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What is W -geometry ?

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ABSTRACT

We address the question of what geometric principle might be related to the W -algebras in the same way as the Virasoro algebra is connected with the diffeomorphisms of a Riemann surface. We suggest that a Kaluza-Klein-type approach on group-manifolds might provide some insight. We exhibit transformations of geometric origin forming an algebra that a) is covariant, b) can also be formulated for dimensions different from two, and c) when restricted to $D = 2$ and chiral transformation parameters, closely resembles the W_N -algebras.

1. Introduction

The most basic principle of gravity is diffeomorphism invariance. Two-dimensional gravity is thus naturally defined on Riemann surfaces. The generator of diffeomorphisms is the energy-momentum tensor, and, if the theory is also scale-invariant, the modes of its holomorphic component yield the (chiral) Virasoro generators.

In 2D conformal field theory, many chiral algebras extending the Virasoro algebra are known, such as the superconformal and the W_N -algebras^[1]. Whereas the former have natural geometric interpretations as remnants of diffeomorphisms of super-Riemann surfaces, we do not know at present what is the geometry of the latter. Such a geometric interpretation of W -symmetry would be particularly valuable in view of the recent progress on 2D-gravity^[2] and of the various conjectures that certain matrix models might correspond to W -gravity^[3-4]. It could also shed some light on the question of whether W -gravity is simply 2D-gravity coupled to a W -algebra extended conformal field theory or something completely different. The first step is to go beyond the chiral algebra and obtain a covariant W -algebra (like the algebra of the generators of diffeomorphisms). Some progress was made recently in ref. 5 where a covariant field theory with W -symmetry was found, but it does not really tell us what W -geometry is.

In the present letter we are not going to give a complete answer to the question raised in the title. However, we will exhibit a set of *geometric* transformations forming an algebra that a) is covariant, b) can be formulated not only in $D = 2$, and c) when restricted to $D = 2$ and chiral transformation parameters, shows a close resemblance to the W_N -algebras. Some problems related to the nonlinear structure of the W_N -algebras remain unsolved. Nevertheless, we hope that our results will be useful for exploring further the geometry underlying the W_N -algebras and will eventually lead to a complete answer to the question "what is W -geometry?"

We will outline an approach that is probably best described as "Kaluza-Klein on group manifolds": we will consider the geometry of a D -dimensional manifold

resulting from a dimensional reduction from a $\dim G$ -dimensional group-manifold. As usual in Kaluza-Klein theories, the basic idea is the following: simple geometric transformations (diffeomorphisms) on the group-manifold give rise to different transformations (i.e. geometry) on the D -dimensional manifold. Some of them are simply diffeomorphisms again, but others are more complicated and have a striking resemblance to W -transformations. Contrary to usual Kaluza-Klein approaches, we obtain *higher-spin* (W -type) transformations from *lower-spin* transformations (i.e. diffeomorphisms) by a dimensional reduction. This is possible due to the presence of the group-invariant tensors $d^{a_1 \dots a_k}$ associated with the Casimir invariants of order k .

2. Geometry on group manifolds

Let us first recall some well-known facts. Consider a Lie group G with Lie algebra $\mathcal{G}$. $\mathcal{G}$ is the set of left-invariant vector fields on G . Its dual $\mathcal{G}^*$ can be identified with the left-invariant 1-forms. We choose a basis $\{T_a\}$ of $\mathcal{G}$ obeying

$$[T_a, T_b] = f^{abc} T_c \quad (1)$$

where the f^{abc} are the structure constants of $\mathcal{G}$. The corresponding equations for the elements of the dual basis $\{e^a\}$ are the Maurer-Cartan equations^{*}

$$de^a = -\frac{1}{2} f^{abc} e^b \wedge e^c. \quad (2)$$

Choosing explicitly coordinates θ^μ on the group manifold, we write $e^a = e^\mu_a d\theta^\mu$

* More precisely, for later convenience, we choose the e^a to be *minus* the elements of the dual basis, such that a minus sign appears on the r.h.s. of eq. (2).

and the Maurer-Cartan equations read in components

$$\partial_\mu e_\nu^a - \partial_\nu e_\mu^a + f^{abc} e_\mu^b e_\nu^c = 0. \quad (3)$$

e_μ^a is a $\dim G$ -bein and the (constant curvature) metric on the group manifold is

$$g_{\mu\nu} = \delta_{ab} e_\mu^a e_\nu^b. \quad (4)$$

We can also think of e_μ^a as a $\mathcal{G}$ -gauge field living on the group-manifold. The Cartan-Maurer equations (2) or (3) then simply state that the associated field strength vanishes, i.e. e_μ^a is pure gauge (we assume G to be a simply-connected covering group). Let us consider the effect of gauge transformations with parameters ξ^a

$$\delta_\xi e_\mu^a = \partial_\mu \xi^a + f^{abc} e_\mu^b \xi^c. \quad (5)$$

Obviously, they leave invariant the Maurer-Cartan equations, but induce a change of the metric (4)

$$\delta_\xi g_{\mu\nu} = e_\nu^a \partial_\mu \xi^a + e_\mu^a \partial_\nu \xi^a. \quad (6)$$

The $\dim G$ -bein e_ρ^a allows us to trade the flat index a for an index ρ of a vector field on the group manifold :

$$\xi^a = e_\rho^a \xi^\rho. \quad (7)$$

Inserting this equation into (6) and using the Maurer-Cartan equations (3) we get

$$\delta_\xi g_{\mu\nu} \equiv \delta_\xi g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + g_{\mu\rho} \partial_\nu \xi^\rho + g_{\nu\rho} \partial_\mu \xi^\rho \quad (8)$$

or directly for the $\dim G$ -bein

$$\delta_\xi e_\mu^a \equiv \delta_\xi e_\mu^a = \xi^\rho \partial_\rho e_\mu^a + e_\rho^a \partial_\mu \xi^\rho. \quad (9)$$

The gauge transformation (5) plus the trading of a Lie algebra index for a vector index result in a diffeomorphism. This is reminiscent of the soldering procedure of

Polyakov^[6] that yields (chiral) 2D diffeomorphisms from a $SL(2, \mathbf{R})$ Kac-Moody symmetry. Note that here we generate covariant diffeomorphisms on a $\dim G$ -dimensional manifold.

Let us note that eq. (7), though natural if we consider e_ρ^a as a $\dim G$ -bein, is rather unnatural from the point of view of gauge theory : the gauge transformation parameter ξ^a is a combination of the gauge field e_ρ^a and a vector field ζ^ρ . In particular, this implies that ξ^a transforms itself non-trivially under gauge transformations. As a consequence, the commutator of two such gauge transformations with parameters $\xi^a = e_\rho^a \zeta^\rho$ and $\tilde{\xi}^a = e_\rho^a \tilde{\zeta}^\rho$ is a gauge transformation *not* with parameter $f^{abc} \xi^b \tilde{\xi}^c$ but instead with parameter $\xi'^a = e_\rho^a \zeta'^\rho$ where

$$\zeta'^\rho = \{\tilde{\zeta}, \zeta\}^\rho = \tilde{\zeta}^\nu \partial_\nu \zeta^\rho - \zeta^\nu \partial_\nu \tilde{\zeta}^\rho \quad (10)$$

is the Lie-bracket of $\tilde{\zeta}$ and ζ . This is of course what one expects for the commutator of two diffeomorphisms generated by the vector fields ζ and $\tilde{\zeta}$.

Another remark is in order. A gauge field e_μ^a is not necessarily an invertible matrix as is a $\dim G$ -bein. This is a non-trivial issue if one tries to interpret a gauge field as a viel-bein. Here, however, the situation is reversed. We start with a $\dim G$ -bein e_μ^a which is invertible. We then interpret it as a gauge field. The question now is whether finite gauge transformations, obtained by integrating eq. (5) preserve the invertibility. Equations (7) and (9) tell us that all such gauge transformations are equivalent to diffeomorphisms. Since the latter preserve the invertibility of the $\dim G$ -bein e_μ^a this is trivially also true for the gauge transformations.

3. Kaluza-Klein and the generalized W -transformations

All preceding equations remain true if we perform a dimensional reduction and consider a D -dimensional submanifold of the group-manifold. Then, all Greek indices only run from 1 to D whereas Latin indices still run from 1 to $\dim G$. In particular, the e_μ^a are no longer invertible. Equations (5) and (7) now generate D -dimensional diffeomorphisms but they only constitute a (D parameter) subset of all possible gauge transformations. What other gauge transformations can we write down? Rewriting eq. (7) as $\xi^a = \delta^{ab} e^b \zeta^\rho$, it is suggestive to replace the G -invariant tensor δ^{ab} (associated with the second-order Casimir invariant) by other G -invariant tensors. To be specific, consider $G = SU(N)$ with invariant tensors $d^{a_1 a_2 \dots a_i}$, $i = 3, \dots, N$, corresponding to the higher order Casimir invariants. Equation (7) can then be replaced by

$$\begin{aligned} \xi^a(\zeta) &= e_\rho^a \zeta^\rho, \\ \xi^a(\Lambda) &= d^{abc} e_\rho^b e_\sigma^c \Lambda^{\rho\sigma}, \\ &\dots \\ \xi^a(\eta) &= d^{ab_1 \dots b_k} e_{\rho_1}^{b_1} \dots e_{\rho_k}^{b_k} \eta^{\rho_1 \dots \rho_k}, \\ &\dots \\ \xi^a(\omega) &= d^{ab_1 \dots b_{N-1}} e_{\rho_1}^{b_1} \dots e_{\rho_{N-1}}^{b_{N-1}} \omega^{\rho_1 \dots \rho_{N-1}}, \end{aligned} \quad (11)$$

where $\zeta, \Lambda, \dots, \eta, \dots, \omega$ are completely symmetric tensors. Altogether they have $\sum_{k=1}^{N-1} \sum_{l=1}^k \binom{k-1}{l-1} \binom{D}{l} = \binom{N+D-1}{D} - 1$ independent components. For $D = 2$, e.g., these are $\frac{1}{2}N(N+1) - 1$ components, whereas $\dim SU(N) = N^2 - 1$. Hence, in $D = 2$, they all generate independent transformations but there are always some gauge transformations that cannot be generated this way.* For $SU(2)$

* For $D > 2$, the situation is different. For $D = 3$, e.g. there are $\frac{1}{6}N(N+1)(N+2) - 1$ independent components. Hence, the transformations (11) contain too many parameters for all N (except for $N = 2$ where $D = \dim G$), i.e. the corresponding gauge transformations are not all independent. This is the generic situation for $D \geq 3$.

only one parameter is missing and it is tempting to associate it with Weyl transformations, whereas for general $SU(N)$ the missing parameters might correspond to generalized Weyl transformations. Let us stress that it is the existence of the invariant tensors $d^{ab_1 \dots b_k}$ that allows us to trade the index a for a rank- k symmetric tensor $\eta^{\rho_1 \dots \rho_k}$. This is different from standard Kaluza-Klein where a vector only yields a vector and scalars.

It is straightforward to write down the transformation laws of e_μ^a under these gauge transformations. Using the Maurer-Cartan equations (3) and the Jacobi identities

$$f^{ab_1 c} d^{cb_2 \dots b_{k+1}} + (\text{cyclic permutations of the } b_i) = 0 \quad (12)$$

we obtain for the transformation involving e.g. the rank k symmetric tensor $\eta^{\rho_1 \dots \rho_k}$

$$\delta_{\xi(\eta)} e_\mu^a = d^{ab_1 \dots b_k} (e_{\rho_1}^{b_1} \dots e_{\rho_k}^{b_k} \partial_\mu \eta^{\rho_1 \dots \rho_k} + k \partial_{\rho_1} e_\mu^{b_1} e_{\rho_2}^{b_2} \dots e_{\rho_k}^{b_k} \eta^{\rho_1 \dots \rho_k}) \quad (13)$$

For $k = 1$ these are again simply D -dimensional diffeomorphisms.

The commutator of two such transformations, involving a rank k tensor $\eta^{\rho_1 \dots \rho_k}$ and a vector ζ^ρ (i.e. a diffeomorphism), respectively, is easily computed. As always, we assume that the gauge transformations only act on the gauge field e_μ^a , as in (13), and not on the tensor fields ζ^ρ , $\eta^{\rho_1 \dots \rho_k}$, etc. We obtain :

$$\left[\delta_{\xi(\zeta)}, \delta_{\xi(\eta)} \right] e_\mu^a = \delta_{\xi(\eta')} e_\mu^a \quad (14a)$$

where

$$\begin{aligned} \eta'^{\rho_1 \dots \rho_k} &= \{\eta, \zeta\}^{\rho_1 \dots \rho_k} \\ &\equiv \eta^{\nu \rho_2 \dots \rho_k} \partial_\nu \zeta^{\rho_1} + (\text{cyclic permutations of the } \rho_1 \dots \rho_k) - \zeta^\nu \partial_\nu \eta^{\rho_1 \dots \rho_k}. \end{aligned} \quad (14b)$$

In particular, if all indices take the same value, we simply have $\eta'^{\rho \dots \rho} = k \eta^{\nu \rho \dots \rho} \partial_\nu \zeta^\rho - \zeta^\nu \partial_\nu \eta^{\rho \dots \rho}$.

How do we interpret this commutation relation? Commuting a diffeomorphism with the rank- k tensor transformation gives back another rank- k tensor transformation. In fact, the commutator (14a,b) is very reminiscent of the OPE of the energy-momentum tensor T and a spin- $(k+1)$ field $W^{(k+1)}$ in 2D conformal field theory. Let us be more explicit. The OPE is

$$T(z)W^{(k+1)}(z') \sim \frac{(k+1)W^{(k+1)}(z')}{(z-z')^2} + \frac{\partial W^{(k+1)}(z')}{z-z'} \quad (15)$$

and hence

$$\left[\oint dz \zeta(z) T(z), \oint dz' \eta(z') W^{(k+1)}(z') \right] = \oint dz \eta'(z) W^{(k+1)}(z) \quad (16a)$$

where

$$\eta' = k\eta\partial_z\zeta - \zeta\partial_z\eta \quad (16b)$$

The analogy between eqs. (14) and (16) is obvious. More precisely, if we consider eq. (14) for $D=2$ and restrict ourselves to chiral transformation parameters, i.e. only $\eta^{z\dots z} \equiv \eta$ and $\zeta^z \equiv \zeta$ are non-zero, the composition laws (14b) and (16b) are identical. Equations (14) however not only hold in $D=2$, and, a priori, do not correspond to a conformal field theory. This suggests identifying the (gauge) transformation $\delta_{\xi(\eta)}$ containing the rank- k symmetric tensor $\eta^{\rho_1\dots\rho_k}$ with a generalization of the (chiral) $W^{(k+1)}$ transformation.

If we really aim to find a covariant generalization of the W -algebra we have to solve two related problems. First, the commutator of two W -generators contains in general polynomials of the other generators, e.g. the commutator of two $W^{(3)}$'s contains the square of the energy-momentum tensor. These nonlinearities are the essential feature of W -algebras and should be present in any covariant generalization. The second problem is a direct consequence of this nonlinear structure of the algebra : the sum of two representations is *not* a representation of the W -algebra. This means that, even if we find a $\delta_{\xi(\eta)}$ such that

$[\delta_{\xi(\eta)}, \delta_{\xi(\tilde{\eta})}]e_\mu^a$ is what one expects to generalize the W -algebra, this is not true e.g. for $[\delta_{\xi(\eta)}, \delta_{\xi(\tilde{\eta})}](e_\mu^a e_\nu^b)$. A way out of this second problem might be to modify the product rule $\delta_{\xi(\eta)}(AB) = A(\delta_{\xi(\eta)}B) + (\delta_{\xi(\eta)}A)B$. Clearly, this is only possible if δ_ξ does not act by commutation on AB .

Here we are not going to solve these problems in general. We will only consider our generalized W -candidates $\delta_{\xi(\eta)}$ and compute their commutator when acting on e_μ^a . The general formula for the commutator of $\delta_{\xi(\omega)}$ for a rank- k symmetric tensor ω (corresponding to a generalized $W^{(k+1)}$) with $\delta_{\xi(\Lambda)}$ for a rank- l symmetric tensor Λ (corresponding to a generalized $W^{(l+1)}$), when acting on e_μ^a , is given in the appendix. To simplify notations, here we only write down the equation for $k=l=2$ corresponding to the commutator of two generalized $W^{(3)}$'s. Using eq. (13) for $k=2$ we obtain :

$$\begin{aligned} \left[\delta_{\xi(\omega)}, \delta_{\xi(\Lambda)} \right] e_\mu^a &= 4\phi^{\lambda\kappa\rho\sigma} d^{abc} d^{bde} (\partial_\rho (\partial_\lambda e_\mu^d e_\kappa^e) e_\sigma^c + \partial_\rho e_\mu^c \partial_\lambda e_\sigma^d e_\kappa^e) \\ &\quad + 4\varphi_\mu^{\lambda\kappa\rho\sigma} (d^{abc} d^{bde} - d^{abe} d^{bdc}) \partial_\rho e_\lambda^d e_\kappa^e e_\sigma^c \\ &\quad + 2d^{abc} d^{bde} (e_\lambda^d e_\kappa^e e_\sigma^c \partial_\mu \psi^{\lambda\kappa\sigma} + (2\partial_\lambda e_\mu^d e_\kappa^e e_\sigma^c + e_\lambda^d e_\mu^e \partial_\sigma e_\kappa^c) \psi^{\lambda\kappa\sigma}) \end{aligned} \quad (17a)$$

where

$$\begin{aligned} \psi^{\lambda\kappa\sigma} &= \partial_\rho \omega^{\lambda\kappa} \Lambda^{\rho\sigma} - \omega^{\rho\sigma} \partial_\rho \Lambda^{\lambda\kappa}, \\ \phi^{\lambda\kappa\rho\sigma} &= \omega^{\lambda\kappa} \Lambda^{\rho\sigma} - \omega^{\rho\sigma} \Lambda^{\lambda\kappa}, \\ \varphi_\mu^{\lambda\kappa\rho\sigma} &= \partial_\mu \omega^{\lambda\kappa} \Lambda^{\rho\sigma} - \omega^{\rho\sigma} \partial_\mu \Lambda^{\lambda\kappa} \end{aligned} \quad (17b)$$

Next we use the following identity for the product of two d -tensors of $SU(N)$ (see appendix)

$$\begin{aligned} d^{abcf} d^{cdef} &= 4\alpha d^{abcd} - \frac{4}{3N(N^2+1)} \left((2N^2-3) (\delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}) - (N^2+6) \delta^{ab}\delta^{cd} \right) \\ &\quad - \frac{1}{3} (f^{ac}f^{bdf} + f^{ad}f^{bfc}) \end{aligned} \quad (18)$$

(α is a constant depending on the normalization of the d^{abcd}) to obtain

$$\begin{aligned}
\left[\delta_{\xi(\omega)}, \delta_{\xi(\Lambda)} \right] e_\mu^a &= 8\alpha d_{abcd} \left(3\partial_\rho e_\mu^b e_\sigma^c e_\lambda^d \tilde{\psi}^{\rho\sigma\lambda} + e_\rho^b e_\sigma^c e_\lambda^d \tilde{\psi}^{\rho\sigma\lambda} \right) \\
&- \frac{16(2N^2 - 3)}{3N(N^2 + 1)} \left(\phi^{\lambda\kappa\rho\sigma} (\partial_\lambda (\partial_\rho e_\mu^b e_\sigma^b) e_\kappa^a + \partial_\lambda e_\mu^a \partial_\rho e_\sigma^b e_\kappa^b) + \varphi_\mu^{\lambda\kappa\rho\sigma} \partial_\rho e_\lambda^b (\partial_\sigma e_\mu^b e_\kappa^a - e_\kappa^b e_\sigma^a) \right. \\
&\quad \left. + e_\lambda^a e_\kappa^b e_\sigma^c \partial_\mu \psi^{\lambda\kappa\sigma} + (\partial_\lambda e_\mu^a e_\sigma^b e_\kappa^b + \partial_\lambda e_\mu^b e_\sigma^b e_\kappa^a + \partial_\sigma e_\mu^b e_\kappa^b e_\lambda^a) \psi^{\lambda\kappa\sigma} \right) \\
&+ \frac{8(N^2 + 6)}{3N(N^2 + 1)} \left(2\phi^{\lambda\kappa\rho\sigma} (\partial_\rho (\partial_\lambda e_\mu^b e_\sigma^b) e_\kappa^a + \partial_\rho e_\mu^a \partial_\lambda e_\sigma^b e_\kappa^b) - 2\phi_\mu^{\lambda\kappa\rho\sigma} \partial_\rho e_\lambda^b (\partial_\sigma e_\mu^b e_\kappa^a - e_\kappa^b e_\sigma^a) \right. \\
&\quad \left. + e_\sigma^a e_\lambda^b e_\kappa^c \partial_\mu \psi^{\lambda\kappa\sigma} + (\partial_\sigma e_\mu^a e_\lambda^b e_\kappa^b + 2\partial_\lambda e_\mu^b e_\sigma^b e_\kappa^a + 2\partial_\lambda e_\mu^b e_\sigma^b e_\kappa^a) \psi^{\lambda\kappa\sigma} \right) \\
&- \frac{2}{3} \left((f^{adj} f^{ce} f + f^{ae} f f^{cd}) \left(2\phi^{\lambda\kappa\rho\sigma} (\partial_\rho (\partial_\lambda e_\mu^d e_\sigma^e) e_\kappa^c + \partial_\rho e_\mu^c \partial_\lambda e_\sigma^d e_\kappa^e) + e_\lambda^d e_\kappa^e e_\sigma^c \partial_\mu \psi^{\lambda\kappa\sigma} \right. \right. \\
&\quad \left. \left. + (\partial_\sigma e_\mu^c e_\lambda^d e_\kappa^e + 2\partial_\lambda e_\mu^d e_\sigma^e e_\kappa^c) \psi^{\lambda\kappa\sigma} \right) + 2f^{adj} f^{ce} f \varphi_\mu^{\lambda\kappa\rho\sigma} \partial_\rho e_\lambda^d e_\sigma^e e_\kappa^c \right) \quad (19)
\end{aligned}$$

where $\tilde{\psi}^{\rho\sigma\lambda}$ denotes $\psi^{\rho\sigma\lambda}$ completely symmetrized in its 3 indices. The first term on the r.h.s. is simply $8\alpha\delta_{\xi(\psi)} e_\mu^a$: a generalized $W^{(4)}$ -candidate acting on e_μ^a , as one expects to appear in the commutator of two generalized $W^{(3)}$ -candidates. To see that this equation is indeed a covariant generalization^{*} of the corresponding commutator of the chiral W_N -algebra, let us consider eq. (19) for $D = 2$ and with chiral transformation parameters, i.e. we suppose that $\omega^{zz} \equiv \omega$ and $\Lambda^{zz} \equiv \Lambda$ are the only non-zero components. In this case, only terms $\sim \psi^{zzz} = \partial_z \omega \Lambda - \omega \partial_z \Lambda$ survive and equation (19) simplifies to yield

$$\left[\delta_{\xi(\omega)}, \delta_{\xi(\Lambda)} \right]_{\text{chiral}} e_\mu^a = 8\alpha\delta_{\xi(\psi^{***})} e_\mu^a - \frac{8(N^2 - 4)}{N(N^2 + 1)} \left(e_z^a e_z^b e_z^c \partial_\mu \psi^{zzz} + (\partial_z e_\mu^a e_z^b e_z^c + 2\partial_z e_\mu^b e_z^c e_\mu^a) \psi^{zzz} \right) \quad (20)$$

where all terms containing the structure constants f^{abc} have dropped out. Note that for $N = 2$ the coefficient of the second term on the r.h.s. vanishes as it

* Note that we did not assume $D = 2$.

should since in this case $W^{(3)} = W^{(4)} = 0$. Let us compare with the corresponding commutator of the W_N -algebra. The OPE is

$$W^{(3)}(z)W^{(3)}(z') \sim \frac{xW^{(4)}(z') + yT^2(z')}{(z - z')^2} + \frac{\partial(xW^{(4)}(z') + yT^2(z'))}{z - z'} + \dots \quad (21)$$

where $+ \dots$ indicate (Virasoro-) descendant fields of the identity (like $T, \partial T, \partial^2 T$) or the identity itself (i.e. the central term). These latter fields are present in the (quantum) OPE but are expected to be absent in the present classical computation. x and y are normalization-dependent constants that can be set equal to 1. Hence

$$\begin{aligned}
&\left[\oint dz \omega(z) W^{(3)}(z), \oint dz' \Lambda(z') W^{(3)}(z') \right] \\
&= \oint dz' \psi(z') \left(xW^{(4)}(z') + yT^2(z') \right) + \dots \quad (22a)
\end{aligned}$$

where

$$\psi = \partial \omega \Lambda - \omega \partial \Lambda \equiv \psi^{zz} \quad (22b)$$

The analogy with eq. (20) is striking. As already mentioned, the first term on the r.h.s. of (20) is a generalized $W^{(4)}$ -transformation (with the rank-3 symmetric tensor $\tilde{\psi}$ as parameter) acting on e_μ^a . What about the T^2 -term? The second term in eq. (20) is almost a diffeomorphism acting on e_μ^a but with field-dependent parameter. This statement becomes exact if we concentrate on $\mu = z$. Then

$$\left[\delta_{\xi(\omega)}, \delta_{\xi(\Lambda)} \right]_{\text{chiral}} e_z^a = 8\alpha\delta_{\xi(\psi)} e_z^a - \frac{8(N^2 - 4)}{N(N^2 + 1)} \delta_{\xi(\zeta)} e_z^a \quad (23a)$$

where the field-dependent parameter ζ of the diffeomorphism is

$$\zeta^z = e_z^b e_z^c \psi^{zzz} \quad (23b)$$

Clearly, at the classical level, we can interpret the non-linear T^2 -term in eq. (22) as a diffeomorphism with parameter $T_{zz} \psi^{zzz}$. The analogy between eqs. (22) and

(23) would be complete if we could identify $e_2^b e_2^b$ (which is the metric of our 2D manifold inherited by the dimensional reduction from the group manifold) with the energy-momentum tensor T_z . We will not explore under which circumstances such an identification could be justified. This point touches upon the intrinsic non-linear structure of the W_N -algebras and clearly needs a better understanding.

4. Conclusion

We started with a reformulation of the diffeomorphisms on a group manifold as gauge-transformations acting on the dim G -bein e_μ^a . Performing a dimensional reduction to a D -dimensional submanifold allowed us to interpret some of these gauge transformations/dim G -dimensional diffeomorphisms still as D -dimensional diffeomorphisms, while others were interpreted as generalized, i.e. covariant and D -dimensional, W_N -transformations. The latter involve rank- k symmetric tensors as transformation parameters ($k = 2, \dots, N$ for $G = SU(N)$). We computed the commutator of two such generalized $W^{(3)}$ -candidates (for general W_N corresponding to $G = SU(N)$) and showed that it indeed generalizes the chiral W_N -algebra. Some problems related to the non-linear structure of the W_N -algebras remained unsolved. In conclusion, it thus seems that the W_N -algebras have a natural geometric interpretation as remnants of the (bigger) diffeomorphism symmetry on the higher-dimensional group manifold. However, more work is required before we can really answer the question “what is W -geometry?”

APPENDIX

In this appendix, we first give the general commutator of $\delta_{\xi(\Lambda)}$ with $\delta_{\xi(\omega)}$, when acting on e_μ^a , where ω and Λ are rank- k , resp. rank- l , symmetric tensors. $\delta_{\xi(\omega)}$ and $\delta_{\xi(\Lambda)}$ correspond to generalized $W^{(k+1)}$ - and $W^{(l+1)}$ -transformations.

$$\begin{aligned} & \left[\delta_{\xi(\omega)}, \delta_{\xi(\Lambda)} \right] e_\mu^a = e_{\sigma_3}^{c_3} \dots e_{\sigma_k}^{c_k} e_{\rho_3}^{b_3} \dots e_{\rho_l}^{b_l} \times \\ & \times \left\{ e_\lambda^{c_2} e_{\sigma_2}^{b_2} \partial_\mu \left(l d^{adb_2} \dots b_l d^{dec_2} \dots c_k \partial_{\rho\omega} \omega^{\lambda\sigma_2} \dots \sigma_k \Lambda^{\rho\rho_2} \dots \rho_l \right) \right. \\ & \quad - k d^{edb_2} \dots b_l d^{dac_2} \dots c_k \omega^{\rho\sigma_2} \dots \sigma_k \partial_\rho \Lambda^{\lambda\rho_2} \dots \rho_l \Big) \\ & + d^{adeb_3} \dots b_l d^{dfg_{c_3}} \dots c_k \partial_{\rho\omega} \omega^{\nu\kappa\sigma_3} \dots \sigma_k \Lambda^{\rho\lambda\rho_3} \dots \rho_l \left(l(l-1) \partial_\lambda e_\mu^e e_\nu^f + l k \partial_\nu e_\mu^f e_\lambda^e \right) e_\kappa^g \\ & - d^{dfg_{b_3}} \dots b_l d^{adec_3} \dots c_k \omega^{\rho\lambda\sigma_3} \dots \sigma_k \partial_\rho \Lambda^{\nu\kappa\rho_3} \dots \rho_l \left(k(k-1) \partial_\lambda e_\mu^e e_\nu^f + l k \partial_\nu e_\mu^f e_\lambda^e \right) e_\kappa^g \\ & + l k \left(d^{adb_2} \dots b_l d^{dec_2} \dots c_k - d^{edb_2} \dots b_l d^{dac_2} \dots c_k \right) \partial_\nu e_\lambda^e e_{\sigma_2}^{b_2} e_{\rho_2}^{c_2} \\ & \quad \times \left(\omega^{\nu\sigma_2} \dots \sigma_k \partial_\mu \Lambda^{\lambda\rho_2} \dots \rho_l + \partial_\mu \omega^{\lambda\sigma_2} \dots \sigma_k \Lambda^{\nu\rho_2} \dots \rho_l \right) \\ & + l k \left(d^{adeb_3} \dots b_l d^{dfg_{c_3}} \dots c_k - d^{fdeb_3} \dots b_l d^{adg_{c_3}} \dots c_k \right) \omega^{\lambda\sigma_2} \dots \sigma_k \Lambda^{\nu\rho_2} \dots \rho_l \\ & \quad \times \left. \left(\partial_\nu \partial_\lambda e_\mu^f e_\nu^g e_{\sigma_2}^e e_{\rho_2}^e + (k-1) \partial_\lambda e_\mu^g \partial_\nu e_\nu^f e_{\sigma_2}^e e_{\rho_2}^e + (l-1) \partial_\lambda e_\mu^f \partial_\nu e_\nu^g e_{\sigma_2}^e e_{\rho_2}^e \right) \right\} \quad (A.1) \end{aligned}$$

Second, we prove the identity (18) needed to reexpress the product of two 3-index d -tensor for general $SU(N)$. For $N = 3$ it can be found in ref. [7]. We normalize the antihermitian generators T_a of $SU(N)$ by $\text{tr} T_a T_b = -\delta_{ab}$. One then defines d_{abc} as $\{T_a, T_b\} = i d_{abc} T_c - \frac{2}{N} \delta_{ab}$. Hence, $-d_{ade} d_{ecd} + \frac{4}{N} \delta_{ab} \delta_{cd} = \text{tr} \{ \{T_a, T_b\}, T_c \} T_d$. Rewriting the r.h.s. as a sum of a completely symmetrized trace $x_{abcd} = \text{tr} T_{(a} T_b T_c T_{d)}$ and commutator terms one gets

$$-d_{ade} d_{ecd} = 4x_{abcd} - \frac{4}{N} \delta_{ab} \delta_{cd} + \frac{1}{3} (f_{ade} f_{bce} + f_{ace} f_{bde}) \quad (A.2)$$

The x_{abcd} is related to the 4-index d -tensor by

$$x_{abcd} = -\alpha d_{abcd} + \frac{2N^2 - 3}{3N(N^2 + 1)} (\delta_{ab} \delta_{cd} + \delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}) \quad (A.3)$$

d_{abcd} is defined such that $d_{aacd} = 0$. This fixes the constant in front of the δ -terms. On the other hand, the constant α depends on the normalization of d_{abcd} . Inserting (A.3) into (A.2) we get the desired identity (18).

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