A NOTE ON THE STRONG COUPLING REGIME OF TWO-DIMENSIONAL QUANTUM GRAVITY

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ABSTRACT

We apply the operatorial results of Bilal-Gervais-Neveu on the existence of a non-critical bosonic string for $D = 7, 13$ and 19 to a path integral formulation *à la* Distler-Kawai, and find $\gamma = \frac{1}{2}$ as the string susceptibility exponent.

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There are two apparently different approaches to two-dimensional quantum gravity (2D-QG): the Liouville field theory approach^[1] and the randomly triangulated approach.^[2] An interesting problem with the Liouville field approach is the breakdown when the central charge (dimension D) of the bosonic string described by the embedding field $X^\mu(z, \bar{z})$, $\mu = 1, \dots, D$, in the target space $\mathbb{R}^D$ is larger than one. The formula for the susceptibility exponent is, for genus-zero string world-sheet

$$\gamma = \frac{D - 1 \pm \sqrt{(D-1)(D-25)}}{12} \quad (1)$$

and it is seen that γ becomes complex when $1 < D < 25$, the so-called “strong coupling regime” (SCR) of 2D-QG.

On the other hand, the randomly triangulated model is well defined in any dimensions D . Unfortunately, no analytical methods are presently available for solving the triangulated models for $d > 1$, and one has to rely on numerical Monte Carlo methods for simulating the surfaces. These numerical methods give γ settled near to $\frac{1}{2}$ ($\gamma = \frac{1}{2}$ in the mean-field theory). Notice that as long as the random triangulated models for large dimensions ($D > 2$) are described by branched polymers, the result $\gamma = \frac{1}{2}$ is exact.

Recently Gervais^[3] has shown in the framework of the canonical formalism that there exists a non-critical bosonic string theory for dimensions $D = 26 - C_{Liouville} = 7, 13, 19$, at least order by order in the expansion over genus. The key point of Ref. 3 is the result that in the SCR of 2D-QG the scaling parameters h_+ and h_- of the Liouville quantum field $\phi = N(\exp(h\phi))$ (here N is the normal ordering symbol) are complex and $h_- = h_+^*$. Thus both values of h must be retained: h_+ and h_- for $1 < D < 25$ are constrained together by the relation (see Ref. 3)

$$h_+ + h_- = s\pi, \quad h_+ h_- = \pi^2, \quad \text{integer } s \quad (2)$$

In this letter we shall show how to take into account (2) in a covariant approach *à la* Distler-Kawai (see Ref. 1). As a consequence we find on the disk that in the regime of validity of (2), i.e. in the strong coupling phase, the string susceptibility exponent γ is $\gamma = \frac{1}{2}$. From a physical point of view this fact means that in the covariant Liouville approach one cannot assume as boundary condition the semiclassical result $D \rightarrow -\infty$. Indeed, the result of Gervais (see Ref. 3), stated in other words, implies that in the SCR of 2D-QG the

underlying quantum group structure is of Sklyanin’s type.^[4] This is a hyperbolic quantum algebra with a real quantum deformation parameter $q = \exp(\lambda/2)$, whose classical limit $\lambda \rightarrow 0$ ($q \rightarrow 1$) is not a Lie algebra, here A_1 .

Following DK (see Ref. 1), Polyakov’s bosonic string partition function may be written in terms of the partition function for fixed genus p and intrinsic area A as:

$$Z_{Polyakov} = \sum_p \int_0^\infty dA Z_p[A] e^{-\beta A} \quad (3)$$

$$Z_p[A] = \int_{\mathcal{M}_p} \mathcal{D}_g \mu \mathcal{D}_g \phi \delta \left[\int d^2 z e^{\alpha \phi} \sqrt{\hat{g}} - A \right] e^{-S_L(\phi, \hat{g})}$$

where $\mathcal{D}_g \mu$ denotes the contribution of the matter fields $X^\mu(z, \bar{z})$ and the reparametrization ghosts which depend only on the moduli m_i of $\mathcal{M}_p$, $\hat{g}(m_i)$ is a background two-metric, ϕ is the Liouville (Weyl) mode defined by the conformal gauge $g = e^{\alpha \phi} \hat{g}(m_i)$ and S_L is the Liouville action without the cosmological constant term:

$$S_L(\phi, \hat{g}) = \frac{1}{2\pi} \int_{\Sigma} d^2 z \left(\partial \phi \bar{\partial} \phi - \frac{1}{4\sqrt{\hat{g}}} \sqrt{\hat{g}} \hat{R}(\hat{g}) \phi \right) \quad (4)$$

In (4), $Q \equiv \frac{1}{\sqrt{\hat{g}}}$ is the background charge defined by the condition:

$$C_{grav.} \equiv C_{Liou.} = \left(1 + \frac{3}{\rho} \right) = (26 - D) \quad (5)$$

where $C_{ghost} = -26$ and $C_{X^\mu} = +D$. As is well known, the calculation of the string susceptibility exponent γ is here obtained by a scaling argument on the partition function $Z_p[A]$. In other words under a constant shift in the Liouville field:

$$\phi \rightarrow \phi + \ln(A)/\alpha \quad (6)$$

the partition function $Z_p[A]$ must satisfy the following scaling law:

$$Z_p[A] = Z_p[1] A^{(Q/2\alpha)-1} \equiv Z_p[1] A^{\gamma-2} \quad (7)$$

where χ is the Euler characteristic of the Riemann surface Γ . Notice that (7) follows by the conformal invariance of the theory if one has also $\hat{g} \rightarrow \frac{1}{A} \hat{g}$. This also means that

the Liouville field operator $\psi \equiv N[\exp(\alpha\phi)]$ in (3) must be a primary conformal field of dimension $(\Delta, \bar{\Delta}) = (1, 1)$, i.e. α is such that:

$$-\frac{1}{2}\alpha\left(\alpha + \frac{1}{\sqrt{\rho}}\right) = 1 \quad (8)$$

Denote by α_+ , α_- the two solutions of (8), i.e.:

$$2\alpha_{\pm} = -\frac{1}{\sqrt{\rho}}\left(1 \pm \sqrt{1-8\rho}\right) = -\frac{1}{\sqrt{3}}\left(\sqrt{25-D} \pm \sqrt{1-D}\right) \quad (9)$$

Here we have used (5). Notice that in the SCR, $1 < D < 25$, α_+ and α_- are complex conjugate, and hence both α_+ and α_- must be treated on the same footing. In order to achieve this result, we define an improved Weyl gauge fixing in (3) as:

$$\int d^2z \sqrt{\hat{g}} F(e^{\alpha\phi}) = A \quad (10)$$

such that F is a suitable functional of the Liouville operator $N(e^{\alpha\phi})$ with conformal dimension $(\Delta_F, \bar{\Delta}_F) = (1, 1)$ iff the coefficient α is given by:

$$\alpha = \alpha_+ + \alpha_- \quad (11)$$

with α_+ and α_- defined by (9). In other words $\delta(\int d^2z \sqrt{\hat{g}} F(e^{\alpha\phi}) - A)$ must rescale as A^{-1} under the shift (6) in which α is defined by (11). The link with the notation of Gervais (see Ref. 3) is obtained by rewriting $\alpha_{\pm}$ in the form:

$$\alpha_{\pm} = -\frac{2\sqrt{\rho}}{\pi}(h_{\mp} + \pi) \quad (12)$$

Thus in Ref. 3 it is shown that there exists a non-trivial physical operatorial solution of the quantum Liouville theory in dimensions $D = 7, 13$ and 19 , i.e. in the strong coupling regime, iff:

$$h_+ + h_- = s\pi, \quad h_+ h_- = \pi^2, \quad \text{integer } s \quad (13)$$

$$Q^2 \equiv \frac{1}{\rho} = 2(s+2)$$

which is equivalent to:

$$C_{\text{grav.}} \equiv C_{\text{Liou.}} = 1 + 6(s+2) \quad (14)$$

(since we are in the SCR $\rho > \frac{1}{8}$, $1 < C_{\text{grav.}} < 25$, there are three solutions with $s = -1, 0, +1$ leading by (5) to: $D = (26 - C_{\text{grav.}}) = 19, 13, 7$). By using Eqs. (11, 12) and the scaling law (7) together with Gervais' results Eqs. (2, 13, 14), one gets that in the case when Σ is a disk $\mathcal{D}$ ($\chi = 1$), γ is given by:

$$\gamma_0 = \frac{1}{2}\frac{Q}{\alpha} + 1 = \frac{1}{2}\frac{Q}{(-Q)} + 1 = \frac{1}{2} \quad (15)$$

in agreement with the random surface models. Of course the above result on the susceptibility exponent γ is exact for any value of D in the SCR for which (2) and Eqs. (13, 14) are satisfied, i.e. for $D = 7, 13, 19$.

Now we shall show a possible solution for the prescription (10) such that $(\Delta_F, \bar{\Delta}_F) = (1, 1)$ is compatible with the ansatz (11). The starting point is to notice that the operator $\hat{\psi} \equiv N(\exp[(\alpha_+ + \alpha_-)\phi])$ has vanishing conformal dimension:

$$-\frac{1}{2}(\alpha_+ + \alpha_-)[(\alpha_+ + \alpha_-) + Q] = -\frac{1}{2}\alpha_+(\alpha_+ + Q) - \frac{1}{2}\alpha_-(\alpha_- + Q) - \alpha_+\alpha_- = 0,$$

since $\alpha_+\alpha_- = 2$. Then to obtain an operator of unit-dimension it is necessary to change $\hat{\psi}(z)$ for the operator $F(z)$ of dimension $\Delta_F = \Delta_{\hat{\psi}} + 1$. A way to achieve this result locally and without changing the ghost sector of 2D-QG, or adding some external conformal matter whose screening vertex operator gives just such an additional unit-dimension, is by taking the derivative ∂_z of $\hat{\psi}(z)$ (similarly for $\hat{\psi}(\bar{z})$). Although $\partial_z \hat{\psi}(z)$ is no longer a primary Liouville conformal field, one should use a covariant derivative instead of ∂_z , in this path integral context we may still regard $\partial_z \hat{\psi}(z)$ as a well defined unit-dimension operator and set F as:

$$F(z, \bar{z}) = \partial_z \partial_{\bar{z}} \hat{\psi}(z, \bar{z}) = \partial_z \partial_{\bar{z}} \left[e^{(\alpha_+ + \alpha_-)\phi(z, \bar{z})} \right] \quad (16)$$

Then the gauge-fixed Polyakov bosonic string partition function (3) by (10) and (16) becomes (at $\chi = 1$ and $\hat{g}_{ab} = \delta_{ab}$):

$$\begin{aligned} Z_0 &= \int_{\mathcal{M}_0} \mathcal{D}\mu \mathcal{D}\phi e^{-S_L[\phi, \delta]} \int_0^\infty dA e^{-\beta A \delta} \left[\int d^2z \left(\partial_z \partial_{\bar{z}} e^{(\alpha_+ + \alpha_-)\phi} \right) - A \right] \\ &= \int_{\mathcal{M}_0} \mathcal{D}\mu \mathcal{D}\phi e^{-S_L[\phi, \delta] - \beta \int \partial_z \partial_{\bar{z}} e^{(\alpha_+ + \alpha_-)\phi}} \end{aligned} \quad (17)$$

Since $\partial_z \partial_z (e^{(\alpha_+ + \alpha_-)K})$ is a total derivative, its integration over the disk $\mathcal{D}$ gives a boundary term.

At the end let us observe that Eq. (16) is a possible solution for the prescription (10) and (11), but clearly it is not unique.

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