rightarrow \bar{z}^{i*}
\end{aligned} \tag{3.1}$$

since this rescaling does not affect the definition of the curvatures and the Bianchi identities.

Requiring i), ii) and iii) leads to the following parametrizations for the vector supermultiplet:

$$\begin{aligned}
F^I &= F_{ab}^I V^a V^b + (f_a^I \bar{\lambda}^{iA} \gamma_a \psi^B \varepsilon_{AB} - f_a^I \bar{\lambda}_A^{i*} \gamma_a \psi_B \varepsilon^{AB}) V^a & (3.2) \\
\nabla \lambda^{iA} &= (\nabla_a \lambda^{iA}) V^a + i P_a^{i*} \gamma^a \psi^A + (G^+)^i_{ab} \gamma^{ab} \psi_B \varepsilon^{AB} + Y^{iAB} \psi_B & (3.3a) \\
\nabla \lambda_A^{i*} &= (\nabla_a \lambda_A^{i*}) V^a + i P_a^{i*} \gamma^a \psi_A + (G^-)^{i*}_{ab} \gamma^{ab} \psi^B \varepsilon_{AB} + Y_{AB}^{i*} \psi^B & (3.3b) \\
dz^i &= Z_a^i V^a + \bar{\psi}_A^i \lambda^{iA} & (3.4a) \\
d\bar{z}^{i*} &= \bar{Z}_a^{i*} V^a + \bar{\psi}^A \lambda_A^{i*} & (3.4b)
\end{aligned}$$

We now introduce a useful nomenclature: given a k-form $\omega^{(k)}$ in superspace, we call its (p,q) sector the part of $\omega^{(k)}$ proportional to $V^{a_1} \dots V^{a_p} \bar{\psi}^{a_1} \dots \bar{\psi}^{a_q}$. Moreover we write the covariant differential of a 0-form ϕ as

$$\nabla \phi \equiv \bar{\psi}^A \nabla_A \phi + \bar{\psi}_A \nabla^A \phi + V^a \nabla_a \phi \tag{3.5}$$

∇^A and ∇_A are called the chiral and antichiral spinor derivatives. The (0,1) part of (3.5) is denoted by

$$(\nabla \phi)_{(0,1)} \equiv \bar{\psi}^A \nabla_A \phi + \bar{\psi}_A \nabla^A \phi$$

In this language, the (0,1) sector of eq.(3.2) defines λ^{iA} and λ_A^{i*} as the spinor derivatives of z^i and $\bar{z}^{i*}$, respectively.

The (0,1) sectors of the eq.s (3.3) have been decomposed on a complete basis of gamma matrices and by i) the spinor derivatives P_a^i , $(G^+)^i_{ab}$, Y^{iAB} must be expressible in terms of the space-time fields λ^{iA} , λ_A^{i*} , z_a^i , $\bar{z}_a^{i*}$ and A^I (and their space-time derivatives $\nabla_a \lambda^{iA}$, $\nabla_a \lambda_A^{i*}$, Z_a^i , $\bar{Z}_a^{i*}$, F_a^I). Here and in the following the superscripts (+) and (-) on an antisymmetric Lorentz tensor denote selfduality and antiselfduality, namely

$$\epsilon^{abcd}(G^+)^i_{cd} = 2i(G^+)^{ijab} \tag{3.6a}$$

$$\epsilon^{abcd}(G^-)^i_{cd} = -2i(G^-)^{ijab} \tag{3.6b}$$

The invariance of the Ansatz (3.2-3) under the global rescaling (3.1) and the local U(1) Kähler transformations determines the corresponding weights of the spinor derivatives (the first entry is the global weight, the second is the U(1) weight):

$$\begin{aligned}
[P_a^i] &= (-1, -1) & [(G^+)^i_{ab}] &= (-1, -1) & [Y^{iAB}] &= (-1, +1) \\
[P_a^{i*}] &= (-1, +1) & [(G^-)^i_{ab}] &= (-1, +1) & [Y_{AB}^{i*}] &= (-1, -1)
\end{aligned} \tag{3.7}$$

In the (1,1) sector of eq. (3.2) the current $\bar{\lambda}^{iA} \gamma^a \psi_B$ and its hermitian conjugate are the only objects compatible with the requirements i), ii), iii). To preserve $\mathcal{M}$ coordinate invariance we have introduced the complex (1,0) and (0,1) covariant vectors f_a^I and f_a^{I*} which, for the time being, are arbitrary functions of the coordinate fields $z^i, \bar{z}^{i*}$. Their scaling assignments are easily deduced to be:

$$[f_a^I] = (+1, +1); \quad [f_a^{I*}] = (1, -1) \tag{3.8}$$

We notice that the F^I parametrizations (3.2) do not contain the (0,2) sectors. This is due to the fact that the definition (2.14a) of the curvature F^I already includes a (0,2) part, namely $\bar{L}^I \varepsilon^{AB} \psi_A \psi_B + c.c.$ Furthermore no $\bar{\psi}^A \gamma_{ab} \psi_B$ or $\bar{\psi}^A \gamma_a \psi_B$ currents can appear in the (0,2) sector of eq. (2.14a); indeed their coefficients should be SO(1,3) and SO(2) tensors constructed out of the physical fields $(z^i, \bar{z}^{i*}, \lambda^{iA}, \lambda_A^{i*})$ such as to match the correct weight assignments, and it is easy to see that no such tensors exist.

We now parametrize the supergravity curvatures. We recall that in absence of matter fields the supergravity curvatures admit the on-shell parametrization [21]:

$$R^a = 0 \tag{3.9}$$

$$\rho_A = \rho_A^{ab} V^a V^b - i(G^+)^{ab} \gamma_b \psi^B V^a \varepsilon_{AB} \tag{3.10a}$$

$$\rho^A = \rho_{ab}^A V^a V^b + i(G^-)^{ab} \gamma_b \psi_A V^a \varepsilon^{AB} \tag{3.10b}$$

$$\begin{aligned}
R^{ab} &= R_{cd}^{ab} V^c V^d - i\bar{\psi}^A (\gamma^a \rho^{b\bar{c}} + \gamma^b \rho^{a\bar{c}} + \gamma^c \rho^{ab\bar{c}} - \gamma^c \rho^{a\bar{c}b}) V^c \\
&\quad - i\bar{\psi}^A (\gamma^a \rho_A^{b\bar{c}} + \gamma^b \rho_A^{a\bar{c}} + \gamma^c \rho_A^{ab\bar{c}}) V_c \\
&\quad - \varepsilon^{AB} \bar{\psi}_A \psi_B (G^-)^{ab} - \varepsilon_{AB} \bar{\psi}^A \psi^B (G^+)^{ab}
\end{aligned} \tag{3.11}$$

$$F = F_{ab} V^a V^b \tag{3.12}$$

$$dQ = 0 \tag{3.13}$$

where $(G^+)^{ab}$ and $(G^-)^{ab}$ are the selfdual and antiselfdual parts of F_{ab} , and the definition of F is given by eq. (2.8d) with $L = \bar{L} = 1$.

For the coupling to vector multiplets we need however the off-shell parametrization. For this purpose we write:

$$R^a = 0 \tag{3.14}$$

$$\rho_A = \rho_A^{(on-shell)} + \rho_{A(1,1)}^{(aux)} \tag{3.15a}$$

$$\rho^A = \rho^{A(on-shell)} + \rho_{(1,1)}^{A(aux)} \tag{3.15b}$$

$$R^{ab} = R^{ab(on-shell)} + R_{(0,2)}^{ab(aux)} \tag{3.16}$$

$$F = F^{(on-shell)} + F_{(1,1)}^{(aux)} \quad (3.17)$$

$$K = K_{(2,0)} + K_{(1,1)} + K_{(0,2)} \quad (3.18)$$

where the extra auxiliary 2-forms $\rho_A^{(aux)}$, $\rho_A^{(aux)}$, $R_{(0,2)}^{ab(aux)}$ must be determined in terms of the physical fields of the matter multiplets.

We set quite generally:

$$\begin{aligned} \rho_{A(1,1)}^{(aux)} &= (M_A^B|_a \psi_B + M_{AB|a} \psi^B) V^a \quad (3.19a) \\ \rho_{(1,1)}^{A(aux)} &= (M^{*A}_{B|a} \psi^B + M^{*AB}_{|a} \psi_B) V^a \quad (3.19b) \\ R_{(0,2)}^{ab(aux)} &= \bar{\psi}^A N_{AB}^{ab} \psi^B + \bar{\psi}^A N_A^{Bab} \psi_B \\ &\quad - \bar{\psi}_A N^{*ABab} \psi_B - \bar{\psi}_A N^{*A}{}^{ab} \psi_B \quad (3.19c) \end{aligned}$$

Notice now that F and F^I have similar definitions (2.8d), (2.14a), and hence similar Bianchi identities. It is therefore convenient to enlarge the range of the index I to $\hat{I}=0,1,\dots,n$ so that

$$F^{\hat{I}} \equiv (F^0, F^I) \quad F^0 \equiv F$$

where $I = 1, \dots, n$ and $\hat{I} = 0, 1, \dots, n$. Similarly we set:

$$L^{\hat{I}} \equiv (L^0, L^I) \quad L^0 \equiv L$$

$$f_i^{\hat{I}} \equiv (f_i^0, f_i^I) \quad f_i^0 \equiv f_i$$

with analogous relations for the conjugate fields.

Then eq. (3.2) with $I \rightarrow \hat{I}$ encompasses the parametrizations of both the graviphoton and the matter vector curvatures.

The Kähler 2-form K has already been fixed in terms of the matter fields by eq. (2.4). In particular using (2.15) we obtain:

$$K_{(2,0)} = ig_{ij} Z_a^i \bar{Z}_b^{j*} V^a V^b \quad (3.20a)$$

$$K_{(1,1)} = ig_{ij} (Z_a^i \bar{\psi}^A \lambda_A^{j*} + \bar{Z}_a^{i*} \bar{\psi}_A \lambda^{iA} V^a) \quad (3.20b)$$

$$K_{(0,2)} = \frac{i}{2} g_{ij} \bar{\lambda}^{iA} \gamma^a \lambda_B^{j*} \bar{\psi}^B \gamma_a \psi_A \quad (3.20c)$$

We note that the supergravitational curvatures (3.14-15) satisfy the constraints:

$$R_{(1,1)}^a = R_{(0,2)}^a = 0 \quad (3.21a)$$

$$\rho_{A(0,2)} = \rho_{(0,2)}^A = 0 \quad (3.21b)$$

Comparing with (3.9) and (3.10) we see these constraints are the same as for pure $N=2$ on-shell supergravity. Notice however that for pure supergravity we have a trivial $U(1)$ connection since $K \equiv dQ = 0$, while for the coupled system Q has been identified by the 1-form (2.3). In fact, if one decomposes the covariant derivative defining ρ_A as follows:

$$\rho_A \equiv \nabla \psi_A = \mathcal{D} \psi_A + \frac{i}{2} Q \psi_A \quad (3.22)$$

then the parametrization of $\mathcal{D} \psi_A$ contains the extra term $-\frac{i}{2} Q \psi_A$ (besides $\rho_{A(1,1)} + \rho_{A(1,1)}^{(aux)}$). This extra term contributes to the $(0,2)$ sector with $\frac{1}{4} \psi_A (G_i \bar{\psi}_A \lambda^{iA} - G_i \bar{\psi}^A \lambda_A^{i*})$. Thus to set $\rho_{A(0,2)} \equiv (\nabla \psi_A)_{(0,2)} = 0$ is equivalent to fix in a particular way the $(0,2)$ sector of $\mathcal{D} \psi_A$.

4 - Bianchi identities

Using the gravitational Bianchi identities we now determine the auxiliary 2-forms $\rho_{A(1,1)}^{(aux)}$ and $R_{(0,2)}^{ab(aux)}$.

Using the Bianchi identity (2.20a) in the (1,2) sector we can easily determine the gamma-matrix structure and the Lorentz and O(2) index structure of the M and N matrices. We find:

$$M_A^B|_a = A_A^B \delta_a^b + A'_A{}^B|_a \gamma^{ab} \quad (4.1a)$$

$$M_{AB}|_a = (iS_{AB}\eta_{ab} + \varepsilon_{AB}(T^+)_{ab})\gamma^b \quad (4.1b)$$

$$N_{AB}^{ab} = S_{AB}\gamma^{ab} - i\varepsilon_{AB}T^{ab} \quad (4.2a)$$

$$N_A^{B|ab} = A'_A{}^B|_c \gamma^c \varepsilon^{abc} f \quad (4.2b)$$

where A_A^B , $A'_A{}^B$ are U(2) - Lie algebra valued vector fields, T^{ab} is a complex field antisymmetric in a,b, and S_{AB} is a complex scalar field symmetric in the O(2) indices A,B.

The explicit dependence of the auxiliary fields A_A^B , $A'_A{}^B$, T_{ab} and S_{AB} on the matter fields can be found in the analysis of the gravitino and graviphoton Bianchi identities (2.20b) and (2.20d). The sector (0,3) of the Bianchi identity (2.20b) takes the form:

$$\begin{aligned} [M_A^B|_a \psi_B + M_{AB}|_a - i(G^+)_{ab} \gamma^b \varepsilon_{AB}) \psi^B] i\tilde{\psi}^C \gamma^a \psi_C + \\ + \frac{1}{4} \gamma_{ab} \psi_A R_{(0,2)}^{ab} - \frac{i}{2} \psi_A K_{(0,2)} = 0 \end{aligned} \quad (4.3)$$

where we have used

$$\rho_{(1,1)} = \rho(1,1)^{(on-shell)} + \rho_{(1,1)}^{(aux)} = -i(G^+)_{ab} \gamma^b \varepsilon_{AB} \psi^B + \rho_{(1,1)}^{(aux)} \quad (4.4)$$

and M_A^B , M_{AB} , $R_{(0,2)}^{ab}$ and $K_{(0,2)}$ are given by eq.s (4.1), (3.19c), (4.2) and (3.20c) respectively.

Equation (4.3) can be further analysed in the (0,3) subsectors:

- i) $\psi_A \psi_B \psi_C$
- ii) $\psi_A^A \psi_B \psi_C$
- iii) $\psi^A \psi_B \psi_C$

In the first subsector the equation (4.3) implies

$$S_{AB} = 0; \quad (T^+)^* = 0 \Rightarrow T^* \equiv (T^-)^* \quad (4.5)$$

The subsector ii) is also satisfied by eq.s (4.5). The subsector iii) gives the equation:

$$\begin{aligned} [i\gamma_{ma} A'_L{}^M \delta_A^B - i\delta_L^M (A_A^B|_a + A'_A{}^B|_a \gamma^{ab}) - \\ - \frac{1}{4} g_{kl} \bar{\lambda}_L^{k*} \gamma_a^{lm} \delta_A^B] \psi_B \tilde{\psi}^L \gamma^a \psi_M = 0 \end{aligned} \quad (4.6)$$

To analyse its content one decomposes the monomial $\psi_B \tilde{\psi}^L \gamma^a \psi_M$ into representations of the $SO(1,3) \times SO(2)$ group according to the formula

$$\begin{aligned} \psi_B \tilde{\psi}^L \gamma^a \psi_M = -\frac{i}{2} (\Xi^+)_{BLM}^a - \frac{3}{8} i\delta_{(BM)} (\Xi^+)_{L}^a + \frac{1}{6} \varepsilon_{B(L} (\Xi^-)_{M)}^a + \frac{1}{4} \varepsilon_{ML} (\Xi^-)_{B}^a + \\ + \frac{1}{16} \varepsilon_{ML} \gamma^a ((\Xi^+)_{B}^A - i(\Xi^-)_{B}^A) - \frac{1}{8} \gamma^a (\varepsilon^{B(M} (\Xi^+)_{B}^{C)} - i\varepsilon^{B(M} (\Xi^-)_{B}^{C)}) \end{aligned} \quad (4.7)$$

where $(\Xi^\pm)_{ABC}$ is symmetric and traceless in the O(2) indices. In eq. (4.7) the $\Xi^\pm$ denote irreducible spinor representations of $SO(1,3) \times SO(2)$ and the superscripts + and - denote their $SO(3)$ parity. Irreducibility implies:

$$\gamma_a \Xi_{ABC}^\pm = \gamma_a (\Xi^\pm)_C^\pm = \gamma_a (\Xi^\pm)_C^\pm = 0 \quad (4.8)$$

Projecting the l.h.s. of eq. (4.6) along the independent irreducible representations $\Xi_{ABC}^\pm$, $(\Xi^+)_{A}^a$, $(\Xi^-)_{A}^a$ and $(\Xi^\pm)_A - i(\Xi^\mp)_A$ we find that the auxiliary fields are completely determined. Precisely we have:

$$A_A^B|_a = -\frac{i}{4} g_{kl} (\bar{\lambda}_A^{k*} \gamma_a^{lm} \lambda^B - \delta_A^B \bar{\lambda}_P^{k*} \gamma_a \lambda^P) \quad (4.9)$$

$$A'_A{}^B|_a = +\frac{i}{4} g_{kl} (\bar{\lambda}_A^{k*} \gamma_a^{lm} \lambda^B - \frac{1}{2} \delta_A^B \bar{\lambda}_P^{k*} \gamma_a \lambda^P) \quad (4.10)$$

so that $A_A^B|_a$ and $A'_A{}^B|_a$ are U(2) and SU(2) Lie algebra valued, respectively.

To complete the solution of the Bianchi identities (2.20a) and (2.20b) we must still analyse the (2,1) sector of (2.20a) and the (1,2) and (2,1) sectors of (2.20b). The former determines $R_{(1,1)}^{ab}$ in terms of $\rho_{A|ab}$ and ρ^A_{ab} giving exactly the same result as for pure supergravity ((1,1) sector of eq. (3.11)), so that $R_{(1,1)}^{ab} = R_{(1,1)}^{ab}(on shell)$, while the latter are uninteresting since they involve the spinor derivatives of the auxiliary fields and of $\rho_{A|ab}$. Recalling eq.s (3.3) and (3.4) we see that the spinor derivative of the auxiliary fields can be computed from the transformation rules of z^i, λ^{iA} and their conjugates; the spinor derivative of $\rho_{A|ab}$ on the other hand can be obtained by covariant differentiation of the $\delta\psi_A$ transformation rule.

Moreover the Bianchi identity of R^{ab} , eq. (2.20e), is automatically satisfied once the Bianchi identities of the physical fields V^a, ψ_A, ψ^A and F are satisfied.

We now extract some consequences from the scalar field Bianchi identity $d^2 z^i = 0$. Using the parametrization (3.4a) we find:

Substituting for $\rho_{B(1,1)}$ its value given by eqs. (3.15a), (3.19a), (4.1a), (4.1b)) and taking into account that only the auxiliary field $A_A^B|_a$ contributes to this subsector we obtain:

$$A_C^C|_a \bar{L}^I = \frac{i}{4} \bar{\lambda}^{\bar{C}} \gamma_a \lambda^{jC} \nabla_j f_{i*}^I \quad (4.19)$$

$$\nabla_i \bar{L}^I = 0 \quad (4.20)$$

$$\nabla_{i*} \bar{L}^I = f_{i*}^I \quad (4.21)$$

Eq. (4.20) tells us that $\bar{L}^I$ is a U(1)-covariantly antiholomorphic function of $z^i, \bar{z}^{i*}$ and eq. (4.21) identifies the (0,1) vector f_{i*}^I with the (0,1) covariant derivative of $\bar{L}^I$. With these informations eq. (4.19) becomes an identity; indeed taking the ∇_j derivative of eq. (4.21) and recalling eq. (2.13b) one has

$$\nabla_j f_{i*}^I = \nabla_j \nabla_{i*} \bar{L}^I = g_{ij*} \bar{L}^I \quad (4.22)$$

Comparing the explicit value of $A_C^C|_a$ given by (4.9) with the r.h.s. of eq. (4.19), we see that (4.19) is identically satisfied.

In the complex conjugate subsector $\psi^A \psi^B V^a$ we find relations which are the complex conjugates of eq.s (4.19-20-21). Therefore we conclude that the U(1) charged scalar functions $L^I, \bar{L}^I$ are covariantly holomorphic and antiholomorphic, respectively, and that f_i^I, f_{i*}^I are their covariant holomorphic and antiholomorphic derivatives.

It is important to realize that the Kähler potential $G(z^i, \bar{z}^{i*})$ is actually determined by eq. (4.20) and its complex conjugate. Indeed recalling eq.s (2.11) we have

$$\partial_{i*} L^I = \frac{1}{2} \partial_{i*} G \bar{L}^I \quad (4.23a)$$

$$\partial_i \bar{L}^I = \frac{1}{2} \partial_i G \bar{L}^I \quad (4.23b)$$

The solution for G is given by

$$G = \ln L^I \bar{L}^I + H^I(z) + \bar{H}^I(\bar{z}) \quad (4.24)$$

where $\bar{I}$ is any value 0,1,...,n and

$$\ln L^I + H^I(z) = \ln \bar{L}^I + \bar{H}^I(\bar{z}) \quad (4.25)$$

with H^I = arbitrary analytic functions. Choosing the index $\bar{I} = 0$ and $H^0 = 0$ for simplicity (this is just a "gauge choice", since the metric is insensitive to the value of H^I in (4.24)), we obtain:

$$G = \ln L^0 \bar{L}^0 = \ln(L^0)^2 \quad (4.26)$$

$$0 = d^2 z^i = d(Z_a^i V^a + \bar{\psi}_A \lambda^{iA}) = \mathcal{D} Z_a^i V^a V^b + (\bar{\psi}^A \nabla_A Z_a^i + \bar{\psi}_A \nabla^A Z_a^i) V^a + i Z_a^i \bar{\psi}^A \gamma_a \psi_A - \bar{\psi}_A \nabla \lambda^{iA} + \bar{\lambda}^{iA} \rho_A \quad (4.11)$$

Inserting the parametrizations of $\nabla \lambda^{iA}$ and ρ_A given in eq.s (3.3a) and (3.19), (4.1) into eq. (4.11) we obtain in the (1,1) sector:

$$\nabla_A Z_a^i = -\varepsilon_{AB} ((T^+)_{ab} - i(G^+)_{ab}) \gamma^b \lambda^{iB} \quad (4.12)$$

$$\nabla^A Z_a^i = \nabla_a \lambda^{iA} - (A_B^A \delta_a^b - A'^A_B \gamma^{ab}) \lambda^{iB} \quad (4.13)$$

In the (0,2) sectors of (4.11) we find:

$$P_a^i = Z_a^i \quad (4.14)$$

$$Y^{iAB} = Y^{iBA} \quad (4.15)$$

We now turn to the vector Bianchi identities. Inserting the parametrization (3.2) into (2.21c) and setting $I \rightarrow \hat{I}$ leads to:

$$\begin{aligned} \varepsilon^{AB} \bar{\psi}_A \psi_B \nabla \bar{L}^{\hat{I}} + \varepsilon_{AB} \bar{\psi}^A \psi^B \nabla L^{\hat{I}} - 2\varepsilon^{AB} \bar{\psi}_A \rho_B - 2\varepsilon_{AB} \bar{\psi}^A \rho_B = \\ - 2i F_{ab}^{\hat{I}} \bar{\psi}_B \gamma^b \psi^B V^a + i \varepsilon_{AB} f_{i*}^{\hat{I}} \bar{\psi}^B \gamma_a (\nabla \lambda^{iA})_{(0,1)} V^a + i \varepsilon^{AB} f_{i*}^{\hat{I}} \bar{\psi}_B \gamma^a (\nabla \lambda^{i*})_{(0,1)} V^a \\ + i \nabla f_{i*}^{\hat{I}} \varepsilon_{AB} \lambda^{iA} \gamma_a \psi^B V^a + i \nabla f_{i*}^{\hat{I}} \varepsilon^{AB} \bar{\lambda}^{iA} \gamma_a \psi_B V^a \end{aligned} \quad (4.16)$$

The ∇ derivatives on $f_{i*}^{\hat{I}}$ and complex conjugate can be expanded as:

$$\nabla f_{i*}^{\hat{I}} = \nabla_j f_{i*}^{\hat{I}} dz^j + \nabla_{j*} f_{i*}^{\hat{I}} d\bar{z}^{j*} \quad (4.17a)$$

$$\nabla f_{i*}^{\hat{I}} = \nabla_j f_{i*}^{\hat{I}} dz^j + \nabla_{j*} f_{i*}^{\hat{I}} d\bar{z}^{j*} \quad (4.17b)$$

and similarly for $\nabla L^{\hat{I}}, \nabla \bar{L}^{\hat{I}}$. The coordinate differentials $dz^j, d\bar{z}^{j*}$ are given by eq.s (3.4). Substituting into eq.s (4.16), and using the parametrizations (3.3) for $(\nabla \lambda^{iA})_{(0,1)}$ and $(\nabla \lambda^{i*})_{(0,1)}$, and the identifications (4.14-15), we obtain an equation which we analyse in its various sectors. The most informative is the (1,2) sector. In the $\bar{\psi}^A \psi^B V^a$ subsector we find:

$$\begin{aligned} - 2\varepsilon^{AB} \bar{\psi}_A \rho_B \bar{L}^{\hat{I}} + \varepsilon^{AB} \bar{\psi}_A \psi_B (\nabla_i \bar{L}^{\hat{I}} Z_a^i + \nabla_{i*} \bar{L}^{\hat{I}} \bar{Z}_a^{i*}) V^a \\ - f_{i*}^{\hat{I}} \bar{Z}_a^{i*} \bar{\psi}_A \psi_B \varepsilon^{AB} V^a + \frac{i}{2} \bar{\lambda}_A^{i*} \gamma^a \lambda^{jC} \nabla_j f_{i*}^{\hat{I}} \bar{\psi}_C \psi_B \varepsilon^{AB} V^a = 0 \end{aligned} \quad (4.18)$$

where the second equality is a consequence of (4.25) when $H^0 = 0$.

We notice also that the condition $\nabla_i L^i = 0$ can be rewritten as:

$$e^{\frac{\sigma}{2}} \partial_i (e^{-\frac{\sigma}{2}} L^i) = 0 \quad (4.27)$$

so that $e^{-\frac{\sigma}{2}} L^i$ are true holomorphic functions of the z^i fields. Therefore the n scalar fields L^i/L^0 are n holomorphic functions of the coordinates. The usual special gauge of the tensor calculus approach can be understood as the identification of these analytic functions with the coordinates of $\mathcal{M}$:

$$\frac{L^i}{L^0}(z^i) \equiv z^i \quad (\text{special gauge}) \quad (4.28)$$

This will be discussed in Sect. 6. Next we analyse the remaining subsectors.

In the subsector $\bar{\psi}_A \gamma_{ab} \psi_B V^b$ of (4.16) we find:

$$\begin{aligned} & [-2i F_{ab}^i + 4i f_i^j (G^+)^i_{ab} - 2((T^+)^i_{ab} - i(G^+)^i_{ab}) \bar{L}^j] \delta_B^C \\ & + [i f_i^j Y^{iAC} \eta_{ab} - \frac{i}{2} \nabla_j f_i^j \lambda^{iA} (\eta_{ab} + \gamma_{ab}) \lambda^{jC}] \varepsilon_{AB} + \text{c.c.} = 0 \end{aligned} \quad (4.29)$$

Let us consider the selfdual projection of this equation in the antisymmetric indices a,b: the resulting equation is different in the cases $\bar{I} = 0$ and $\bar{I} = 1, \dots, n$. Indeed if $\bar{I} = 0$ we have $(F^+)^0_{ab} \equiv (F^+)^i_{ab} = (G^+)^i_{ab}$ since $(G^+)^i_{ab}$ appearing in the parametrizations (3.10-11) coincides with the selfdual part of the graviphoton field strength. On the other hand the $(G^+)^i_{ab}$ are quantities yet to be determined. The selfdual projection of (4.29), in the cases $\bar{I} = 0, \bar{I} = 1, \dots, n$, becomes respectively:

$$(T^+)^i_{ab} = \frac{1}{2\bar{L}^0} [-2i(G^+)^i_{ab}(1 - \bar{L}^0) + 4i f_i^j (G^+)^i_{ab} - \frac{i}{4} \nabla_j f_i^j \bar{\lambda}^{iL} \gamma_{ab} \lambda^{jM} \varepsilon_{LM}] \quad (4.30)$$

$$(f_i^j - \frac{\bar{L}^j}{\bar{L}^0} f_i)(G^+)^i_{ab} = -\frac{1}{2} \frac{\bar{L}^j}{\bar{L}^0} (G^+)^i_{ab} + \frac{1}{2} (F^+)^j_{ab} + \frac{1}{16} (1 - \frac{\bar{L}^j}{\bar{L}^0}) \nabla_j f_i^j \bar{\lambda}^{iL} \gamma_{ab} \lambda^{jM} \varepsilon_{LM} \quad (4.31)$$

The antiselfdual projection gives the corresponding complex conjugate equations for $(T^-)^i_{ab}$ and $(G^-)^i_{ab}$.

The (B,C)-antisymmetric part of eq. (4.29) gives the relation

$$\nabla_{[i} f_{j]}^i \equiv \nabla_{[i} \nabla_{j]} L^j = 0$$

which is satisfied on a Kähler manifold.

Finally we examine the (a,b)-symmetric content of eq. (4.29). One has:

$$i f_i^j Y^{iAC} \varepsilon_{AB} + i f_i^j Y^{i*AC} \varepsilon_{AB} - \frac{i}{2} \nabla_j f_i^j \bar{\lambda}^{iA} \lambda^{jC} \varepsilon_{AB} - \frac{i}{2} \nabla_j f_i^j \bar{\lambda}^{i*A} \lambda^{j*C} \varepsilon^{AC} = 0 \quad (4.32)$$

Taking into account the scaling weights of Y^{iAC} , Y_{AC}^{i*} (cf. eq. (3.7)) and the symmetry in the O(2) indices (A,C) we can write the following Ansatz for the so far undetermined auxiliary fields of the vector multiplets:

$$Y^{iAC} = C^i_{jk} \bar{\lambda}^{jA} \lambda^{kC} + C^i_{j^*k^*} \bar{\lambda}^{j*A} \lambda^{k*C} \varepsilon^{CM} \quad (4.33a)$$

$$Y_{AC}^{i*} = C^i_{jk} \bar{\lambda}^{jL} \lambda^{kM} \varepsilon_{AL} \varepsilon_{CM} + C^i_{j^*k^*} \bar{\lambda}^{j*L} \lambda^{k*M} \varepsilon_{AC} \quad (4.33b)$$

where C^i_{jk} , $C^i_{j^*k^*}$, $C^{i*}_{jk} \equiv (C^i_{j^*k^*})^*$, and $C^{i*}_{j^*k^*} \equiv (C^i_{jk})^*$ are tensors on $\mathcal{M}$ of the type indicated by their indices. We note that the symmetry in the O(2) indices of Y^{iAC} and Y_{AC}^{i*} implies symmetry in the lower two indices of the C -tensors:

$$C^i_{jk} = C^i_{kj} \quad (4.34a)$$

$$C^i_{j^*k^*} = C^i_{k^*j^*} \quad (4.34b)$$

and the complex conjugate relations. Their U(1) weights are:

$$\begin{aligned} [C^i_{jk}] &= 0 & [C^i_{j^*k^*}] &= -2 \\ [C^{i*}_{jk}] &= 0 & [C^{i*}_{j^*k^*}] &= +2 \end{aligned} \quad (4.35)$$

Inserting the Ansatz (4.33) into eq. (4.32) results in the following relations between the C -tensors and the f_i^j -vectors:

$$f_i^j C^i_{j^*k^*} + f_i^j C^i_{j^*k^*} = \frac{1}{2} \nabla_{(j^*} f_{k^*)}^j \quad (4.36a)$$

$$f_i^j C^i_{jk} + f_i^j C^i_{jk} = \frac{1}{2} \nabla_{(j} f_{k)}^j \quad (4.36b)$$

We have thus exhausted the analysis of the (1,2) sector of the F^i Bianchi identities. The remaining superspace sectors are uninteresting: indeed in the (0,3)-sector one finds again eqs. (4.20-21), while the (2,1) sector determines the spinor derivative (and therefore the supersymmetry transformations) of the F^i field strengths in terms of the other fields. However this is not a new information, since the supersymmetry transformation of A^i is already known from the F^i parametrizations.

The last Bianchi identity to be analysed is the gaugino Bianchi identity, eq. (2.21b). In the next Section we show how the restriction on the geometry of $\mathcal{M}$ comes out as a consequence of this Bianchi identity.

5 - The gaugino Bianchi identity and the special geometry of $\mathcal{M}$

In the gaugino Bianchi identity (2.21b) only the (0,2) sector is relevant, since the (1,1) sector just determines the spinor derivative of $\nabla_a \lambda^{iA}$. To find the explicit form of the (0,2) sector we need the spinor derivatives of the fields Z_a^i , $(G^+)^i_{ab}$ and Y^{iAB} , appearing in the parametrization of $\nabla \lambda^{iA}$, eq. (3.3a).

The spinor derivatives of Z_a^i are given in eq.s (4.12-13) and the spinor derivatives of the auxiliary field Y^{iAB} are computed presently. The spinor derivative of $(G^+)^i_{ab}$ can be obtained from eq. (4.31); however its explicit form will not be needed.

After use of the parametrizations of $\rho_{(1,1)}$, $R_{(0,2)}^{ab}$, $K_{(0,2)}$ and $(dz^i)_{(0,1)}$ given in Sect. 3, and some Fierz rearrangements, the (0,2) sector of eq. (2.21b) yields three equations corresponding to the subsectors $\bar{\psi}_A \psi_B$, $\bar{\psi}_A \gamma_{ab} \psi_B$ and $\bar{\psi}^A \gamma^m \psi_B$, respectively:

$$\gamma^{ab} \varepsilon^{A(B} \nabla^{C)} (G^+)^i_{ab} + \nabla^{(C} Y^{B)Ai} = 0 \quad (5.1)$$

$$\gamma^{ab} \gamma^{mn} \varepsilon^{A(B} \nabla^{C)} (G^+)^i_{ab} + \gamma^{mn} \nabla^{(C} Y^{B)Ai} = 0 \quad (5.2)$$

$$\begin{aligned} & 2i \nabla_m \lambda^{i(A} \delta_M^{L)} + \frac{i}{2} \gamma^m \nabla \lambda^{iL} \delta_M^A - \frac{i}{2} \varepsilon^{AL} \gamma^m \gamma_m \nabla_M (G^+)^i_{ab} \\ & - \frac{1}{2} \gamma_m \nabla_M Y^{ALi} + \frac{i}{2} \gamma^e \gamma_m (A^L_B|_a - 3A'^L_B|_a) \delta_M^A \lambda^{iB} \\ & - \left(\frac{1}{2} R^i_{jk*} + \frac{1}{8} \delta^i_{jk*} \right) \gamma_m \lambda_M^{k*} \bar{\lambda}^{iL} \lambda^{jA} \\ & + \left(\frac{1}{8} R^i_{jk*} + \frac{1}{32} \delta^i_{jk*} \right) \gamma^e \gamma_m \lambda_M^{k*} \bar{\lambda}^{iL} \gamma_{ab} \lambda^{jA} - i \gamma_{am} \lambda^{iA} A'^L_M|_a = 0 \end{aligned} \quad (5.3)$$

We note that the two further sectors $\bar{\psi}^A \psi^B$ and $\bar{\psi}^A \gamma_{ab} \psi^B$ are absent since the terms proportional to $\bar{\psi}^A \psi^B$ cancel identically, while the single term in the sector $\bar{\psi}^C \gamma_{lm} \psi^D$:

$$-\frac{i}{8} \gamma^a \gamma^{lm} \gamma^b \lambda^{iB} \varepsilon_{CB} ((T^+)^i_{ab} - i(G^+)^i_{ab}) \psi^C \gamma^{lm} \psi^A$$

is identically zero because of the antiselfduality of $\psi^C \gamma^{lm} \psi^A$.

We first consider eq.s (5.1-2). An immediate consequence is

$$\nabla^{(C} Y^{AB)i} = 0 \quad (5.4)$$

After contraction of (5.2) with γ_{mn} , we can eliminate $\nabla^C (G^+)^i_{ab}$ by taking appropriate linear combinations of eq.s (5.1), (5.2) and contractions of the O(2) indices, obtaining:

$$\nabla^L Y_L^{Ai} + \frac{1}{2} \nabla^A Y_L^{Li} = 0 \quad (5.5)$$

The implications of the two constraints (5.4-5) are easily analysed by using the explicit form of Y^{ABi} as given in eq. (4.33a). To compute the spinor covariant derivative $\nabla^A Y^{BCi}$ from eq. (4.33a) we use

$$\begin{aligned} (\nabla C^i_{jk})_{(0,1)} &= \nabla_l C^i_{jk} (dz^l)_{(0,1)} + \nabla_l C^i_{jk} (dz^l)^*_{(0,1)} \equiv \\ & \nabla_l C^i_{jk} \bar{\psi}_A \lambda^{lA} + \nabla_l C^i_{jk} \bar{\psi}^{lA} \lambda_A^{i*} \end{aligned} \quad (5.6)$$

and similarly for the other C -tensors; furthermore, using the (0,1)-sector of $\nabla \lambda^{iA}$ and $\nabla \lambda^{i*A}$ given by eq.s (3.3), we find from eq. (5.4):

$$(\nabla_l C^i_{jk}) - C^i_{m(l} C^m_{jk}) \lambda^{lA} \bar{\lambda}^{jB} \lambda^{kC} = 0 \quad (5.7)$$

$$\nabla_l C^i_{j^*k^*} \lambda^{lA} \bar{\lambda}^{j*} \lambda_M^{k*} \varepsilon^{BLCM} = 0 \quad (5.8)$$

where complete symmetrization of the indices A,B,C is understood.

Similarly from eq. (5.5) one finds:

$$\nabla_l C^i_{j^*k^*} (3\lambda^{lB} \bar{\lambda}^{j*} \lambda_R^{k*} - 2\lambda^{lL} \bar{\lambda}^{j*} \lambda_B^{k*}) = 0 \quad (5.9)$$

$$(\nabla C^i_{jk} - 2C^i_{jm} C^m_{kl}) \bar{\lambda}^{B(l} \bar{\lambda}^{j)l} \lambda^{kL} = 0 \quad (5.10)$$

The common solution of these equations is

$$C^i_{jk} = 0 \quad (5.11)$$

$$\nabla_l C^i_{j^*k^*} = 0 \quad (5.12)$$

implying also the complex conjugate relations

$$C^{i*}_{j^*k^*} = 0 \quad (5.13)$$

$$\nabla_l C^{i*}_{j^*k^*} = 0 \quad (5.14)$$

Eq. (4.36b), after setting $C^i_{jk} = 0$, becomes

$$f_{i*}^j C^{i*}_{jk} = \frac{1}{2} \nabla_{(j} f_{k)}^i \quad (5.15)$$

Inverting the $n \times n$ submatrix f_{i*}^j , we find

$$C^{i*}_{jk} = \frac{1}{2} f_{i*}^l \nabla_{(j} f_{k)}^l \quad (5.16)$$

where $f_{i*}^j = (f_{i*}^j)^{-1}$. Similarly from (4.36a) one obtains the complex conjugate relation

$$C^i_{j^*k^*} = \frac{1}{2} f_{i*}^l \nabla_{(j^*} f_{k^*)}^l \quad (5.17)$$

Substituting the solution (5.16) into the $\hat{I} = 0$ equation of system (5.15) yields the consistency condition

$$f_i^0 f_j^* \nabla_i^* \nabla_j^* f_k^I = \nabla_i f_k^0 \quad (5.18)$$

Eq. (5.18) is a complex constraint relating the $2(n+1)$ vectors $f_i^0, f_i^*, f_i^I, f_i^I$.

We now analyse the consequences of eq. (5.12). By lowering the upper index and taking the complex conjugate we find that C_{ijk} is $U(1)$ -covariantly constant

$$\nabla_i C_{ijk} = 0 \quad (5.19)$$

i.e. C_{ijk} is a holomorphic section of a $U(1)$ bundle with $U(1)$ connection $Q_i = -\frac{i}{2} \partial_i G$. Substituting the solution for C_{ijk}^* given by (5.16) into (5.19) yields an integrability condition for the f_i^I -vectors:

$$\begin{aligned} 0 &= 2\nabla_i C_{ijk} = \nabla_i \{f_j^* \nabla_i^* \nabla_j^* f_k^I\} = \\ &= \nabla_i f_j^* \nabla_i^* \nabla_j^* f_k^I + f_j^* \nabla_i^* \nabla_i \nabla_j^* f_k^I = \\ &= \nabla_i f_j^* \nabla_i^* \nabla_j^* f_k^I + \frac{1}{2} f_j^* (\nabla_j \nabla_i^* f_k^I + \nabla_k \nabla_i^* f_j^I) \\ &\quad + \frac{1}{2} f_j^* \{[\nabla_i^*, \nabla_j^*] f_k^I + [\nabla_i^*, \nabla_k^*] f_j^I\} \end{aligned} \quad (5.20)$$

Recalling that ∇_i and ∇_i^* are covariant with respect to general holomorphic and antiholomorphic coordinate transformations on $\mathcal{M}$, and also with respect to $U(1)$ -Kähler transformations, we have:

$$[\nabla_i^*, \nabla_j^*] f_k^I = 2R_{kij}^m f_m^I + g_{ij} f_k^I \quad (5.21)$$

Moreover

$$\nabla_j \nabla_i^* f_k^I = \nabla_j (g_{ik} L^I) = g_{ik} f_j^I \quad (5.22)$$

Therefore eq. (5.20) becomes

$$\nabla_i f_j^* \nabla_i^* \nabla_j^* f_k^I + f_j^* \nabla_i^* \nabla_i \nabla_j^* f_k^I + 2R_{kij}^m f_m^I + f_j^* \nabla_i^* \nabla_i f_k^I = 0 \quad (5.23)$$

Multiplying by $f_j^* f_i^K$ and using

$$f_i^K \nabla_i^* f_j^* = -f_j^* \nabla_i^* f_i^K \quad (5.24)$$

we find

$$R_{kij}^I = -g_{i(j} \delta_{k)}^I + \frac{1}{2} f_i^* \nabla_i^* f_j^I f_j^* \nabla_i^* f_k^I \quad (5.25)$$

or

$$R_{jir}^I = -g_{i(j} \delta_{r)}^I + 2C_{ir}^i C_{kj}^* \quad (5.26)$$

Eq. (5.26) is a constraint on the Riemann tensor of $\mathcal{M}$. The Kähler geometry restricted by this constraint is called "special Kähler geometry".

Lowering the index i with by means of the metric we can rewrite (5.26) as:

$$R_{i^* j^* k} = -g_{i^*(j} g_{k)^*} + 2 C_{i^* i^* s} C_{ikj} g^{s^* i} \quad (5.27)$$

We see that the symmetry of the Riemann tensor in the (j, k) and (i^*, i^*) indices implies the same symmetry on the r.h.s. Taking into account the original symmetry of the C -tensors in the lower two indices (eq. (4.34)), we conclude that C_{ijk} and $C_{i^* j^* k^*}$ are completely symmetric (3,0) and (0,3) tensors respectively.

Furthermore, from the Bianchi identity of the Riemann curvature two-form:

$$\nabla R^i_j = 0 \quad (5.28)$$

that is

$$\nabla_m R^i_{jkl} dz^m dz^k dz^l = 0 \quad (5.29a)$$

$$\nabla_m R^i_{jkl} dz^m dz^k dz^l = 0 \quad (5.29b)$$

we find

$$\nabla_{[m} C_{ijk]} = 0 \quad (5.30a)$$

$$\nabla_{[m} C_{i^* j^* k^*]} = 0 \quad (5.30b)$$

Owing to the complete symmetry of C_{ijk} and $C_{i^* j^* k^*}$, the relations (5.30) imply that the tensor C_{ijk} ($C_{i^* j^* k^*}$) is the covariant derivative of a function S (S^*) of $U(1)$ -Kähler weight $= 2$ (-2) [7]:

$$C_{ijk} = \nabla_i \nabla_j \nabla_k S \quad (5.31)$$

Notice that the constraint (5.27) can be rewritten as

$$R_{i^* j^* k} = -g_{i^*(j} g_{k)^*} + \frac{1}{2} e^{2G} T_{i^* i^* s}(\bar{z}) T_{tkj}(z) g^{s^* i} \quad (5.32)$$

in terms of a truly holomorphic tensor

$$T_{ijk}(z) = 2 e^{-G} C_{ijk} \quad (5.33)$$

Indeed recalling that C_{ijk} has $U(1)$ -weight $= 2$ we have

$$\nabla_i C_{ijk} \equiv (\partial_i - G_i) C_{ijk} = e^G \partial_i (e^{-G} C_{ijk}) = 0 \quad (5.34)$$

so that T_{ijk} depends only on z . Eq. (5.32) is now identical to the constraint found in the non-covariant approach of refs [1,2a] where, however, T_{ijk} is the third ordinary derivative of an analytic function, and therefore not a "good" $\mathcal{M}$ tensor. In the special

gauge (4.36) our T_{ijk} will indeed reduce to the third ordinary derivative of an analytic function, as we show in the next Section.

We note that the consistency condition (5.18) implies no further restriction on the geometry of $\mathcal{M}$. Indeed taking the ∇_{I^*} -derivative of both sides gives:

$$2f_m^0 C_{I^* I^*}^m C^{i^* i^*}_{jk} = \frac{1}{2} \nabla_{I^*} \nabla_{(j} f_{k)} \quad (5.35)$$

where we have used eq.s (5.18-19). Commuting the covariant derivatives in (5.35) yields

$$2 f_m^0 C_{I^* I^*}^m C^{i^* i^*}_{jk} = f_{(j}^0 g_{k)}^{i^* i^*} + R_{k^* i^*}^j f_{j^*}^0 \quad (5.36)$$

which is identically satisfied by virtue of the constraint (5.27).

To complete our analysis we still have to show that the (0,2) subsector proportional to $\bar{\psi}^L \gamma_m \psi_M$ of the gaugino Bianchi identity does not imply any further restriction on the geometry of $\mathcal{M}$.

We multiply eq. (5.3) by γ_m , thus obtaining:

$$\begin{aligned} & 2i \nabla_L \lambda^{(A} \delta_M^{L)} - 2 \nabla_M Y^{iAL} - i \gamma^a (A_B^L|_a - 3 A_B^L|_a) \delta_M^A \lambda^{iB} - \\ & - 3i \gamma^a \lambda^{iA} A_M^L|_a + (2R_{jk^*i}^i + \frac{1}{2} \delta_j^i g_{k^*i}) \lambda_M^{k^*} \bar{\lambda}^{kL} \lambda^{Aj} = 0 \end{aligned} \quad (5.37)$$

If we antisymmetrize in A, L and use the values of $A_B^L|_a$ and $A'^B_A|_a$ given by eq.s (4.9-10) we find $0 = 0$. Tracing (A,L) and (M,L) we obtain two equations combining which $\nabla \lambda^{iA}$ can be eliminated:

$$\begin{aligned} & -6 \nabla_M Y^{iL}_L + 4 \nabla_L Y^{iLM} + 6(-R_{jk^*i}^i + \frac{1}{4} \delta_j^i g_{k^*i}) \lambda_M^{k^*} \bar{\lambda}^{kL} \lambda^{jL} \\ & - 4(-R_{jk^*i}^i + \frac{1}{4} \delta_j^i g_{k^*i}) \lambda_L^{k^*} \bar{\lambda}^{iL} \lambda^{jM} + 2i \gamma^a A_L^M|_a \lambda^{iL} - 9i \gamma^a A_M^L|_a \lambda^{iL} = 0 \end{aligned} \quad (5.38)$$

We compute now the spinor derivatives of Y^{iAB} as before and substitute the values of $A_L^M|_a$ and $A'^L_M|_a$ given in eq.s (4.9-10). After Fierz rearrangements we find that all the terms can be grouped as coefficients of the following trilinear gaugino structures:

- i) $2\lambda_M^{[i} \bar{\lambda}^{k^*]} \lambda_R^{j^*} + \frac{1}{4} \gamma^{ab} \lambda_M^{(k^*} \bar{\lambda}^{i^*]} \gamma_{ab} \lambda_R^{j^*}$
- ii) $\lambda_M^{k^*} \bar{\lambda}^{iL} \lambda^{jL}$
- iii) $\lambda_P^{k^*} \bar{\lambda}^{iM} \lambda^{jP}$

From i) we obtain the condition (5.30), and from ii) and iii) we find again the constraint (5.26).

We conclude this Section by observing that the Bianchi identities also imply the equations of motion of the physical fields. Indeed from eq. (5.37), for example, we can retrieve the gaugino equation of motion. A supersymmetry transformation (to be read off the parametrization of the curvatures) gives in turn the bosonic equations of motion, namely the scalar and gauge-vector equations. Similarly the sector (2,1) of the gravitino Bianchi identity yields the gravitino equation and, by supersymmetry, also the graviton and graviphoton equations. The Lagrangian leading to these equations can be obtained using the constructive methods of refs [20]. We point out that this Lagrangian, unlike the one presented in refs [1,2a], is by construction invariant under diffeomorphisms on $\mathcal{M}$. These further developments are left to a future publication.

6 - The metric of $\mathcal{M}$

We have already noticed in Sect. 4 (see eq. (4.24)) that the Kähler potential is essentially determined by the gauge vector Bianchi identities.

In this Section we show that the metric of $\mathcal{M}$ derived from eq. (4.26) does provide a solution for eq. (5.26). We use the modified index convention $I, J, K.. = 0, 1, ..., n$ and $A, B, C.. = 1, ..., n$.

We start by rewriting eq. (4.26) as follows:

$$G = -\ln \frac{L^I \bar{L}^J N_{IJ}}{L^0 \bar{L}^0} \quad (6.1)$$

where

$$N_{IJ} = F_{IJ}(L) + \bar{F}_{IJ}(\bar{L}) \quad (6.2)$$

$$F_{IJ} = \frac{\partial}{\partial \bar{L}^I} \frac{\partial}{\partial \bar{L}^J} F(L) \quad (6.3)$$

and $F(L)$ is a homogeneous holomorphic function of degree 2 in the $n+1$ variables L . Eq. (6.1) is equivalent to (4.26) if the $n+1$ complex functions L^I satisfy the (real) constraint:

$$L^I \bar{L}^J N_{IJ} = 1 \quad (6.4)$$

From the definition

$$g_{ij^*} = \partial_i \partial_{j^*} G = \nabla_i \nabla_{j^*} G \quad (6.5)$$

where the last equality holds because G has $U(1)$ Kähler weight = 0, we find:

$$g_{ij^*} = \frac{-f_i^I f_{j^*}^J (N_{IJ} N_{KN} - N_{IK} N_{JN}) \bar{L}^K L^N}{(L^I \bar{L}^J N_{IJ})} \quad (6.6)$$

We have used

$$F_{IJK}L^I = 0 \quad (F_{IJK} \equiv \frac{\partial}{\partial L^I} \frac{\partial}{\partial L^J} \frac{\partial}{\partial L^K} F(L)) \quad (6.7)$$

implied by the homogeneity of F . Moreover, taking the ∇_i derivative of (6.4) gives

$$\bar{L}^I f_i^I N_{IJ} = 0 \quad (6.8)$$

so that the metric (6.6) reduces to:

$$g_{ij} = -f_i^I f_j^I N_{IJ} \quad (6.9)$$

Let us now compute the curvature corresponding to (6.9). We first notice that

$$2R_{i^*jm^*k} = g_{i^*} \partial_{m^*} \Gamma_{jk}^i = g_{i^*} \nabla_{m^*} \Gamma_{jk}^i = \nabla_{m^*} (g_{i^*} \Gamma_{jk}^i) \quad (6.10)$$

Next, in our conventions the Christoffel connection is

$$\Gamma_{jk}^i = -g^{i^*} (\partial_j g_{k i^*}) \quad (6.11)$$

and the last expression in (6.10) becomes

$$\begin{aligned} \nabla_{m^*} [g_{i^*} \Gamma_{jk}^i] &= \nabla_{m^*} [-\partial_j g_{k i^*}] = \\ &= \nabla_{m^*} [(\partial_j f_k^I f_{i^*}^I) N_{IJ} - \frac{G_j}{2} g_{k i^*} + f_k^I f_{i^*}^I f_j^K F_{IJK}] = \\ &= \nabla_{m^*} [(\nabla_j^{U(1)} f_k^I) f_{i^*}^I N_{IJ} + f_k^I f_{i^*}^I f_j^K F_{IJK}] = \\ &= \nabla_{m^*} (\nabla_j^{U(1)} f_k^I) f_{i^*}^I N_{IJ} + (\nabla_j^{U(1)} f_k^I) (\nabla_{m^*} f_{i^*}^I) N_{IJ} + \\ &+ (\nabla_j^{U(1)} f_k^I) f_{i^*}^I \bar{F}_{IJK} f_{m^*}^K + f_k^I (\nabla_{m^*} f_{i^*}^I) f_j^K F_{IJK} \\ &+ 2(\nabla_{m^*} f_k^I) f_{i^*}^I f_j^K F_{IJK} \end{aligned} \quad (6.12)$$

The last term vanishes because of the complex conjugate of (4.22) and (6.8), the first is just equal to $-2g_{m^*}(jg_k)_{i^*}$ and we finally find

$$\begin{aligned} 2R_{i^*jm^*k} &= -2g_{m^*}(jg_k)_{i^*} - 4C_{jk}^{i^*} C_{m^*i^*}^i g_{i^*} + \\ &+ 2f_k^I f_{i^*}^I f_{m^*}^K \bar{F}_{IJK} C_{jk}^{i^*} + 2C_{m^*i^*}^i f_k^I f_{i^*}^I f_j^K F_{IJK} \end{aligned} \quad (6.13)$$

after use of (5.16) and (5.17).

Now consider a particular choice of the functions $L^I(z, \bar{z})$ such that

$$\frac{L^I}{L^0} = z^I \quad (z^0 = 1) \quad (6.14)$$

Notice that

$$\nabla_{i^*} \frac{L^I}{L^0} = \partial_{i^*} \frac{L^I}{L^0} = 0 \quad (6.15)$$

so that the choice (6.14) makes sense. We call (6.14) the special gauge. In this gauge, it is easy to find that:

$$f_i^I = \nabla_i L^I = (\partial_i + \frac{G_i}{2})(z^I L^0) = \delta_i^I L^0 + G_i L^0 z^I \quad (6.16)$$

and

$$f_i^I f_j^K F_{IJK} = (L^0)^3 F_{ijk} = (L^0)^2 Q_{ijk} \quad (6.17)$$

since the second term in (6.15) never contributes when contracted with F_{IJK} . Q_{ijk} is defined as the third ordinary derivative of $f(z)$, i.e. $Q_{ijk} \equiv \partial_i \partial_j \partial_k f(z)$, where $F(L) = (L^0)^2 f(z)$. Therefore, in the special gauge, eq. (6.13) reduces to

$$\begin{aligned} 2R_{i^*jm^*k} &= -2g_{m^*}(jg_k)_{i^*} - 4C_{jk}^{i^*} C_{m^*i^*}^i g_{i^*} + \\ &+ 2(L^0)^2 C_{jk}^{i^*} \bar{Q}_{i^*i^*m^*} + 2(L^0)^2 Q_{kji} C_{m^*i^*}^i \end{aligned} \quad (6.18)$$

On the other hand, we know that the curvature in the special gauge is also equal to

$$R_{i^*jm^*k} = -g_{m^*}(jg_k)_{i^*} + \frac{1}{2} e^{\phi} Q_{jk}^{i^*} \bar{Q}_{m^*i^*}^i g_{i^*} \quad (6.19)$$

as computed in ref.s [1,2a]. This means that in the special gauge we have:

$$C_{ijk} \equiv \frac{1}{2} g_{i^*} f_{i^*}^I \partial_j f_k^A = \frac{1}{2} (L^0)^2 Q_{ijk} \quad (6.20)$$

Consider now both eq.s (6.17) and (6.20). These imply that in the special gauge the tensor

$$f_i^I f_j^K F_{IJK} - 2C_{ijk} = 0 \quad (6.21)$$

vanishes. Since it is a tensor, it vanishes in any coordinate system. We have therefore found an expression for the symmetric tensor C_{ijk} :

$$C_{ijk} = \frac{1}{2} f_i^I f_j^K F_{IJK} \quad (6.22)$$

Substituting (6.22) back into eq. (6.13) proves that the metric (6.9) indeed satisfies the covariant constraint (5.26). While proving it, we have also seen that in the special gauge (6.14) the constraint (5.26) reduces to the (non-covariant) constraint of ref.s [1,2a], which is to be expected since the metric itself becomes identical to the special metric of ref.s [1,2a] (indeed the Kähler potential (6.1) in the special gauge coincides with (1.1)).

Next we want to identify the function S of eq. (5.31).
Theorem: the function

$$S = -\frac{1}{4} L^I L^J N_{IJ} \quad (6.23)$$

satisfies

$$C_{ijk} = \nabla_i \nabla_j \nabla_k S \quad (6.24)$$

Proof: successive covariant derivatives of S yield:

$$\nabla_k S = -\frac{1}{2} f_k^I L^J N_{IJ} \quad (6.25)$$

$$\begin{aligned} \nabla_j \nabla_k S &= -\frac{1}{2} \nabla_j f_k^I L^J N_{IJ} - \frac{1}{2} f_k^I f_j^J N_{IJ} = \\ &= -f_k^I C_{jk}^* L^J N_{IJ} - \frac{1}{2} f_k^I f_j^J N_{IJ} = \\ &= -\frac{1}{2} f_k^I f_j^J N_{IJ} \end{aligned} \quad (6.26)$$

The last equality is due to the conjugate of eq. (6.8). The third covariant derivative reads:

$$\begin{aligned} \nabla_i \nabla_j \nabla_k S &= -(\nabla_i f_k^I f_j^J N_{IJ} - \frac{1}{2} f_k^I f_j^J f_i^K F_{IJK} \\ &= -2 f_k^I C_{ik}^* f_j^J N_{IJ} - \frac{1}{2} f_k^I f_j^J f_i^K F_{IJK} = \\ &= +2 C_{ijk} - \frac{1}{2} f_k^I f_j^J f_i^K F_{IJK} = \\ &= C_{ijk}. \end{aligned} \quad (6.27)$$

where we used (6.22) in the last equality. This proves the theorem.

7 - Conclusions

We have presented a geometrical formulation of N=2, D=4 Supergravity coupled to abelian vector multiplets. By construction we have $\mathcal{M}$ coordinate invariance at all stages of our computations, and we have been able to characterize the so-called "special geometry" of $\mathcal{M}$ in a coordinate independent way. More specifically, we have found the covariant "special metric" of $\mathcal{M}$.

It is clear that our work can be extended to include also the matter hypermultiplets, and we may expect to deduce covariant "special quaternionic" constraints on $\mathcal{M}$. This is left for future investigations.

Table

Kähler geometry

Covariant derivative of a covariant vector W_j : $\nabla_i W_j \equiv \partial_i W_j + \Gamma_{ij}^i W_i$

Connection: $\Gamma_{ij}^i \equiv -g^{ik*} (\partial_i g_{jk*})$

U(1)-covariant derivative of a field with U(1)-weight= p :

$$\nabla_i^{U(1)} \Phi = \partial_i \Phi + \frac{p}{2} G_i \Phi \quad (G_i \equiv \partial_i G)$$

$$\nabla_{i*}^{U(1)} \Phi = \partial_{i*} \Phi - \frac{p}{2} G_{i*} \Phi \quad (G_{i*} \equiv \partial_{i*} G)$$

U(1)-Connection: $Q = -\frac{i}{2} (\partial_i G d\bar{z}^i - \partial_{i*} G d\bar{z}^{i*})$

Curvature: $2 R^i_{jm*k} = \partial_{m*} \Gamma_{jk}^i$

$\mathcal{M}$ special geometry

Covariant constraint: $R_{i*j*}^i k = -g_{i*} (g_{k*} + 2 C_{i*}{}^{l*} C_{lk*} g^{s*l})$

Covariantly holomorphic symmetric tensor:

$$C_{ijk} = \frac{1}{2} g_{i*} f_j^* \nabla_i f_k^* = \nabla_i \nabla_j \nabla_k S = \frac{1}{2} f_i^I f_j^J f_k^K F_{IJK}$$

with $f_i^I = (f^I)^{-1}$, $f_i^I = \nabla_i L^I$, $S = -\frac{1}{2} L^I L^J (F_{IJ} + \bar{F}_{IJ})$, $F_{IJ} \equiv \frac{\partial}{\partial L^I} \frac{\partial}{\partial L^J} F(L)$
 $F(L)$ = homogeneous holomorphic function of degree 2 in L .

Special Kähler potential: $G = -\ln \frac{L^I L^J (F_{IJ} + \bar{F}_{IJ})}{L^0 \bar{L}^0}$

Special gauge: $\frac{L^I}{L^0} = z^I$

N=2 D=4 Supergravity $\oplus$ vector multiplets:

Curvature definitions

$$R^a = dV^a - \omega^{ab} V^b - i\psi_A \gamma^a \psi^A \equiv DV^a - i\psi_A \gamma^a \psi^A$$

$$\rho_A = d\psi_A - \frac{1}{4} \omega^{ab} \gamma_{ab} \psi_A + \frac{i}{2} Q \psi_A \equiv \nabla \psi_A$$

$$\rho^A = d\psi^A - \frac{1}{4} \omega^{ab} \gamma_{ab} \psi^A - \frac{i}{2} Q \psi^A \equiv \nabla \psi^A$$

$$F = dA + \varepsilon_{AB} L^A \bar{\psi}^B + \varepsilon^{AB} \bar{L}^A \psi_B$$

$$R^{ab} = d\omega^{ab} - \omega^{ac} \omega^{cb}$$

$$\begin{aligned}
F^I &= d\lambda^I + \epsilon_{AB} L^I \bar{\psi}^A \psi^B + \epsilon^{AB} \bar{L}^I \bar{\psi}_A \psi_B \\
\nabla \lambda^{iA} &= d\lambda^{iA} - \frac{1}{4} \omega^{ab} \gamma_{ab} \lambda^{iA} - \frac{i}{2} Q \lambda^{iA} - \Gamma^i{}_j \lambda^{jA} \\
\nabla \lambda^{i*}_A &= d\lambda^{i*}_A - \frac{1}{4} \omega^{ab} \gamma_{ab} \lambda^{i*}_A + \frac{i}{2} Q \lambda^{i*}_A - \Gamma^{i*}{}_{j*} \lambda^{j*}_A \\
dM^I &= \partial_t M^I(z) dz^i; \quad d\bar{M}^I = \partial_{\bar{t}} \bar{M}^I(\bar{z}) d\bar{z}^{\bar{i}}
\end{aligned}$$

Parametrizations of the curvatures*

$$\begin{aligned}
R^c &= 0 \\
\rho_A &= \rho^a_A V^a V^b - i(G^+)^{ab} \gamma_b \psi^B V^a \epsilon_{AB} + (M^B{}_A)_a \psi_B + M_{AB|a} \psi^B V^a \\
\rho^A &= \rho^a_{ab} V^a V^b + i(G^-)^{ab} \gamma_b \psi_A V^a \epsilon^{AB} + (M^{*A}{}_B)_a \psi^B + M^{*AB}{}_a \psi_B V^a \\
R^{ab} &= R^{ab}{}_{cd} V^c V^d - i\bar{\psi}_A (\gamma^a \rho^{A|bc} + \gamma^b \rho^{A|ca} - \gamma^c \rho^{A|ab}) V^c \\
&\quad - i\bar{\psi}^A (\gamma^{a,bc} - \gamma^{b,ca} + \gamma^{c,ab}) V_c \\
&\quad - \epsilon^{AB} \bar{\psi}_A \psi_B (G^-)^{ab} - \epsilon_{AB} \bar{\psi}^A \psi^B (G^+)^{ab} \\
&\quad + \bar{\psi}^A N_{AB}^{ab} \psi^B + \bar{\psi}^A N_A^{Bab} \psi_B \\
&\quad - \bar{\psi}_A N^{*ABab} \psi_B - \bar{\psi}_A N^{*A}{}^B{}_{ab} \psi^B \\
K &= ig_{ij} Z^i \bar{Z}^j V^a V^b + ig_{ij} (Z^i \bar{\psi}^A \lambda^j_A + \bar{Z}^j \psi_A \lambda^{iA}) V^a + \\
&\quad + \frac{i}{2} g_{ij} \bar{\lambda}^{iA} \gamma^a \lambda^{j*}_B \bar{\psi}^B \gamma_a \psi_A
\end{aligned}$$

$$\begin{aligned}
F^I &= F^I_{ab} V^a V^b + (f^I{}_a \bar{\lambda}^{iA} \gamma^a \psi^B \epsilon_{AB} - f^I{}_a \bar{\lambda}^{i*}_A \gamma^a \psi_B \epsilon^{AB}) V^a \\
\nabla \lambda^{iA} &= (\nabla_a \lambda^{iA}) V^a + i P^i{}_a \gamma^a \psi^A + (G^+)^i{}_{ab} \gamma^{ab} \psi_B \epsilon^{AB} + Y^{iAB} \psi_B \\
\nabla \lambda^{i*}_A &= (\nabla_a \lambda^{i*}_A) V^a + i P^{i*}{}_a \gamma^a \psi_A + (G^-)^i{}_{ab} \gamma^{ab} \psi^B \epsilon_{AB} + Y^{i*}_{AB} \psi^B \\
dz^i &= Z^i_a V^a + \bar{\psi}_A \lambda^{iA} \\
d\bar{z}^{\bar{i}} &= \bar{Z}^{\bar{i}}_{\bar{a}} V^{\bar{a}} + \bar{\psi}^{\bar{A}} \lambda^{\bar{i}}_{\bar{A}}
\end{aligned}$$

with:

* the supersymmetry variations are obtained by replacing ψ by the supersymmetry parameter ϵ . $\delta_{susy}\psi$ contains the additional term $\nabla\epsilon$, since ψ is the "gauge field" of local supersymmetry.

$$\begin{aligned}
M^B{}_A|_a &= A^B{}_A|_a \delta^b_a + A'^B{}_A|_a \gamma^{ab} \\
M_{AB}|_a &= (iS_{AB}\eta_{ab} + \epsilon_{AB}(T^+)^{ab})\gamma^b \\
N^a{}_B &= S_{AB}\gamma^{ab} - i\epsilon_{AB}T^{ab} \\
N^{B|ab} &= A'^B{}_A|_a \gamma^b \epsilon^{abc} f
\end{aligned}$$

Auxiliary fields in terms of the physical fields:

$$\begin{aligned}
A^B{}_A|_a &= -\frac{i}{4} g_{kl} (\bar{\lambda}^{k*}_A \gamma_a \lambda^{lB} - \delta^B_A \bar{\lambda}^{k*}_P \gamma_a \lambda^{lP}) \\
A'^B{}_A|_a &= +\frac{i}{4} g_{kl} (\bar{\lambda}^{k*}_A \gamma_a \lambda^{lB} - \frac{1}{2} \delta^B_A \bar{\lambda}^{k*}_P \gamma_a \lambda^{lP}) \\
(T^+)^{ab} &= \frac{1}{2\bar{L}^0} [-2i(G^+)^{ab}(1 - \bar{L}^0) + 4i f_i (G^+)^i{}_{ab} - \frac{i}{4} \nabla_j f_i \bar{\lambda}^{iL} \gamma_{ab} \lambda^{jM} \epsilon_{LM}] \\
(f^i{}_j - \frac{\bar{L}^i}{\bar{L}^0} f_i)(G^+)^i{}_{ab} &= -\frac{1}{2} \frac{\bar{L}^i}{\bar{L}^0} (G^+)^{ab} + \frac{1}{2} (F^+)^i{}_{ab} + \frac{1}{16} (1 - \frac{\bar{L}^i}{\bar{L}^0}) \nabla_j f_i \bar{\lambda}^{iL} \gamma_{ab} \lambda^{jM} \epsilon_{LM}
\end{aligned}$$

all other auxiliary fields vanish.

Bianchi identities

$$\begin{aligned}
\mathcal{D}R^a + R^{ab} V_b - i\bar{\psi}^A \gamma^a \rho_A + i\bar{\rho}^A \gamma^a \psi_A &= 0 \\
\nabla \rho_A + \frac{1}{4} \gamma_{ab} R^{ab} \psi_A + \frac{1}{2} g_{ij} dz^i d\bar{z}^{\bar{j}} \psi_A &= 0 \\
dF - \epsilon_{AB} \nabla L \bar{\psi}^A \bar{\psi}^B + 2\epsilon_{AB} L \bar{\psi}^A \rho^B - \epsilon^{AB} \nabla \bar{L} \bar{\psi}_A \psi_B + 2\epsilon^{AB} \bar{L} \bar{\psi}_A \psi_B &= 0 \\
\mathcal{D}R^{ab} &= 0
\end{aligned}$$

$$\begin{aligned}
d^2 z^i &= 0 \\
\nabla^2 \lambda^{iA} + \frac{1}{4} R^{ab} \gamma_{ab} \lambda^{iA} + \frac{i}{2} R[Q] \lambda^{iA} + R[\Gamma]^i{}_j \lambda^{jA} &= 0 \\
dF^I - \epsilon_{AB} \nabla L^I \bar{\psi}^A \bar{\psi}^B + 2\epsilon_{AB} L^I \bar{\psi}^A \rho^B - \epsilon^{AB} \nabla \bar{L}^I \bar{\psi}_A \psi_B + 2\epsilon^{AB} \bar{L}^I \bar{\psi}_A \psi_B &= 0
\end{aligned}$$

References

- [1] B. de Wit, P.G. Lowers, R. Philippe, S. Q. Su and A. Van Proeyen, Phys. Lett. B134 (1984) 37; B. de Wit and A. Van Proeyen, Nucl. Phys. B245 (1984); B. de Wit, P. G. Lowers and A. Van Proeyen, Nucl. Phys. B255 (1985) 569; J. P. Derendinger, S. Ferrara, A. Masiero and A. Van Proeyen, Nucl. Phys. B140 (1984) 307.
- [2a] E. Cremmer, C. Kounnas, A. Van Proeyen, J. P. Derendinger, S. Ferrara, B. de Wit and L. Girardello, Nucl. Phys. B250 (1985) 385.
- [2b] L. Castellani, R. D' Auria and S. Ferrara, DFTT-29/89 and CERN-TH-5635/90, subm. to Phys. Lett. B.
- [3] N. Seiberg, Nucl. Phys. B303 (1988) 286.
- [4] S. Ferrara, S. Cecotti and L. Girardello, Int. Journal of Modern Physics A, vol. 4 (1989) 2475; Phys. Lett. B213 (1988) 443.
- [5] P. Candelas, G. Horowitz, A. Strominger and E. Witten, Nucl. Phys. B258 (1985) 46.
- [6] A. B. Zamolodchikov, JETP Lett. 43 (1986) 731.
- [7] L. Dixon, V. S. Kaplunovsky and J. Louis, Nucl. Phys. B239 (1990) 27.
- [8] S. Ferrara and A. Strominger, CERN-TH-5291/89, UCLA-89-TEP/6, (1989)
- [9] P. Candelas, P. S. Green and T. Hubsch, UTTG-17-89 (Austin report) (1989).
- [10] V. Perwal and A. Strominger, UCSB preprint NSF-ITP-89-144, (1989)
- [11] R. D' Auria and P. Fré, Phys. Lett. B200 (1988) 63; R. D' Auria, P. Fré, M. Raciti and F. Riva, Int. Jour. of Mod. Phys. A3 (1988) 953.
- [12] M. Dine, P. Huf and N Seiberg, Nucl. Phys. B322 (1989) 301.
- [13] A. Strominger, Phys. Rev. Lett. 55 (1985) 2547; A. Strominger and E. Witten, Comm. Math. Phys. 101 (1985) 341; A. Strominger, in Proceedings of the Santa Barbara Workshop, "Unified String Theory", World Scientific (edited by M. Green and D. Gross), (1985).
- [14] P. Candelas, Nucl. Phys. B298 (1988) 458.
- [15] R. Dijkgraaf, E. Verlinde and H. Verlinde, preprint THU-87/30 (Princeton report); V. P. Nair, A. Shapere, A. Strominger and F. Wilczek, Nucl. Phys. B287 (1987) 402; A. Shapere and F. Wilczek, Nucl. Phys. B320 (1989) 669; A. Giveon, E. Rabinovici and G. Veneziano, Nucl. Phys. B322 (1989) 167; S. Ferrara, D. Lüst, A. Shapere and S. Theisen, Phys. Lett. 225B (1989) 363; J. Lauer, J. Mas, H. P. Nilles, Phys. Lett. B226 (1989) 251; W. Lerche, D. Lüst and N. P. Warner, Phys. Lett. B231 (1989) 417; E. J. Chun, J. Mas, J. Lauer, and H. P. Nilles, Phys. Lett. B233 (1989) 141; C. Vafa, Harvard preprint HUTP-89/A021(1989); S. Ferrara, D. Lüst, and S. Theisen, Phys. Lett. B233 (1989) 147; B. Greene, A. Shapere, C. Vafa and S.T.Yau, Harvard Preprint HUTP-89/A047 and IASSNS-HEP-89/47; J. H. Schwarz, Caltech preprint CALT-68-1581/19.
- [16] A. Strominger, in preparation.
- [17] J. Bagger and E. Witten, Phys. Lett. 115B (1982) 202.
- [18] J. Bagger and E. Witten, Nucl. Phys. B222 (1983) 1.
- [19] B. Zumino, Phys. Lett. 87B (1979) 203.
- [20] L. Castellani, R. D' Auria and P. Fré, Seven Lectures on the group manifold approach to supergravity and the spontaneous compactification of extra dimensions, Proc. XIX Winter School, Karpacz 1983, ed. B. Milewski (World Scientific, Singapore); L. Castellani, R. D' Auria and P. Fré, "Supergravity and Superstrings : a geometric perspective", World Scientific (1989), in press.
- [21] R. D' Auria, P. Fré, E. Maina and T. Regge, Ann. of Physics 135 (1981) 237.