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$SL(2)_k$ WZNW Models, Perturbations and Integrable Models

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ABSTRACT:

Perturbations of the $SL(2)_k$ WZNW model by relevant operators are studied. Nontrivial off-critical conservation laws in these perturbed systems are constructed. For the rational level k , one of the perturbations is shown to be an analogue of the $(1, 2j + 1)$ operator in the WZNW model. Linear systems (Lax pairs) are defined whose zero curvature condition give rise to the classical versions of the perturbed equations of motion. The classical r -matrix for these linear systems is found.

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1 Introduction

The discovery by Zamolodchikov^[1] that perturbing some minimal rational conformal field theories by means of specific relevant operators, leads to infinite conserved charges, has brought a lot of interest into this subject and the related field of integrable models [2--6]. The first problem indeed consists in recognizing what integrable model corresponds to the above infinite conserved charges at the lowest order perturbation theory.

In a sense, since minimal conformal field theories can be described by means of integrable models (the Liouville model and relevant generalizations), it is not unmotivated to speculate that crucial tools in describing certain systems at and away from criticality, are integrable models. This speculation must be complemented by the remark that the conformal invariance keeps playing a fundamental role even off criticality. Of this one can find many traces in the literature [7]. In a recent paper it has been suggested [8] that this nice interplay between conformal invariance and integrability may be implemented in a theory which is both conformal invariant and integrable: in such a framework, off-critical integrable models can be recast as special (conformal symmetry breaking) limits. These conformal invariant and integrable theories are based on Kac-Moody algebras, and among other things, they introduce a natural spectral parameter into the theory.

The above concerns minimal conformal field theories and their integrable perturbations. It is interesting to see whether this can be extended to other models. In this paper we are concerned with the $SL(2)_k$ WZNW model and some of its perturbations. The problem of perturbing WZNW models is interesting in itself, but it is the more relevant because they bear a direct relation to the Chern-Simons theories in 2+1 dimensions^[9,1]. It is still an open problem how to introduce a spectral parameter in the Chern-Simons theories. Perturbing WZNW models is perhaps a clue to solving this problem.

This paper is organized as follows. In section 2, we summarize the free field representation of the WZNW models, which is basic for our construction. In section 3, we collect many examples of perturbations of $SL(2)_k$ WZNW model by means of relevant operators. We construct a few non-

trivial off-critical conserved charges in these perturbed theories, to be eventually identified as integrals of motion of suitable integrable models. Thus we find the perturbed equations of motion of the basic fields. In a recent paper^[5], Eguchi and Yang have given a heuristic recipe to recognize the integrable models corresponding to certain perturbations of minimal rational CFT's. In section 4 we follow the same recipe to define new models, possessing all the properties of integrability, inspired by perturbations of the $SL(2)_k$ WZNW model: we define linear systems whose zero curvature conditions lead to equations of motion for the basic fields which are identical to the equations of motion of section 3. We find the r -matrix of these linear systems; it turns out to be the r -matrix of the sinh-Gordon model. We also define the affine extension as in ref.[8].

2 Free Field Representation of $SL(2)_k$ WZNW Model

The $SL(2)_k$ currents $J^\pm(z), J^0(z)$ and the stress tensor $T(z)$ satisfy the following OPE

$$\begin{aligned} T(z)T(z') &= \frac{c/2}{(z-z')^4} + \frac{2T(z)}{(z-z')^3} + \frac{\partial T(z')}{(z-z')^2} + \text{RT} \\ T(z)J^a(z') &= \frac{J^a(z')}{(z-z')^2} + \frac{\partial J^a(z')}{(z-z')} + \text{RT} \\ J^a(z)J^b(z') &= \frac{k/2\eta^{ab}}{(z-z')^2} + \frac{f_c^{ab}J^c(z')}{(z-z')} + \text{RT} \end{aligned} \quad (1)$$

where k is the level, $c = \frac{3k}{k+2}$ is the central charge and

$$\begin{aligned} a, b &= 0, \pm 1, \\ q^{00} &= 1, \quad q^{+-} = q^{-+} = 2 \\ f_0^{+-} &= 2, \quad f_+^{+0} = f_-^{0-} = -1 \end{aligned} \quad (2)$$

the rest of the f_c^{ab} vanish. The well-known free field realization of the $SL(2)_k$ currents in terms of a scalar field ϕ with a background charge and a free (β, γ) system is given by [9,10]

$$\begin{aligned} J^+ &= \beta \\ J^0 &= \beta\gamma - \frac{i}{2\alpha_0}\partial\phi \\ J^- &= -\beta\gamma^2 + \frac{i}{\alpha_0}\gamma\partial\phi + k\partial\gamma \end{aligned} \quad (3)$$

where $\alpha_0 = \frac{1}{\sqrt{2(k+2)}}$ and the normalizations are such that $\beta(z)\gamma(z') = \frac{1}{z-z'}$ and $\phi(z)\phi(z') = -\log(z-z')$. The stress tensor can be represented in the Sugawara form

$$\begin{aligned} T(z) &= \frac{1}{2(k+2)}(J^+(z)J^-(z) + J^-(z)J^+(z) + 2J^0(z)J^0(z)) \\ &= \beta\partial\gamma - \frac{1}{2}(\partial\phi)^2 - i\alpha_0\partial^2\phi \end{aligned} \quad (4)$$

For the Fourier components J_n^a and L_n ,

$$\begin{aligned} J^a(z) &= \sum_n z^{-1-n} J_n^a, \quad T(z) = \sum_n z^{-2-n} L_n \\ J_n^a &= \oint dz z^n J^a(z), \quad L_n = \oint dz z^{n+1} T(z) \end{aligned}$$

one has the following commutation relations

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0} \\ [L_n, J_m^a] &= -mJ_{n+m}^a \\ [J_n^a, J_m^b] &= f_c^{ab}J_{n+m}^c + \frac{k}{2}n\eta^{ab}\delta_{n+m,0} \end{aligned} \quad (5)$$

and the relation between generators

$$L_n = \frac{1}{2(k+2)} \sum_m : (J_m^+ J_{n-m}^- + J_m^- J_{n-m}^+ + 2J_m^0 J_{n-m}^0) : \quad (6)$$

where the symbol $::$ denotes the conventional normal ordering: the operators J_n with negative n are always placed to the left of the operators with $n > 0$.

The primary fields $\Phi_j^m(z)$ of the $SL(2)_k$ WZNW model are labelled by spin j and weights m and are represented as

$$\Phi_j^m(z) = \gamma^{j-m} e^{i2\alpha_0 j \phi} \quad (7)$$

which have the conformal dimensions $\Delta(\Phi_j^m) = \frac{j(j+1)}{k+2}$. It is easy to check that the primary fields $\Phi_j^m(z)$ have the following OPE with the currents

$$\begin{aligned} J^+(z)\Phi_j^m(z') &= \frac{j-m}{z-z'}\Phi_j^{m+1}(z') + \text{RT} \\ J^0(z)\Phi_j^m(z') &= \frac{m}{z-z'}\Phi_j^m(z') + \text{RT} \\ J^-(z)\Phi_j^m(z') &= \frac{j+m}{z-z'}\Phi_j^{m-1}(z') + \text{RT} \end{aligned} \quad (8)$$

From(6), the following equation

$$L_n \Phi_j^m = \frac{1}{2(k+2)} \sum_l : (J_l^+ J_{n-l}^- + J_l^- J_{n-l}^+ + 2J_l^0 J_{n-l}^0) : \Phi_j^m \quad (9)$$

is satisfied with any primary field Φ_j^m . Hence any primary field in WZNW model is degenerate.

The screening operator of the $SL(2)_k$ WZNW model is

$$\begin{aligned} Q_+ &= \oint \Phi_+(z) \\ \Phi_+(z) &= \beta e^{-i2\alpha_0\phi} \end{aligned} \quad (10)$$

one can easily verify that

$$\begin{aligned} J^+(z)\Phi_+(z') &= \text{RT} \\ J^0(z)\Phi_+(z') &= \text{RT} \\ J^-(z)\Phi_+(z') &= (k+2)\partial\left(\frac{1}{z-z'}e^{-i2\alpha_0\phi(z')}\right) + \text{RT} \end{aligned} \quad (11)$$

Therefore, the screening operator commutes with the currents.

We can bosonize the (β, γ) system in terms of two free scalar fields (u, v)

$$\beta = -i\partial v e^{-u+iv}, \quad \gamma = e^{u-iv} \quad (12)$$

with

$$T_{u,v} = -\frac{1}{2}(\partial u)^2 - \frac{1}{2}\partial^2 u - \frac{1}{2}(\partial v)^2 + \frac{i}{2}\partial^2 v \quad (13)$$

where $u(z)u(z') = -\log(z-z')$ and $v(z)v(z') = -\log(z-z')$. Thus, in terms of ϕ, u and v one can locally express the currents as

$$\begin{aligned} J^+ &= -i\partial v e^{-u+iv}, \\ J^0 &= -\partial u - i\sqrt{\frac{k+2}{2}}\partial\phi, \\ J^- &= [i\sqrt{2(k+2)}\partial\phi + (k+2)\partial u - i(k+1)\partial v]e^{u-iv} \end{aligned} \quad (14)$$

and the primary fields (7) and the screening operator (10) as

$$\begin{aligned} \Phi_j^m &= \gamma^{j-m} e^{i2\alpha_0 j\phi} = e^{i2\alpha_0 j\phi + (j-m)(u-iv)} \\ \Phi_+ &= \beta e^{-i2\alpha_0\phi} = -i\partial v e^{-i2\alpha_0\phi - u+iv} \end{aligned} \quad (15)$$

3 Perturbing the $SL(2)_k$ WZNW Model

We will consider the WZNW model formulated by means of free fields, perturbed by several relevant operators. In particular we will consider the most symmetric perturbations, in which a $U(1)$ conserved current is left unperturbed: there are three cases corresponding to the perturbations $\Psi =$ (i) Φ_j^j , (ii) Φ_j^0 and (iii) Φ_j^{-j} . Perturbing the WZNW model amounts to adding to the action the term $\mu \int dw d\bar{w} \Psi(w, \bar{w})$, where μ is a dimensional coupling constant so that the $\mu \int dw d\bar{w} \Psi(w, \bar{w})$ is dimensionless. Due to eq.(8) above, we have

$$\begin{aligned} J^+(z)\Phi_j^j(z') &= \text{RT} \\ J^0(z)\Phi_j^0(z') &= \text{RT} \\ J^-(z)\Phi_j^{-j}(z') &= \text{RT} \end{aligned} \quad (16)$$

So, although $SL(2)_k$ current algebra symmetry is violated under the relevant perturbations, there still exists a residual $U(1)$ current algebra symmetry in the perturbed systems, generated by $J^+(z)$ in the case (i), by $J^0(z)$ in the case (ii), and by $J^-(z)$ in the case (iii), respectively. The other nontrivial off-critical conserved quantities can be found as follows. Let $P_s(z)$ be a conserved current of the unperturbed WZNW model with spin s , namely $\bar{\partial}P_s(z) = 0$. The relevant deformed correlation functions in the presence of the perturbation Ψ are given by

$$\langle P_s(z, \bar{z}) \dots \rangle_\mu = \langle P_s(z) \dots \rangle_0 + \mu \int dw d\bar{w} \langle P_s(z) \Psi(w, \bar{w}) \dots \rangle_0 + O(\mu^2)$$

In our case, $P_s(z) = P_s(J, \partial J, T, \partial T, \dots)(z)$ and thus the spin s is an integer, and the $P_s(z)$ is local with respect to $\Psi(w, \bar{w})$

$$P_s(z)\Psi(w, \bar{w}) = \sum_{n=-\infty}^0 (z-w)^{-s-n} (P_s)_n(w) \Psi(w, \bar{w}) \quad (17)$$

where

$$(P_s)_n(w) = \oint dz (z-w)^{n+s-1} P_s(z)$$

Thus, to the lowest order of the perturbation theory the above equations lead to the perturbed counterpart of the conformal conservation laws

$$\bar{\partial} P_+(z, \bar{z}) = \mu X(z, \bar{z}) + \mu \partial(\text{something}), \quad (18)$$

where $X = (P_+)_{-+1} \Psi(z, \bar{z})$. The existence of the off-critical conservation laws of higher spin depends on whether or not X is a total derivative w.r.t. z . If X is a total derivative of the form $X = \partial Q$, then the integral $\oint dz P_+(z, \bar{z})$ gives a conserved charge since

$$\frac{d}{d\bar{z}} \oint dz P_+(z, \bar{z}) = 0 \quad (19)$$

Hence the conserved charge density can be obtained by looking for $P_+(z)$ with certain spin s so that the quantity $(P_+)_{-+1} \Psi(w, \bar{w})$ in (18) is a total derivative. That $(P_+)_{-+1} \Psi(w, \bar{w})$ is a total derivative means that eq.(17) is equivalent to

$$\left[\oint dz P_+(z), \oint dw \Psi(w, \bar{w}) \right] = 0 \quad (20)$$

Since the anti-holomorphic part is irrelevant in the commutation relation (20), we will henceforth suppress the antiholomorphic coordinate.

One may regard $\oint dw \Psi(w)$ as the Hamiltonian of the system and $\oint dz P_+(z)$ as a conserved charge commuting with the Hamiltonian. We remark however that the screening operator $\Phi_+(w)$, eq.(10) has dimension one, so that $[T, \oint \Phi_+(w)] = 0$, and from (11) one also has $[J^a, \oint \Phi_+(w)] = 0$, $a = 0, \pm 1$. Therefore the effective quantum Hamiltonian describing the deformation of the $SL(2)_k$ WZNW model under the relevant operator Ψ can be written as

$$H = \oint dw (\Psi(w) + \Phi_+(w)) \quad (21)$$

In the following, when calculating the quantum equations of motion of the perturbed system, we will always assume a Hamiltonian as in eq.(21).⁴

⁴While this is certainly allowed, it is less clear why one must add the screening charge to the Hamiltonian in order to obtain equations of motion whose classical limits leads to integrable models. However, that the screening operator leaves a trace in the classical limit of the model is clear from many examples: the Liouville theory on one side and the examples of ref.[5] on the other side. The origin of this modification of the Hamiltonian is likely to be connected to the fact that when representing the WZNW by means of free fields one should impose the condition that $\beta \exp(2\phi)$ is a total derivative, (see [9]). Renormalization group arguments lead to analogous conclusions^[12] on the necessity of adding the screening charge.

In the remaining part of this section we present a few deformations of the $SL(2)_k$ WZNW model away from criticality by using some relevant perturbations and show that there exist corresponding nontrivial off-critical conservation laws of higher spin. Moreover we propose the Hamiltonians describing the perturbed systems and derive the perturbed quantum equations of motion. The classical versions of these equations may be a clue to understand what are the integrable models that mimic the behaviour of our WZNW systems near criticality.

$$(i) \Psi(z) = \Phi_j^j(z)$$

The perturbation $\Phi_j^j(z)$ has the dimension $\frac{j(j+1)}{k+2}$. We assume sufficiently large k so that $\frac{j(j+1)}{k+2} < 1$. In this case the perturbation is relevant. Take the Hamiltonian

$$H = \oint (\Phi_j^j(z) + j\Phi_+(z)) = \oint (e^{i2\alpha_0 j\phi} + j\beta e^{-i2\alpha_0\phi}) \quad (22)$$

It is easy to see that $[J^+, H] = 0$ or, $\bar{\partial} J^+ = 0$. Namely the Hamiltonian has a $U(1)$ current algebra symmetry generated by $J^+(z)$.

The first few off-critical conserved quantities are found to be

$$\begin{aligned} P_1^{(1)} &= \oint dz T(z) \\ P_1^{(2)} &= \oint dz [(J^0 J^0)(z) + (J^- J^+)(z)] \\ P_2 &= \oint dz \left\{ (T J^+)(z) + \frac{1}{6j} [j(j+1)(c-3) + 6] (J^0 \partial J^+)(z) \right\} \\ P_3 &= \oint dz \left\{ (T^2)(z) - \frac{1}{3j} [j(j+1)(c-3) + 6] (T \partial J^0)(z) \right. \\ &\quad \left. - \frac{1}{36j^2} [j(j+1)(c-3) + 6]^2 (J^0 \partial^2 J^0)(z) \right\} \\ P_4 &= \oint dz \left\{ (T(T J^+))(z) - \frac{1}{3j} [j(j+1)(c-3) + 6] (T(J^+ \partial J^0))(z) \right. \\ &\quad \left. - \frac{1}{36j^2} [j(j+1)(c-3) + 6]^2 (J^+ (J^0 \partial^2 J^0))(z) \right. \\ &\quad \left. + \frac{1}{108j} [j(j+1)(c-3)(j(j+1)(c-3) - 18) \right. \end{aligned}$$

$$-126] (J^0 \partial^3 J^+)(z) \} \quad (23)$$

where $c = \frac{3k}{k+2}$ is the central charge of the $SL(2)_k$ WZNW model and (AB) stands for the normal ordered product of A and B ,

$$(AB)(z) = \oint_{C_z} \frac{dz'}{z' - z} A(z') B(z).$$

and the degenerate nature of the perturbation Φ_j^i ,

$$L_n \Phi_j^i = \frac{1}{2(k+2)} \sum_m : (J_m^+ J_m^- + J_m^- J_m^+ + 2J_m^0 J_m^0) : \Phi_j^i$$

with $n = -1$ has been also used. We conjecture that the perturbed system has an infinite series of commuting conserved quantities, so that the perturbation is integrable.

To the lowest order perturbation theory we have the following equations of motion

$$\begin{aligned} \bar{\partial} \partial \phi &= \mu_{(j)} [\partial \phi, H] \\ &= -i2\alpha_0 j \mu_{(j)} (e^{i2\alpha_0 j \phi} - \beta e^{-i2\alpha_0 \phi}) \\ \bar{\partial} \gamma &= \mu_{(j)} [\gamma, H] = -\mu_{(j)} j e^{-i2\alpha_0 \phi} \\ \bar{\partial} \beta &= \mu_{(j)} [\beta, H] = 0 \end{aligned} \quad (24)$$

In the Appendix we discuss the meaning of this perturbation for the rational level case.

$$(ii) \quad \Psi(z) = \Phi_j^0(z)$$

The perturbation $\Phi_j^0(z)$ has dimension $\frac{i(j+1)}{k+2}$ and is relevant for sufficiently large k . The Hamiltonian is taken to be

$$H = \oint (\Phi_j^0(z) + j \Phi_+(z)) = \oint (\gamma^j e^{i2\alpha_0 j \phi} + j \beta e^{-i2\alpha_0 \phi}) \quad (25)$$

Since $\bar{\partial} J^0 = [J^0, H] = 0$, the Hamiltonian (25) has a $U(1)$ current algebra symmetry generated by $J^0(z)$.

The off-critical conserved charges are

$$\begin{aligned} P_1^{(1)} &= \oint dz T(z) \\ P_1^{(2)} &= \oint dz (J^- J^+)(z) \\ P_2 &= \oint dz \left[(T J^0)(z) + \frac{1}{6}(c-3)(J^0(J^- J^+))(z) \right] \\ P_3 &= \oint dz \left[(T^2)(z) + \frac{1}{12}(c-3)(J^- \partial^2 J^+)(z) \right. \\ &\quad \left. + \frac{1}{6}(c-3)(T(J^0 J^0 + J^- J^+ + \partial J^0))(z) \right] \\ P_4 &= \oint dz \left\{ (T(T J^0))(z) + \frac{1}{6}(c-3) \left[T(J^0(J^0 J^0 + J^- J^+)) \right. \right. \\ &\quad \left. \left. + \frac{1}{2} J^-(J^0 \partial^2 J^+) + \frac{1}{2} J^+(J^0 \partial^2 J^-) \right] (z) \right. \\ &\quad \left. - \frac{1}{6} \frac{j(j+1)(c-3) + 18}{j(j+1)(c-3) + 6} (T \partial^2 J^0)(z) \right. \\ &\quad \left. + \frac{5}{36}(c-3)(J^- \partial^3 J^+)(z) \right\} \end{aligned} \quad (26)$$

where the degenerate nature of $\Phi_j^0(z)$,

$$L_n \Phi_j^0 = \frac{1}{2(k+2)} \sum_m : (J_m^+ J_m^- + J_m^- J_m^+ + 2J_m^0 J_m^0) : \Phi_j^0$$

with $n = -1, -2, -3$ and -4 has been used.

The quantum equations of motion are

$$\begin{aligned} \bar{\partial} \partial \phi &= \mu_{(j)} [\partial \phi, H] \\ &= -i2\alpha_0 j \mu_{(j)} (\gamma^j e^{i2\alpha_0 j \phi} - \beta e^{-i2\alpha_0 \phi}) \\ \bar{\partial} \gamma &= \mu_{(j)} [\gamma, H] = -\mu_{(j)} j e^{-i2\alpha_0 \phi} \\ \bar{\partial} \beta &= \mu_{(j)} [\beta, H] = j \mu_{(j)} \gamma^{j-1} e^{i2\alpha_0 j \phi} \end{aligned} \quad (27)$$

$$(iii) \quad \Psi(z) = \Phi_j^{-j}(z)$$

For the relevant perturbation $\Phi_j^{-j}(z)$ (we assume sufficiently large k),

the Hamiltonian is

$$H = \oint (\Phi_j^{-j}(z) + j\Phi_+(z)) = \oint (\gamma^{2j}e^{i2\alpha_0j\phi} + j\beta e^{-i2\alpha_0\phi}) \quad (28)$$

which has a $U(1)$ symmetry generated by $J^-(z)$ since $\delta J^- = [J^-, H] = 0$.

The off-critical conserved charges are now

$$\begin{aligned} P_1^{(1)} &= \oint dz T(z) \\ P_1^{(2)} &= \oint dz [(J^0 J^0)(z) + (J^- J^+)(z)] \\ P_2 &= \oint dz \left\{ (T J^-)(z) - \frac{1}{6j} [j(j+1)(c-3) + 6] (J^0 \partial J^-)(z) \right\} \\ P_3 &= \oint dz \left\{ (T^2)(z) + \frac{1}{3j} [j(j+1)(c-3) + 6] (T \partial J^0)(z) \right. \\ &\quad \left. - \frac{1}{36j^2} [j(j+1)(c-3) + 6]^2 (J^0 \partial^2 J^0)(z) \right\} \\ P_4 &= \oint dz \left\{ (T(T J^-))(z) + \frac{1}{3j} [j(j+1)(c-3) + 6] (T(J^- \partial J^0))(z) \right. \\ &\quad \left. - \frac{1}{36j^2} [j(j+1)(c-3) + 6]^2 (J^- (J^0 \partial^2 J^0))(z) \right. \\ &\quad \left. - \frac{1}{108j} [j(j+1)(c-3)(j(j+1)(c-3) - 18) \right. \\ &\quad \left. - 126] (J^0 \partial^3 J^-)(z) \right\} \end{aligned} \quad (29)$$

The quantum equations of motion are given by

$$\begin{aligned} \bar{\partial} \partial \phi &= \mu_{(j)} [\partial \phi, H] \\ &= -i2\alpha_0 j \mu_{(j)} (\gamma^{2j} e^{i2\alpha_0 j \phi} - \beta e^{-i2\alpha_0 \phi}) \\ \bar{\partial} \gamma &= \mu_{(j)} [\gamma, H] = -\mu_{(j)} j e^{-i2\alpha_0 \phi} \\ \bar{\partial} \beta &= \mu_{(j)} [\beta, H] = 2j \mu_{(j)} \gamma^{2j-1} e^{i2\alpha_0 j \phi} \end{aligned} \quad (30)$$

$$(iv) \quad \Psi(z) = \Phi_j^m(z)$$

We now consider the asymmetric perturbation $\Phi_j^m(z)$, $m \neq j, 0$, and $-j$. In this case, no residual symmetry is preserved in the perturbed system. We still assume sufficiently large k so that the perturbation Φ_j^m is relevant.

Take the Hamiltonian

$$H = \oint (\Phi_j^m(z) + j\Phi_+(z)) = \oint (\gamma^{j-m} e^{i2\alpha_0 j \phi} + j\beta e^{-i2\alpha_0 \phi}) \quad (31)$$

The off-critical conserved quantities are found to be

$$\begin{aligned} P_1 &= \oint dz T(z) \\ P_2 &= \oint dz \left\{ (T J^0)(z) - \frac{1}{6} m(c-3)(J^- \partial J^+)(z) \right. \\ &\quad \left. + \frac{1}{3}(c-3)(J^0(J^0 J^0 + J^- J^+))(z) \right. \\ &\quad \left. + \frac{1}{12m} [(j(j+1) + m(2m+1))(c-3) + 6] (J^0 \partial J^0)(z) \right\} \\ P_3 &= \oint dz \left\{ (T^2)(z) - \frac{1}{3m} [j(j+1)(c-3) + 6] (T \partial J^0)(z) \right. \\ &\quad \left. + \frac{1}{36j^2} [j(j+1)(c-3) + 6]^2 (J^0 \partial^2 J^0)(z) \right\} \end{aligned} \quad (32)$$

This perturbed model too, although less symmetric than the previous ones seems to allow for infinite conserved quantities.

The equations of motion defined by (31) read

$$\begin{aligned} \bar{\partial} \partial \phi &= \mu_{(j)} [\partial \phi, H] \\ &= -i2\alpha_0 j \mu_{(j)} (\gamma^{j-m} e^{i2\alpha_0 j \phi} - \beta e^{-i2\alpha_0 \phi}) \\ \bar{\partial} \gamma &= \mu_{(j)} [\gamma, H] = -\mu_{(j)} j e^{-i2\alpha_0 \phi} \\ \bar{\partial} \beta &= \mu_{(j)} [\beta, H] = (j-m) \mu_{(j)} \gamma^{j-m-1} e^{i2\alpha_0 j \phi} \end{aligned} \quad (33)$$

4 Integrable Models

In this section we introduce new classical models whose equations of motion coincide with those for the basic fields found in the previous section. We introduce these new models by means of suitable linear systems whose zero curvature conditions correspond to the equations of motion.

Model (a)

In this subsection we want to reproduce the following equations of motion,

$$\begin{aligned}\partial_+ \partial_- \phi &= \beta e^{2\phi} - \gamma e^{-2\phi} \\ \partial_+ \beta &= e^{-2\phi} \\ \partial_+ \gamma &= -e^{2\phi}\end{aligned}\quad (34)$$

which are the classical versions of eqs.(27) with $j = 1$. Here for simplicity we ignore the coupling constants and consider the hyperbolic version of the equations. Moreover we use light-cone variables $z_{\pm} = x \pm t$, (so $\partial_{\pm} = \frac{1}{2}(\partial_x \pm \partial_t)$).

The Lie algebra underlying eqs.(34) is the loop algebra $\widehat{sl}(2) = C(\lambda, \lambda^{-1}) \otimes \widehat{sl}(2)$. Here for later use, we introduce also the Kac-Moody algebra $\widehat{sl}(2) = \widehat{sl}(2) \oplus C\hat{c} \oplus C\hat{d}$. The generators of $\widehat{sl}(2)$ are $H_m, E_m^{\pm}, \hat{c}$ and $\hat{d}$, where $X_m = X \otimes \lambda^m$ and $\hat{d} = \lambda \frac{d}{d\lambda}$. The commutation relation are

$$\begin{aligned}[H_m, E_n^{\pm}] &= \pm 2E_{m+n}^{\pm} \\ [H_m, H_n] &= 2m\delta_{m+n}\delta\hat{c} \\ [E_m^+, E_n^-] &= H_{m+n} + m\delta_{m+n}\delta\hat{c} \\ [\hat{d}, E_m^{\pm}] &= mE_m^{\pm} \\ [\hat{d}, H_m] &= mH_m\end{aligned}\quad (35)$$

the remaining ones being zero. The loop algebra $\widehat{sl}(2)$ is obtained by ignoring $\hat{c}$ and $\hat{d}$.

Consider the following linear system

$$\begin{aligned}(\partial_{\pm} + \mathcal{A}_{\pm})\mathcal{T} &= 0, \\ \mathcal{A}_+ &= \frac{1}{2}\partial_+ \phi H_0 + e^{\phi} E_0^+ + e^{-\phi} E_{-1}^- \\ \mathcal{A}_- &= -\frac{1}{2}\partial_- \phi H_0 + \beta e^{\phi} E_0^- + \gamma e^{-\phi} E_{-1}^+ - \frac{1}{2}H_{-1}\end{aligned}\quad (36)$$

Here $\mathcal{T}$ belongs to a group whose Lie algebra is the loop algebra $\widehat{sl}(2)$. Then the zero curvature condition

$$\mathcal{F}_{+-} = \partial_+ \mathcal{A}_- - \partial_- \mathcal{A}_+ + [\mathcal{A}_+, \mathcal{A}_-] = 0$$

implies the eqs.(34)

Equations (34) come from the action

$$S = - \int (\partial_+ \phi \partial_- \phi + \beta \partial_+ \gamma + \gamma \partial_- \beta + \beta \gamma e^{2\phi} + \gamma \beta e^{-2\phi}) \quad (37)$$

So the conjugate momenta are

$$\Pi_{\phi} = \partial_{\phi} \phi, \quad \Pi_{\beta} = \gamma$$

Poisson brackets are defined by

$$\{\Pi_{\phi}(x), \phi(y)\} = h\delta(x-y), \quad \{\Pi_{\beta}(x), \beta(y)\} = h\delta(x-y)$$

Now write

$$\mathcal{A} = \mathcal{A}_+ + \mathcal{A}_- = \frac{1}{2}\Pi_{\phi} H_0 + e^{\phi}(E_0^+ + \beta E_0^-) + e^{-\phi}(E_{-1}^- + \Pi_{\beta} E_{-1}^+) - \frac{1}{2}H_{-1} \quad (38)$$

Then,

$$\begin{aligned}\{\mathcal{A}(x) \circ \mathcal{A}(y)\} &= \frac{h}{2} [e^{\phi}(H_0 \otimes E_0^+ - E_0^+ \otimes H_0) + \beta e^{\phi}(H_0 \otimes E_0^- \\ &\quad - E_0^- \otimes H_0) + e^{-\phi}(E_{-1}^- \otimes H_0 - H_0 \otimes E_{-1}^-) \\ &\quad - \Pi_{\beta} e^{-\phi}(H_0 \otimes E_{-1}^+ - E_{-1}^+ \otimes H_0) \\ &\quad - 2(E_0^- \otimes E_{-1}^+ - E_{-1}^+ \otimes E_0^-)] \delta(x-y)\end{aligned}$$

Defining the classical r -matrix,

$$r = \frac{\hbar}{2} \left[E_0^+ \otimes E_0^- - E_0^- \otimes E_0^+ + \frac{1}{2} \sum_{m \geq 1} (H_m \otimes H_{-m} - H_{-m} \otimes H_m) + \sum_{m \geq 1} (E_m^+ \otimes E_{-m}^- - E_{-m}^- \otimes E_m^+ + E_m^- \otimes E_{-m}^+ - E_{-m}^+ \otimes E_m^-) \right] \quad (39)$$

and introducing $\mathcal{A}_1 = \mathcal{A} \otimes 1$ and $\mathcal{A}_2 = 1 \otimes \mathcal{A}$, then one obtains the following Poisson brackets

$$\left\{ \mathcal{A}(x) \otimes \mathcal{A}(y) \right\} = [r, \mathcal{A}_1 + \mathcal{A}_2] \delta(x - y) \quad (40)$$

Model (b)

We want to find a linear system corresponding to the classical versions of eqs.(30) with $j = 1$ (with the same simplifications as above)

$$\begin{aligned} \partial_+ \partial_- \phi &= \beta e^{2\phi} - \gamma^2 e^{-2\phi} \\ \partial_+ \beta &= \gamma e^{-2\phi} \\ \partial_+ \gamma &= -\frac{1}{2} e^{2\phi} \end{aligned} \quad (41)$$

Consider the following linear system in the framework of the loop algebra $\tilde{sl}(2)$,

$$\begin{aligned} (\partial_\pm + \mathcal{A}_\pm) \mathcal{T} &= 0, \\ \mathcal{A}_+ &= \frac{1}{2} \partial_+ \phi H_0 + e^\phi E_0^+ + e^{-\phi} E_{+1}^- \\ \mathcal{A}_- &= -\frac{1}{2} \partial_- \phi H_0 + \beta e^\phi E_0^- + \gamma^2 e^{-\phi} E_{-1}^+ \\ &\quad - \frac{1}{2} \gamma H_{-1} - \frac{1}{4} e^\phi E_{-1}^- \end{aligned} \quad (42)$$

Then the zero curvature condition gives rise to eqs.(41).

The action is

$$S = - \int (\partial_+ \partial_- \phi + 2\beta \partial_+ \gamma + \beta e^{2\phi} + \gamma^2 e^{-2\phi}) \quad (43)$$

The conjugate momenta are

$$\Pi_\phi = \partial_t \phi, \quad \Pi_\beta = 2\gamma$$

Poisson brackets are

$$\{\Pi_\phi(x), \phi(y)\} = \hbar \delta(x - y), \quad \{\Pi_\beta(x), \beta(y)\} = \hbar \delta(x - y)$$

Now,

$$\begin{aligned} \mathcal{A} = \mathcal{A}_+ + \mathcal{A}_- &= \frac{1}{2} \Pi_\phi H_0 + e^\phi (E_0^+ + \beta E_0^-) + e^{-\phi} (E_{+1}^- + \frac{1}{4} (\Pi_\phi)^2 E_{-1}^+) \\ &\quad - \frac{1}{4} \Pi_\beta H_{-1} - \frac{1}{4} e^\phi E_{-1}^- \end{aligned}$$

Then one has the following Poisson brackets

$$\left\{ \mathcal{A}(x) \otimes \mathcal{A}(y) \right\} = [r, \mathcal{A}_1 + \mathcal{A}_2] \delta(x - y) \quad (44)$$

with the same r -matrix as above.

Conformal Affine Extension of Model (a)

In this subsection, following ref.[8], we present a model which has the properties of integrable models and is conformal invariant. It reduces to the model (a) above under a specific limiting condition. Following the general construction discussed above, we define the linear system in the Kac-Moody algebra $sl(2)$

$$\begin{aligned} (\partial_\pm + \mathcal{A}_\pm) \mathcal{T} &= 0, \\ \mathcal{A}_\pm &= \pm \partial_\pm \Phi + e^{\pm i \alpha \phi} \mathcal{E}_\pm \end{aligned} \quad (45)$$

where

$$\begin{aligned} \Phi &= \frac{1}{2} \phi H + \eta \tilde{d} + \frac{1}{2} \xi \tilde{c} \\ \mathcal{E}_+ &= E_0^+ + E_1^- \\ \mathcal{E}_- &= \beta E^- + \gamma E_{-1}^+ - \frac{1}{2} H_{-1} \end{aligned}$$

The zero curvature condition can be worked out using the algebra structure of $\widehat{sl}(2)$. We get

$$\begin{aligned}\partial_+ \partial_- \phi &= \beta e^{2\phi} - \gamma e^{-2\phi+2\eta} \\ \partial_+ \beta &= e^{-2\phi+2\eta} \\ \partial_+ \gamma &= -e^{2\phi} \\ \partial_+ \partial_- \eta &= 0 \\ \partial_+ \partial_- \xi &= \gamma e^{-2\phi+2\eta}\end{aligned}\quad (46)$$

The above equations of motion are invariant under the transformations

$$z_+ \longrightarrow f(z_+), \quad z_- \longrightarrow g(z_-) \quad (47)$$

if the fields are transformed as follows

$$\begin{aligned}\phi &\longrightarrow \phi - \frac{1}{2} \log f' \\ \beta &\longrightarrow (g')^{-1} \beta, \quad \gamma \longrightarrow \gamma, \\ \eta &\longrightarrow \eta - \log f' - \frac{1}{2} \log g', \\ \xi &\longrightarrow \xi - A \log(f'g')\end{aligned}\quad (48)$$

where A is an arbitrary parameter. Moreover, by taking the limit $\eta \rightarrow const$, we recover the eqs.(34).

The equations of motion (46) can be obtained from the action

$$S = - \int \left(\partial_+ \phi \partial_- \phi + \beta \partial_+ \gamma + \partial_+ \eta \partial_- \xi + \partial_- \eta \partial_+ \xi + \beta e^{2\phi} + \gamma e^{-2\phi+2\eta} \right) \quad (49)$$

The conjugate momenta can be read off from the action,

$$\begin{aligned}\Pi_\phi &= \partial_t \phi, & \Pi_\beta &= \gamma, \\ \Pi_\eta &= \partial_t \xi, & \Pi_\xi &= \partial_t \eta\end{aligned}$$

The Poisson brackets are now defined by

$$\begin{aligned}\{\Pi_\phi(x), \phi(y)\} &= h\delta(x-y), \quad \{\Pi_\beta(x), \beta(y)\} = h\delta(x-y) \\ \{\Pi_\eta(x), \eta(y)\} &= h\delta(x-y), \quad \{\Pi_\xi(x), \xi(y)\} = h\delta(x-y)\end{aligned}\quad (50)$$

Let

$$\begin{aligned}\mathcal{A} &= \mathcal{A}_+ + \mathcal{A}_- = \frac{1}{2} \Pi_\phi H_0 + \Pi_\xi \dot{\phi} + \frac{1}{2} \Pi_\eta \dot{\eta} + e^\phi E_0^+ \\ &\quad + \beta e^\phi E_0^- + e^{-\phi+\eta} E_{+1}^- + \Pi_\beta e^{-\phi+\eta} E_{-1}^+ - \frac{1}{2} e^\eta H_{-1}\end{aligned}$$

then, it is easy to check, using the Poisson brackets (50), that this $\mathcal{A}$ satisfies the Poisson bracket

$$\left\{ \mathcal{A}(x) \otimes \mathcal{A}(y) \right\} = [\tau, \mathcal{A}_1 + \mathcal{A}_2] \delta(x-y) \quad (51)$$

with the same r-matrix, eq.(39), as above.

From the above results we can easily arrive at a classical exchange algebra. Let us start from the remark that the second equation in (45) can be rewritten as

$$(\partial_- + e^{-2\phi+2\eta} \mathcal{E}_-) e^{-\frac{\eta}{2}} \mathcal{T} = 0 \quad (52)$$

For any chosen highest weight vector $|\Psi_r\rangle$ of $\widehat{sl}(2)$, let us define

$$\xi^{(r)}(x) = \langle \Psi_r | e^{-\frac{\eta}{2}(x)} \mathcal{T}(x)$$

Now, using (52) and the properties of the highest weight vectors, we find

$$\partial_- \xi^{(r)}(s) = 0$$

Next, using standard techniques, it is not hard to derive the following Poisson brackets (classical exchange algebra)

$$\{\xi^{(r)}(x) \otimes \xi^{(r')}(y)\} = \xi^{(r)}(x) \otimes \xi^{(r')}(y) [\theta(x-y)r^+ + \theta(y-x)r^-] \quad (53)$$

Here we have introduced

$$\begin{aligned}r^+ &= \frac{h}{4} \left[\mathcal{H} + 4E_0^+ \otimes E_0^- + 2 \sum_{k \geq 1} H_k \otimes H_{-k} \right. \\ &\quad \left. + 4 \sum_{k \geq 1} (E_k^+ \otimes E_{-k}^- + E_k^- \otimes E_{-k}^+) \right] \\ r^- &= -\frac{h}{4} \left[\mathcal{H} + 4E_0^- \otimes E_0^+ + 2 \sum_{k \geq 1} H_{-k} \otimes H_k \right. \\ &\quad \left. + 4 \sum_{k \geq 1} (E_{-k}^+ \otimes E_k^- + E_{-k}^- \otimes E_k^+) \right]\end{aligned}$$

where

$$\mathcal{H} = H \otimes H + 2\hat{c} \otimes \hat{d} + 2\hat{d} \otimes \hat{c}$$

These two r -matrices differ from the r -matrix introduced above, eq.(39), by the Casimir of $\mathfrak{sl}(2)$. Therefore they can replace r in eq.(51). But unlike r , they both satisfy the classical Yang-Baxter equation.

5 Concluding Remarks

We have seen that for the $SL(2)_k$ WZNW model, we can construct off-critical conservation laws of higher spin for several relevant perturbations. We have assumed that there are in fact an infinite number of conserved quantities and tried to identify some classical integrable models that mimic the behaviour of the corresponding perturbed theories near criticality.

The main reason for studying these models lies in the well-known relation between WZNW models and braid group representations, on the one hand, and WZNW models and Chern-Simons theory on the other hand.

It is now well understood that one way to construct linear representations of the braid group is to start from the monodromies of the conformal blocks of a suitable WZNW model. One of the main problems in this regard consists in introducing a spectral parameter in these representations in such a way as to obtain a solution of the Yang-Baxter equation: the so-called "baxterization"^[13]. This program is being carried out essentially from an algebraic point of view. By perturbing with suitable relevant operators a WZNW model, as we have done before, one is naturally led to an affine extension (see above) which automatically contains a spectral parameter. Therefore by way of perturbing WZNW models we are perhaps in condition of developing from a field theoretic point of view the "baxterization program".

There is more to that. It has been shown recently^[14] in the framework of the canonical quantization of the Chern-Simons theory, that the quantum Gauss constraint, for the group $SL(2)$ say, has as a solution the wave functional associated to a WZNW model with Kac-Moody group $\widehat{SL}(2)_k$. Therefore by means of the method outlined in this paper one may hope to be able to introduce a spectral parameter into the Chern-Simons theory by considering the "exchange algebra" or, equivalently, the monodromies of the perturbed wave functionals of the Gauss constraint.

Summarizing this discussion, we regard the WZNW model as an interesting laboratory to analyze the relationship between braid group/knot theory and integrable models from the point of view of field theory.

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6 Appendix. Meaning of the Perturbation $\Phi_j^i(z)$

In this Appendix we discuss the meaning of the perturbation $\Phi_j^i(z)$. We will show that it is an analogue of the $(1, 2j+1)$ operator in the $SL(2)_k$ WZNW model. By this we mean that under the Hamiltonian reduction Φ_j^i turns into the $(1, 2j+1)$ operator of the minimal model.

We start with introducing the Hamiltonian reduction of the WZNW model (see [10]). The Hamiltonian reduction is obtained by imposing the constraints $J^+ = 1$ and $J^0 = 0$ on the theory. However, in doing so, we must pay attention to the normal-ordering since we have to solve non-linear quantum equation like $\beta\gamma = i\sqrt{\frac{k+2}{2}}\partial\phi$. To avoid this we bosonize the β and γ fields[10] in a slightly different way from section 2,

$$\beta = \exp(-u - iv), \quad \gamma = i\partial v \exp(u + iv). \quad (54)$$

with the same $T_{u,v}$ as in section 2,

$$T_{u,v} = -\frac{1}{2}(\partial u)^2 - \frac{1}{2}\partial^2 u - \frac{1}{2}(\partial v)^2 + \frac{i}{2}\partial^2 v \quad (55)$$

Then the constraints $J^+ = 1$ and $J^0 = 0$ become linear equations, $u + iv = 0$ and $u + i\sqrt{\frac{k+2}{2}}\phi = 0$. Moreover making the following linear transformation from basis (u, v, ϕ) to new one (U, V, Φ) ,

$$\begin{aligned} U &= i\sqrt{\frac{k+2}{2}}\phi + \left(\sqrt{\frac{2}{k}} + \sqrt{\frac{k}{2}}\right)u + i\sqrt{\frac{k}{2}}v \\ V &= \sqrt{\frac{k+2}{k}}\phi - i\sqrt{\frac{2}{k}}u \\ \Phi &= \phi - i\sqrt{\frac{k+2}{2}}(u + iv) \end{aligned} \quad (56)$$

the Hamiltonian reduction simply corresponds to putting $U = 0 = V$ [10].

In terms of U, V and Φ ,

$$T = -\frac{1}{2}[(\partial U)^2 + (\partial V)^2 + (\partial\Phi)^2] - \sqrt{\frac{k}{2}}\partial^2 U + i\frac{k+1}{\sqrt{2(k+2)}}\partial^2\Phi, \quad (57)$$

and screening operator of the $SL(2)_k$ WZNW model is represented as

$$\begin{aligned} Q_+ &= \oint \Phi_+ \\ \Phi_+ &= \beta \exp(-i\sqrt{\frac{2}{k+2}}\phi) = \exp(-i\sqrt{\frac{2}{k+2}}\Phi) \end{aligned} \quad (58)$$

Thus in terms of the U , V and Φ fields the screening operator becomes independent of U and V ; however, as we shall see below, the primary fields become more complicated than usual.

Now we consider a rational level $k+2 = \frac{p'}{p}$ with non-negative integer p and coprime integer p' satisfying $p' - 2 \geq 0$. Integral k corresponds to setting $p = 1$. The primary fields are labelled by isospin $J = j - j'(k+2)$ and weight $M = J, J-1, \dots, -J+1, -J$ (where, $j' = 0, \frac{1}{2}, 1, \dots, \frac{p-1}{2}$ and $j = 0, \frac{1}{2}, 1, \dots, \frac{p'-2}{2}$), and are represented as

$$\begin{aligned} \Phi_J^M &= \gamma^{J-M} e^{i\sqrt{\frac{2}{k+2}}J\phi} = \left(i\sqrt{\frac{k+2}{2}}\partial\Phi - \sqrt{\frac{k}{2}}\partial U \right)^{J-M} \\ &\quad \times e^{i\sqrt{\frac{2}{k+2}}J\Phi + i\sqrt{\frac{1}{2}}MV - \sqrt{\frac{1}{2}}MU} \end{aligned} \quad (59)$$

The conformal dimensions of the primary fields Φ_J^M are shown to be given by the usual formulae: $\frac{J(J+1)}{k+2}$. The highest weight representation (i.e. $M = J$) reads from (59)

$$\Phi_J^J = \exp(i\sqrt{\frac{2}{k+2}}J\Phi + i\sqrt{\frac{2}{k}}JV - \sqrt{\frac{2}{k}}JU) \quad (60)$$

Putting $V = 0 = U$, we obtain the reduced $SL(2)_k$ WZNW model:

$$\begin{aligned} T_{\text{red}}[\Phi] &= -\frac{1}{2}(\partial\Phi)^2 + i\frac{k+1}{\sqrt{2(k+2)}}\partial^2\Phi, \\ C_{\text{red}} &= \frac{3k}{k+2} - 6k - 3 = 1 - 6\frac{(p'-p)^2}{p'p}, \\ (\Phi_J^J)_{\text{red.}} &= \exp(i\sqrt{\frac{2}{k+2}}J\Phi). \end{aligned} \quad (61)$$

The conformal dimensions $\Delta_{\text{red.}}$ of $(\Phi_J^J)_{\text{red.}}$ are

$$\Delta_{\text{red.}} = \frac{J(J+1)}{k+2} - J = \frac{[(2j'+1)p' - (2j+1)p]^2 - (p'-p)^2}{4p'p} \quad (62)$$

which corresponds to $\Delta_{(2j'+1, 2j+1)}$ of the ordinary minimal model $M_{p'p}$.

So, by means of the Hamiltonian reduction we have precisely reproduced the central charge and the spectrum of dimensions of the ordinary minimal model $M_{p'p}$ from $SL(2)_k$ WZNW model at the rational level $k = \frac{p'}{p} - 2$; in addition, (62) implies that the primary fields $(\Phi_J^J)_{\text{red.}}$ can be identified as $(2j'+1, 2j+1)$ operator $\psi_{(2j'+1, 2j+1)}$ of the $M_{p'p}$ model.

Taking $j' = 0$, we get the $(1, 2j+1)$ operator of the $M_{p'p}$ model,

$$\psi_{(1, 2j+1)} \equiv (\Phi_J^J)_{\text{red.}} = \exp(i\sqrt{\frac{2}{k+2}}j\Phi) \quad (63)$$

with

$$\Delta_{(1, 2j+1)} = \frac{j(j+1)}{k+2} - j = \frac{[p' - (2j+1)p]^2 - (p'-p)^2}{4p'p}$$

It follows from eqs.(60) and (63) that the $(1, 2j+1)$ operator analogue in the $SL(2)_k$ WZNW model is given by

$$\exp(i\sqrt{\frac{2}{k+2}}j\Phi + i\sqrt{\frac{2}{k}}jV - \sqrt{\frac{2}{k}}jU)$$

which is just the bosonized version of the perturbation Φ_1^j . Accordingly Φ_1^j is an analogue of the $(1, 3)$ operator in the WZNW model and is a relevant operator for any positive k . In terms of fields Φ , U and V , eq.(22) becomes

$$H = \oint \left(e^{i\sqrt{\frac{2}{k+2}}j\Phi + i\sqrt{\frac{1}{2}}jV - \sqrt{\frac{1}{2}}jU} + j e^{-i\sqrt{\frac{2}{k+2}}\Phi} \right) \quad (64)$$

Under the Hamiltonian reduction, H becomes

$$H_{\text{red.}} = \oint \left(e^{i\sqrt{\frac{2}{k+2}}j\Phi} + j e^{-i\sqrt{\frac{2}{k+2}}\Phi} \right), \quad (65)$$

which, for $j = 1$, is just the effective Hamiltonian for the perturbed Z_2 invariant system (i.e. the minimal model perturbed by the $(1,3)$ operator)^[5]. The equations of motion (24) now become

$$\begin{aligned}
\bar{\partial}\partial\Phi &= \mu_{(j)} [\partial\Phi, H] = -i\sqrt{\frac{2}{k+2}} j\mu_{(j)} \\
&\quad \times \left(e^{i\sqrt{\frac{2}{k+2}}j\Phi + i\sqrt{\frac{2}{k+2}}jV - \sqrt{\frac{2}{k+2}}jU} - e^{-i\sqrt{\frac{2}{k+2}}j\Phi} \right) \\
\bar{\partial}\partial U &= \mu_{(j)} [\partial U, H] \\
&= \sqrt{\frac{2}{k}} j\mu_{(j)} e^{i\sqrt{\frac{2}{k+2}}j\Phi + i\sqrt{\frac{2}{k+2}}jV - \sqrt{\frac{2}{k+2}}jU} \\
\bar{\partial}\partial V &= \mu_{(j)} [\partial V, H] \\
&= -i\sqrt{\frac{2}{k}} j\mu_{(j)} e^{i\sqrt{\frac{2}{k+2}}j\Phi + i\sqrt{\frac{2}{k+2}}jV - \sqrt{\frac{2}{k+2}}jU}.
\end{aligned}$$

In fact, these equations are transformed into eqs (24) by the change of variables

$$\begin{aligned}
\phi &= \Phi + \sqrt{\frac{k+2}{k}} V + i\sqrt{\frac{k+2}{k}} U, \\
\beta &= \exp(-i\sqrt{\frac{2}{k}}(U - iV)), \\
\gamma &= \left(i\sqrt{\frac{k+2}{2}} \partial\Phi - \sqrt{\frac{k}{2}} \partial U \right) e^{i\sqrt{\frac{k}{2}}(U - iV)}
\end{aligned}$$

References

1. A.B.Zamolodchikov, JETP Lett.43(1986) 730, 46(1987) 160, and Sov.J.Nucl.Phys.46(1987) 1090, 44(1987) 539.
2. V.A.Fateev and A.B.Zamolodchikov, Int.J.Mod.Phys.A5(1990) 1025; A.B.Zamolodchikov, Int.J.Mod.Phys.A3(1988) 746; J.Cardv and G.Mussardo, Phys.Lett.B225(1989) 275.
3. A.W.W.Ludwig and J.L.Cardv, Nucl.Phys.B285[FS19](1987) 687; D.A.Kastor, E.J.Martinec and S.H.Shenker, Nucl.Phys.B316(1989) 590; E.J.Martinec, Phys.Lett.B217(1989) 431; C.Vafa and N.Warner, Phys.Lett.B218(1989) 51; G.v.Gehlen, Nucl.Phys.B330(1990) 741.
4. J.L.Cardv and G.Mussardo, Phys.Lett.B225(1989) 275; G.Sotkov and C.J.Zhu, Phys.Lett.B229(1989) 391; G.Mussardo and G.Sotkov, ISAS -117-89/EP.
5. T.Eguchi and S.-K.Yang, Phys.Lett.B224(1989) 373.
6. B.A.Kupersmidt and P.Mathieu, Phys.Lett.B227(1989)245; T.J.Hollowood and P.Mansfield, Phys.Lett.B226(1989)73; A.B.Zamolodchikov, Int.Jour.Mod.Phys. A4(1989)4235.
7. N.Reshetikhin and F.Smirnov, Preprint HUTMP 89/B246 ; T.Eguchi and S.-K.Yang, UT-554 (1989) University of Tokio preprint; D.Bernard and Le Clair, Preprint
8. O.Babelon and L.Bonora, SISSA-36/90/EP.
9. V.Dotsenko and V.Fateev, Nucl.Phys.B240[FS 12](1984) 312; V.S.Dotsenko, Nucl.Phys.B(1990) (to appear); A.Gerasimov, A.Marshakov, A.Morozov, M.Olshanetsky and S.Shatashvili, ITEP Preprints, 64-89, 70-89, 73-89, and 74-89; D.Bernard and G.Felder, ETH-TH/89-26(1989).
10. A.Gerasimov, A.Marshakov and A.Morozov, Phys.Lett.B236(1990) 269; M.Bershadsky and H.Ooguri, Comm.Math.Phys.126(1989) 49.

11. E.Witten, Comm.Math.Phys.121(1989) 351, and Nucl.Phys.B322 (1989) 629.
12. M.Grisaru, A.Lerda, S.Penati and D.Zanon, CTP 1803 89
13. V.F.R.Jones, CMA-R23-89
14. G.V.Dunne, R.Jackiw and C.A.Trugenberger, CPT/1711
15. K.Kobayashi and T.Uematsu KUCP-0022/90 February 1990