NON-PERTURBATIVE TWO-DIMENSIONAL
QUANTUM GRAVITY, KdV HIERARCHY AND
SOLUTIONS OF THE STRING EQUATIONS

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ABSTRACT

We investigate the relation between the multicritical one-matrix random model and the moduli space parametrizing the solutions of the KdV hierarchy and give a geometrical interpretation of the non-perturbative free parameters of 2D-QG. From our viewpoint the string equations, in the asymptotic limit, give the boundary conditions of the stationary generalized KdV equation which contains all the information about the theory. We thus study those stationary solutions, matching the asymptotic behaviour of the solutions of the string equations, which are real and free of singularities in the scaling variable. We also discuss the consequences of these requirements on the geometry of such solutions.

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Two-dimensional quantum gravity (2D-QG) may be formulated as a functional integral over two-dimensional internal geometries by first performing the path integral for fixed topology (genus) and then summing over the genus.^[1] It is commonly believed that all moduli spaces of closed Riemann surfaces of arbitrary genus contribute to the 2D-QG partition function $Z = \sum_g c_g Z_g$ and hence that Z might also be defined by a suitable measure over the so-called universal moduli space $\mathcal{M}_\infty$. However the sum over topologies is very badly behaved, the series is non-Borel summable^[2] and, on the other hand, we have not a unique model for $\mathcal{M}_\infty$.

In the last years, a completely different approach to 2D-QG and string theories has been proposed.^[3] The key point is to discretize the sum over geometries as a sum over random triangulations belonging to a simplicial complex lattice and to map onto the dual lattice. This allows one to approximate the 2D geometry of fixed genus by a dense Feynman graph of the same topology, so that the discretized 2D-QG path integral becomes an integral over random Hermitian matrices. The topological genus expansion is now taken into account by the large N expansion of the corresponding matrix model, where N is the dimension of the matrix.

Very recently, by analyzing the multicritical random one-matrix model,^[4] it has been shown that the specific heat $u(x)$ obeys, in a suitable double scaling limit, a universal differential equation, the string equation, which for the case of pure gravity is the Painlevé type I. Due to this fact the theory presents some non-perturbative ambiguities which are not completely fixed by the Cauchy data. Some progress has been recently made to circumvent this problem.^[5]

Furthermore a connection between 2D-QG and the KdV hierarchy (KdV-H) has been established.^[6]

The motivation of this letter is to investigate in a more direct way the relation between the random matrix model and the KdV-H to try to shed some light on the non-perturbative issues. More in detail we consider all the asymptotic information of 2D-QG to be contained in the solutions of the stationary KdV-H. The role of the string equation is to impose certain constraints on these solutions such as: reality, asymptotic behaviours etc.

As is well known the boundary conditions characterize different classes of solutions of the KdV-H. In the inverse scattering problem^[7,8] they are the potentials of a certain Hamiltonian operator as we will later discuss in more detail. The potentials which are periodic or vanish

at ∞ give rise to solutions which are characterized by θ -functions associated to hyperelliptic curves. These are the only exactly known solutions up to now. 2D-QG leads to a different asymptotic behaviour which can be analysed with approximate methods.

The plan of the letter is the following: we first recall few formulae of the multicritical one-matrix model approach and illustrate how the solutions of the associated Sturm-Liouville problem are in correspondence with auxiliary* Riemann surfaces. We then proceed to analyse the 2D-QG case and exhibit solutions of the string equations which are real and singularity free in the asymptotic limit of the scaling variable corresponding to the planar approximation.

Following Ref. 4 we introduce orthogonal polynomial $P_n(\Phi) = \Phi^n + \dots$ satisfying the recursion equation:

$$\Phi P_n(\Phi) = P_{n+1}(\Phi) + R_n P_{n-1}(\Phi), \quad (1)$$

where R_n are some coefficients. (1) can be rewritten as an operator-valued recursion relation on the Hilbert space spanned by the orthonormal states:

$$|n\rangle = \frac{1}{\sqrt{h_n}} |P_n\rangle, \quad (2)$$

as:

$$\hat{\Phi} = \sqrt{R(\hat{n})} e^{\hat{\Phi}} + e^{-\hat{\Phi}} \sqrt{R(\hat{n})}, \quad (3)$$

where:

$$\begin{aligned} e^{\pm \hat{\Phi}} |n\rangle &= |n \pm 1\rangle, \\ R(\hat{n}) |n\rangle &= R_n |n\rangle, \end{aligned} \quad (4)$$

and the h_n of (2) are just normalization constants.

The recursion coefficients R_n are determined by the equations:

$$\begin{aligned} \frac{n}{\beta} &= -\frac{1}{2\beta} + \frac{1}{2} \langle n | \hat{\Phi} U'(\hat{\Phi}) | n \rangle, \\ 0 &= \langle n | U'(\hat{\Phi}) | n \rangle. \end{aligned} \quad (5)$$

where $U(\hat{\Phi})$ is the symmetric potential of the matrix model. These equations are valid for any N . Since we are interested in the double scaling limit $N \mapsto \infty$, $\beta/N \mapsto 1$, keeping fixed

* By auxiliary we mean that there is no straightforward connection among the Riemann surfaces entering the Polyakov's path integral of 2D-QG and those parametrizing the KdV solutions.

$N^2(\beta/N - 1)^k$ (k being the parameter describing the order of the criticality), we define the continuous variable $z \equiv n/N$ and write $R(z) = R_n$.

Alvarez and Windey^[9] showed that the general matrix elements $\langle n | \hat{\Phi}^s | m \rangle$, $N \rightarrow \infty$, $\langle z | \hat{\Phi}^s | z' \rangle \equiv K(z, z'; s)$ satisfy a difference equation which is a discrete one-dimensional random walk in the space of orthogonal polynomials with hopping amplitude $\sim \sqrt{R(z)}$ (K is the associated "discrete" diffusion kernel).

Introduce now a lattice spacing a , a renormalized string coupling constant $\lambda \equiv a^{-\frac{2s+1}{2s}}$, ϵ ($\epsilon \equiv 1/N$) finite in the continuum limit $a \rightarrow 0$ and large N expansion $N \rightarrow \infty$, and a continuous variable x defined at the critical behaviour $\beta \rightarrow \beta_c$ ($\beta_c/N = 1$, $N \rightarrow \infty$) by: $1 - z = a^2 x$.

Therefore as $N \rightarrow \infty$ and $a \rightarrow 0$ one has the following replacements: $n \rightarrow z \rightarrow x$ and $R_n \rightarrow R(x) \rightarrow R(z)$. In these limits the above discrete diffusion equation becomes a stationary Schrödinger equation in the Euclidean time y (diffusion time) defined by:

$$y \equiv sa^2/m \frac{\lambda^2}{2}, \quad s \in \mathbb{Z}_+. \quad (6)$$

Furthermore:

$$\langle n | \hat{\Phi}^s | m \rangle \xrightarrow{N \rightarrow \infty, a \rightarrow 0} K(x, x'; y) = \langle x | e^{-yL} | x' \rangle, \quad (7)$$

where the heat kernel K satisfies the one-dimensional (in x) stationary ($\partial_y u = 0$) Schrödinger equation:

$$-\partial_y K(x, x'; y) = L[u(x)]K(x, x'; y), \quad (8)$$

with diffusion Hamiltonian L given by the Sturm-Liouville operator:

$$L[u(x)] \equiv -\partial_x^2 + u(x). \quad (9)$$

Here the potential $u(x)$ is defined by the following ansatz for $R(x)$:

$$2\sqrt{R(x)} = \left\{ 1 - \frac{1}{2N^2 a^4} u(x) \right\}. \quad (10)$$

Gelfand and Diki^[10] have shown that the diagonal elements of the heat kernel (7) have

an asymptotic expansion:

$$\langle x | e^{-y(-\partial_x^2 + u(x))} | x \rangle \sim \frac{1}{\sqrt{4\pi y}} \sum_{q=0}^{\infty} y^q R_q(u(x)), \quad (11)$$

where $R_q(u)$ is a differential polynomial in u .

If one introduces a new time parameter t on which u depends parametrically, i.e. $u(x) \equiv u(x; t)$, one could show that the quantity $\partial_x R_q[u(x; t)]$ defines the q -th Hamiltonian vector field (corresponding to $R_q(n)$) of the q -th member of the KdV hierarchy, $\partial_t u = \partial_x R_q[u]$. Therefore the potential u of (9) plays also the role of KdV potential.

Let's now recall how a smooth real periodic potential $u(x)$ leads to soliton-type solutions which are genus dependent. Here the genus, as we will immediately discuss, is connected to the number of gaps of $u(x)$. Let u be a periodic function of period T : $u(x + T) = u(x)$. The spectrum of the operator L is given by the eigenvalue problem:

$$L\psi(x; E) = E\psi(x; E), \quad (12)$$

such that:

$$\begin{aligned} \psi(x + T; E) &= e^{ik(E)T} \psi(x; E) \equiv w(E)\psi(x; E), \\ \psi(0; E) &= 1. \end{aligned} \quad (13)$$

The spectrum of L consists of a set of closed intervals, called bands, $[E_0, E_1]$, $[E_2, E_3]$, $[E_4, E_5]$, ... with $E_0 < E_1 < E_2 < \dots$. The open intervals $(-\infty, E_0)$, (E_1, E_2) , (E_2, E_3) , ... are called gaps and they accumulate at infinity, i.e. $E_{2n} - E_{2n-1} \rightarrow 0$ as $n \rightarrow \infty$.

The set Γ of pairs $P \equiv (E, w(E))$ for which there are Bloch eigenfunctions of the Sturm-Liouville operator L is called the spectral Floquet set. It turns out that Γ is isomorphic to a hyperelliptic Riemann surface Σ_g of genus g and irrationality $\sqrt{\Delta^2 - 1}$ with Δ the discriminant of L . In other words, Σ_g is a two-sheeted covering of the (complex) E -plane with branch points $E_0, E_1, E_2, \dots$. Therefore the genus g of Σ_g is given by the number of gaps of $\text{Spec}(L)$; the holes are located over $n^2\pi^2$ ($n = 1, 2, 3, \dots$), or nearly so, and diminish rapidly near ∞ , so they are not very noticeable far from the point $P_0 = \infty$ covering the infinite point of the E -plane, and Σ_g is compactified by P_0 . P_0 is a distinguished point of Σ_g with local uniformizer $k^{-1} = k^{-1}(P)$ in its neighbourhood such that $k_0^{-1} = 0$. Indeed the Bloch eigenfunctions $\psi(x; P)$, $P = (E, w(E))$ coincide with the so-called Baker-Akhiezer functions^[11] which have the following properties:

1. $\psi(z, P)$ is meromorphic in E outside P_0 , i.e. has exactly one pole γ_n and one zero γ_n^+ ($n = 1, \dots, g$) over each gap. The poles of ψ do not depend on z . Furthermore, ψ has an essential singularity at P_0 .

2. in a neighbourhood of P_0 , $k^{-1}(P_0) = 0$, $\psi(z, P)$ has the form:

$$\psi(z, P) = e^{kz} \left(1 + \sum_{s=1}^{\infty} \xi(z) k^{-s} \right), \quad k = k(P) \equiv \sqrt{-E}. \quad (14)$$

Notice that a function $\psi(z, P)$ associated to the data $(\Sigma_g, P_0, k, \gamma_1, \dots, \gamma_g)$, where Σ_g is a (here, hyperelliptic) Riemann surface, P_0 a distinguished (here, Weierstrass) point with local uniformizer k , $\gamma_1 + \dots + \gamma_g$ a divisor in general position, is uniquely determined by the above analytic properties 1 and 2. Notice also that the complex structure of Σ_g is specified by the positions (moduli) of the $2g + 2$ branch point E_i . As a consequence the flows, commuting with the KdV-H which move the points E_i of Γ , generate the deformations of the complex structure of $\Sigma_g \mapsto \Gamma$.

After this introduction we begin the study of those solutions of the KdV-H which, after a rescaling, satisfy the asymptotics of the string equation.

The string equation for the k -critical one-matrix random model is:

$$z = R_k[u(z)], \quad (15)$$

where z is the scaling variable and the Gel'fand-Dikii differential polynomials $R_k[u(z)]$ are normalized here as:

$$\begin{aligned} R_0 &= 1, \\ R_1 &= u, \\ R_2 &= u^2 - \frac{1}{3}u'', \\ R_3 &= u^3 - uu'' - \frac{1}{2}(u')^2 + \frac{1}{10}u''', \\ &\vdots \end{aligned} \quad (16)$$

The continuum scaling limit of the multicritical matrix model is thus reduced to the mathematical problem of finding solutions of (15).

For pure gravity ($k = 2$) the existence of a real solution of (15) with double poles on the real axis is not physically admissible, since a pole in $u(z)$ for $z = z_0 \in \mathbb{R}$ implies a zero of the partition function. Furthermore, in Ref. 5 David has argued that only solutions which are analytic along the whole real axis give loop amplitudes satisfying the Dyson-Schwinger equations. So, it is quite important to see if (15) admits real solutions without poles on the real axis and with the correct asymptotic behaviour at infinity:

$$u(z) \sim z^{1/k}. \quad (17)$$

In the following we reconsider (15) and show that it is asymptotically related to the Novikov equation (see Ref. 7):

$$1 = R_k[v(\xi)]. \quad (18)$$

The problem at this point is to look for a real solution of (18) analytic in $\xi \in \mathbb{R}$. Of course, as a consequence, one also gets a classification of the physical solutions of (15).

The starting point is the transformation^[12]

$$\begin{aligned} u(z) &= x^{1/k} v[\xi(z)], \\ \xi(z) &= \frac{2k}{2k+1} x^{1+1/2k}. \end{aligned} \quad (19)$$

Substituting (19) in (15) and discarding positive powers in $1/\xi$, i.e. for large ξ , we obtain (18). Notice that this approach is equivalent to the Whitham method^[13] of perturbation of an integrable system. Indeed by differentiating (15) once, one gets:

$$\frac{d}{dx} R_k[u(x)] = 1. \quad (20)$$

But (20) is just a perturbation of the k -th equation of the stationary KdV-hierarchy. The Whitham method allows us to find the asymptotic solution of (20) whose leading term is again a solution of (18) depending on the slow variable ϵx , with small ϵ . The validity of this approach is for z of order $1/\epsilon$, and hence for large z as $\epsilon \ll 1$.

* Upon differentiating once, this equation becomes the k -th stationary KdV.

As a concrete example we consider the case of pure gravity, and hence setting $k = 2$ in (18), one has:

$$v'' = 3v^2 - 3. \quad (21)$$

The general real solution of (21) without singularities on the real axis is given by the two-parameter family:[†]

$$v(\xi) = -2a^2 \frac{d^2}{d\xi^2} \ln \theta_{00}(a(\xi - \xi_0)|\tau) + \text{const}, \quad (22)$$

where θ_{00} is the one-dimensional theta function with $(0, 0)$ characteristic, associated to the complex torus $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ and:

$$a^4 = \frac{3}{40h(\tau)}, \quad h(\tau) = \frac{\pi^4}{15} - (2\pi)^4 \sum_{n=1}^{\infty} n^3 \frac{e^{2n\pi i\tau}}{1 - e^{2n\pi i\tau}}. \quad (23)$$

where, in order to assure reality, one requires that:

$$\tau = i\tau, \quad \tau \in \mathbb{R}, \quad h(\tau) > 0. \quad (24)$$

The free parameters are (a, ξ_0) , or equivalently (τ, ξ_0) . From a geometrical viewpoint the manifold of solutions of type (22) is a cylinder which is a subvariety of the moduli space $\mathcal{M}_1$, fibered with the jacobian varieties J_1 . Notice that ξ_0 is a point on one of the two real components J_1^{Re} of J_1 .

To summarize, we have shown that any physical solution of the $k = 2$ string equation behaves for large x as:

$$u(x) \sim \sqrt{x} v\left(\frac{4}{5}x^{5/4}|\tau, \xi_0\right). \quad (25)$$

Eq. (25) implies that one can classify the physical non-perturbative solutions of $2D - QG$ in terms of a suitable set of solutions of the Novikov equation (18). In this context the free-parameters (τ, ξ_0) are true non-perturbative parameters as the ϑ -angle in QCD. Clearly the relation (25) becomes exact in the planar (spherical topology) approximation of the one-matrix model. The generalization of this approach to higher k -critical models

[†] One of these two parameters can be fixed by using the spectrum translational invariance $E_i = E_i + \text{const}$, thus recovering the standard result.

is straightforward. Now the relevant Novikov solution of (18) is associated to an auxiliary hyperelliptic surface Γ_{k-1} and it has the form:^[14]

$$v_k(\xi) = -2 \frac{d^2}{d\xi^2} \ln \theta(\bar{a}\xi + \bar{b} + \bar{\chi}|\bar{\Omega}) + \text{const}, \quad (26)$$

where $\bar{a} = \bar{a}(\Omega) \in \mathbb{R}^n$, $n = k - 1$ is the vector of the b -periods of the normalized meromorphic differential with a second order pole at the infinity, $\bar{b}$ is an arbitrary real vector and $\chi \in \frac{1}{2}\omega\mathbb{Z}^n$ is an arbitrary characteristic. Here $\Omega = \Omega(\bar{T})$ is the period matrix of an n -parameter family of hypersurfaces Γ_n selected by the requirement that $v_{n+1}(\xi)$ be a real solution of (18). The dependence $\bar{a} = \bar{a}(\Omega(\bar{T}))$ may be inverted and one may use $\bar{a}$ in the place of $\bar{T}$ as n -free real parameters. For a generic choice of $(\bar{a}, \bar{b})$ the solutions (26) are real and analytic on the real axis. Again the above non-perturbative free parameters $(\bar{a}, \bar{b})$ have a clear geometrical interpretation: $\bar{a}$ parametrizes an n -(real) dimension subvariety of the moduli space $\mathcal{M}_n$ and $\bar{b} + \bar{\chi}$ is a point over one of the 2^n real components of the jacobian variety $J(\Gamma_n)$.

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