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LOOP EQUATIONS AND KdV HIERARCHY IN 2-D QUANTUM GRAVITY

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ABSTRACT

A derivation of the loop equation for two-dimensional quantum gravity from the KdV equations and the string equation of the one-matrix model has been recently given. The loop equation was found to be equivalent to an infinite set of linear constraints on the square root of the partition function satisfying the Virasoro algebra. Starting from the equations expressing these constraints we are able to rederive the equations of the KdV hierarchy using the vertex operator construction of the $A_1^{(1)}$ infinite dimensional twisted Kac-Moody algebra. From these considerations it follows that the solutions of the string equation of the one-matrix model are given by a subset of the solutions of the KdV hierarchy.

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Two-dimensional quantum gravity (from now on 2D-QG)^[1] has been recently formulated as the double scaling limit of a matrix model,^[2-4] elaborating on some previous work on random surfaces.^[5]

The double scaling limit allows a resummation of the topological expansion of the path integral leading to non-perturbative equations, called string equations, which are non-linear differential equations in the specific heat of the theory $u(x)$, where x is the scaling variable. For the k -critical model with $k = 2$ (pure gravity) this differential equation is the Painlevé type I.

The solutions of the differential equations of the type:

$$\frac{\partial^2 u(x)}{\partial x^2} = F(u(x), \frac{\partial u(x)}{\partial x}, x), \quad (1)$$

where F is rational in $\frac{\partial u(x)}{\partial x}$, algebraic in $u(x)$ and analytic in x , can be classified according to their singularities.^[6] The Painlevé school found that there were only fifty equations whose solutions had no movable essential singularities. Out of these fifty equations only six are of a new type and they are known in the mathematical literature as Painlevé transcendents. These solutions, though, allow for an infinite number of movable double poles. This can potentially lead to a disaster for our theory, since it could imply that the partition function of 2-D QG has zeroes. Besides, we need to interpret the parameters appearing in this solution, as was first pointed out in Ref.2.

Some progress has been made to solve these problems. A numerical solution^[7] has been exhibited which for large x (the spherical limit) behaves according to the scaling limit and is singularities-free on the real axis. There has also been some analytic progress.^[8,9]

In this letter we will argue that the solutions of the string equation are a subset of the solutions of the KdV hierarchy of partial differential equations (from now on KdV-H), thus extending the range of validity of the results of Ref.9 (which were obtained in the large x limit).

The string equation for a general massive model is:^[10]

$$x = \sum_{k=1}^{\infty} (k + \frac{1}{2}) t_k R_k[u(x, t_1, \dots)], \quad (2)$$

where the t_k 's are certain sources and the Gel'fand-Dikii:^[11] differential polynomials $R_k[u]$

are:

$$\begin{aligned} R_0 &= \frac{1}{2}, \\ R_1 &= -\frac{1}{4}u, \\ R_2 &= \frac{1}{16}(3u^2 - u''), \\ R_3 &= -\frac{1}{64}(10u^3 - 10uu'' - 5(u')^2 + u'''), \\ &\vdots \end{aligned} \quad (3)$$

The k -th KdV flow can now be obtained from:

$$\frac{\partial u}{\partial t_k} = \frac{\partial}{\partial x} R_{k+1}[u]. \quad (4)$$

From these formulas it follows that:

$$Z(t_0, t_1, \dots) = \tau^2(t_0, t_1, \dots). \quad (5)$$

This statement is equivalent to saying that the solution of (4) is $u = -2\frac{\partial^2}{\partial x^2} \log \tau$ where $\tau(t_0, t_1, \dots)$ is the τ -function of the KdV-H.

As is well known, an alternative way to define the theory is through its Schwinger-Dyson or loop equations.^[12]

From this input, two different groups^[13,14] have independently been able to formulate the loop and string equations as the conditions:

$$L_n \tau = 0, \quad n \geq -1, \quad (6)$$

where the L_n 's are the usual Virasoro generators.

Before proceeding further we stop here for two remarks: in the first place (6) has to be considered merely as a translation of (4) and (2). David^[15] has in fact shown that the loop equation for pure gravity does not satisfy the string equation. Furthermore, the usual string equation is obtained from a matrix model with a potential which, for simplicity's

sake, is even. In the general case^[16] the solution of the string equation becomes the sum of solutions of two non-linear differential equations. This changes (6) to:

$$L_n \tau^2 = 0, \quad n \geq -1. \quad (7)$$

We will later show that both (6) and (7) lead to the KdV-H.

We now recall a few mathematical notions which we will need later on to perform our computation.

One of the most elegant achievements of the theory of integrable models is the description of the solitonic solutions of hierarchies of partial differential equations in terms of the orbit of a certain vacuum.^[17] This orbit is called the τ -function. This observation is crucial as it allows one to connect the τ -functions to the theory of infinite dimensional Kac-Moody algebras and, via the Borel-Weil theorem, to algebraic geometry.^[18]

Now let $\mathcal{G}$ be a Kac-Moody algebra, G its associated group and $\mathcal{R}(\Lambda)$ an integrable highest weight representation of $\mathcal{G}$ with highest weight Λ and highest weight vector $|\Lambda\rangle$.^{*} Given two bases $\{T_n\}$ and $\{T_{-n}\}$ of $\mathcal{G}$, dual with respect to the bilinear form $(\cdot, \cdot)$ on $\mathcal{G}$, a non-zero vector $\tau \in G \cdot |\Lambda\rangle$ if and only if^[19-21]

$$\Omega \tau \otimes \tau = \sum_{n \in \mathbb{Z}} T_n \tau \otimes T_{-n} \tau = (\Lambda|\Lambda) \tau \otimes \tau, \quad (8)$$

where $\otimes$ is the symbol for the tensor product of two vector spaces. We remark that Ω commutes with $\mathcal{G}$ and hence with G .

Before showing that (8) leads to the equation of the KdV-H we pause for few comments.

(8) is the statement that the action of the Casimir of an infinite dimensional Kac-Moody algebra gives the equation of a certain hierarchy of partial differential equations with soliton-type solutions. Let us call it the Casimir Ω . Following Goddard and Olive^[22] we can write

* We use here the notation of bras and kets to be closer to physicist's notations. The reader should be warned that in the future we might use different realisations of the generators of $\mathcal{G}$ keeping the same notation for the vacuum.

the Kac-Moody level k algebra in a modified Cartan-Weyl basis as:

$$[H_m^i, H_n^j] = km \delta^{ij} \delta_{m+n,0}, \quad (9)$$

$$[H_m^i, E_n^\alpha] = \alpha^i E_{m+n}^\alpha, \quad (10)$$

$$[E_m^\alpha, E_n^\beta] = \begin{cases} \epsilon(\alpha, \beta) E_{m+n}^{\alpha+\beta}, & \text{if } \alpha + \beta \text{ is a root,} \\ \frac{2}{\alpha^2} (\alpha \cdot H_{m+n} + km \delta_{m+n,0}), & \text{if } \alpha = -\beta, \\ 0, & \text{otherwise,} \end{cases} \quad (11)$$

$$[k, E_n^\alpha] = [k, H_n^j] = [k, d] = 0, \quad (12)$$

$$[d, E_n^\alpha] = n E_n^\alpha, \quad [d, H_n^j] = n H_n^j, \quad (13)$$

where α, β are roots and for level one we can set $k = I$, I being the identity matrix. The Cartan subalgebra is given by the set $\{H_0^i, k, d\}$, where i runs from one to the rank of the finite-dimensional Lie algebra associated to $\mathcal{G}$, k is the central extension (whose eigenvalue is the level of the algebra) and d is the energy operator. From these brackets and those given by the Virasoro generators in the Sugawara construction we can define:

$$\Omega = L_0 + kd. \quad (14)$$

One can immediately check that:

$$[\Omega, E_n^\alpha] = [\Omega, H_n^j] = [\Omega, d] = [\Omega, k] = 0, \quad (15)$$

that is, Ω is the Casimir of the infinite-dimensional Kac-Moody algebra. The product of two vectors of simultaneous eigenvalues of the generators of the Cartan subalgebra $\Lambda =$

(m, k, m_d) , $\Lambda' = (m', k', m'_d)$ is:

$$(\Lambda|\Lambda') = m \cdot m' + km'_d + k'm_d. \quad (16)$$

From:

$$\begin{aligned} E_n^\alpha |\Lambda\rangle &= 0, & n &\geq 0, \\ H_n^j |\Lambda\rangle &= 0, & n &> 0, \\ k |\Lambda\rangle &= |\Lambda\rangle, \\ d |\Lambda\rangle &= m_d |\Lambda\rangle, \\ H_0^i |\Lambda\rangle &= m |\Lambda\rangle, \end{aligned} \quad (17)$$

and:

$$L_0 = \sum_{n \in \mathbb{Z}} E_n^\alpha E_{-n}^{-\alpha} + \sum_{n \in \mathbb{Z}/\{0\}} H_n^i H_{-n}^i + H_0^i H_0^i, \quad (18)$$

we can infer the value of $(\Lambda|\Lambda)$ in (8).

Redefining:

$$\begin{aligned} E_n^\alpha &\mapsto E_n^\alpha + mk, \\ d &\mapsto d - m_d, \end{aligned} \quad (19)$$

(8) becomes:

$$\Omega\tau \otimes \tau = 0. \quad (20)$$

Now in the exercises at the end of Chapter 14 of Ref.21 it is proved that if $L_0|\Lambda\rangle = 0$, then $L_0|\Lambda\rangle \otimes |\Lambda\rangle = 0$ is also true. This could also be seen by decomposing $|\Lambda\rangle \otimes |\Lambda\rangle$ à la Clebsch-Gordan.

Recalling that a highest weight state of the Virasoro algebra is also a highest weight state of the Kac-Moody algebra and given (6) and (7) we see that (20) is tautological.

The last thing that is now left to do is to get the KdV-H from (20). This computation is performed in Ref.19 but here we repeat it in physicist's notations. Since we aim at the KdV-H we choose to give the principal realization (twisted in physicist's language) of $A_1^{(1)}$.

This allows us to simplify notations, since $\text{rank} = 1$ and we have one simple root:

$$E(z) = \frac{1}{2} e^{\sum_{j>0} z_j z^{-j} - 2 \sum_{j>0} \frac{1}{2} \frac{z}{z_j} z^{-j}}, \quad (21)$$

$$H_j = \frac{\partial}{\partial x_j}, \quad H_{-j} = jx_j, \quad (22)$$

$$d = - \sum_{j \in \mathbb{Z}} jx_j \frac{\partial}{\partial x_j}, \quad (23)$$

where all the sums run over odd j . This is equivalent to taking half-integer modes. To go from one notation to the other, expand E with respect to $z^{\frac{1}{2}}$. For a discussion of twisted algebras in a physicist's context see also Ref.23. We introduce now Shur polynomials, defined as:

$$\sum_{j \in \mathbb{Z}} p_j z^j = e^{\sum_{j=1}^{\infty} x_j z^j}. \quad (24)$$

Then from (24) and changing variables:

$$x = \frac{1}{2}(x' + x''), \quad y = \frac{1}{2}(x' - x''), \quad (25)$$

we get:

$$\sum_{n \in \mathbb{Z}} E_n E_{-n} = z^0 \text{ term of } \frac{1}{4} e^{\sum_{j>0} y_j z^{-j} - 2 \sum_{j>0} \frac{1}{2} \frac{y}{y_j} z^{-j}} = \frac{1}{4} \sum_{n \geq 1} p_n(4y_j) p_n(-\frac{2}{j} \frac{\partial}{\partial y_j}). \quad (26)$$

The tensor product $\tau(x') \otimes \tau(x'')$ is now realized as $\mathbb{C}[x'_1, x'_2, \dots] \otimes \mathbb{C}[x''_1, x''_2, \dots]$, that is, the product of two spaces of polynomials in infinite variables. Furthermore we need:

$$\sum_{j \in \mathbb{Z}} H_j H_{-j} + kd = \sum_{j \in \mathbb{Z}} jx'_j \frac{\partial}{\partial x''_j} + jx''_j \frac{\partial}{\partial x'_j} - \sum_{j \in \mathbb{Z}} (jx'_j \frac{\partial}{\partial x'_j} + jx''_j \frac{\partial}{\partial x''_j}) = - \sum_{j \in \mathbb{Z}} 2jy_j \frac{\partial}{\partial y_j}. \quad (27)$$

Substituting (18), (26), (27) in (20) leads to:

$$\begin{aligned} &(- \sum_{j \in \mathbb{Z}} jy_j \frac{\partial}{\partial y_j} + \frac{1}{8} \sum_{n \geq 1} p_n(4y_j) p_n(-\frac{2}{j} \frac{\partial}{\partial y_j})) \tau(x+y) \otimes \tau(x-y) \\ &= (- \sum_{j \in \mathbb{Z}} jx_j D_j + \frac{1}{8} \sum_{n \geq 1} p_n(4y_j) p_n(-\frac{2}{j} D_j)) e^{\sum_{j \geq 1} y_j D_j} \tau(x+\xi) \otimes \tau(x-\xi) \Big|_{\xi=0} = 0, \end{aligned} \quad (28)$$

where $D_j = \frac{\partial}{\partial \xi_j}$. Expanding the exponential in series (28) leads to an infinite number of

* We could also rewrite this formula as $\oint \frac{dx}{2\pi i} \frac{E(x)E'(x)}{x}$. This is to point out that (20) is also known in the mathematical literature as the Hirota bilinear equation.

equations given by the coefficients of the y_j terms which have to vanish term by term. Now the first Shur polynomials in the odd variables (we remind the reader that j is always odd) are:

$$\begin{aligned} p_1(x) &= x_1, \\ p_2(x) &= \frac{1}{2}x_1^2, \\ p_3(x) &= \frac{1}{6}x_1^3 + x_3, \\ p_4(x) &= \frac{1}{24}x_1^4 + x_1x_3, \\ &\vdots \end{aligned} \quad (29)$$

Plugging (29) in (28) leads, for the coefficient of y_1y_3 ,[†] to:

$$(D_1^4 - 4D_1D_3)\tau(x + \xi) \otimes \tau(x - \xi) \Big|_{\xi=0} = 0, \quad (30)$$

which is the KdV equation in Hirota's form. In fact, changing variables once again (we drop now the $\otimes$ sign for the tensor product):

$$\xi_1 = \frac{1}{2}(x' - x''), \quad \xi_3 = -2(t' - t''), \quad (31)$$

we get:

$$\begin{aligned} & \left[\left(\frac{\partial}{\partial x'} - \frac{\partial}{\partial x''} \right)^4 + \left(\frac{\partial}{\partial x'} - \frac{\partial}{\partial x''} \right) \left(\frac{\partial}{\partial t'} - \frac{\partial}{\partial t''} \right) \right] \tau(x', t') \tau(x'', t'') \Big|_{x'=x''=x, t'=t''=t} \\ &= 2(\tau_{xxx}\tau - 4\tau_{xx}\tau_x + 3\tau_x^2 + \tau_x\tau_t - \tau_x\tau_t) = 0. \end{aligned} \quad (32)$$

Defining $v = -2\frac{\partial}{\partial x}\log\tau$ (32) leads to:

$$\frac{\partial}{\partial x}(v_{xxx} - 3v_x^2 + v_t) = 0, \quad (33)$$

and setting $u = v_x = -2\frac{\partial^2}{\partial x^2}\log\tau$, we finally get:

$$u_{xxx} - 6uv_x + u_t = 0, \quad (34)$$

the KdV equation.

[†] Other coefficients lead to other equations of the KdV-H.

Conclusions

2D-QG is given by (6) and (7). $L_{-1}\tau = 0$ gives the string equation, while $L_0\tau = 0$ gives the equations of the KdV-H. It is tempting to interpret the equations with $n \geq 1$ as constraints on the solutions.

It is also remarkable to recover the $A_1^{(1)}$ symmetry of the theory which was first found by Polyakov. Here the setting is different, though: in Polyakov's work this symmetry was stemming out of the classical equation of motion while here it is derived from the quantum loop equations. Furthermore we deduce that this symmetry is realized in its principal realization, that is the algebra is twisted. This could not be realized by proving certain commutation relations among generators, as the twist is an inner automorphism of the algebra. In fact the twisted and untwisted algebra are isomorphic (a redefinition of the generators sends one algebra into the other, see Ref.22).

The very last remark concerns Ref.14 and the conjecture about W-gravity: in the general picture drawn by Kac and Wakimoto a hierarchy of differential equations is assigned to a realization of a certain Kac-Moody algebra. We have seen that the principal realization of $A_1^{(1)}$ is connected to the KdV-H. Much in the same way the non-linear Schrödinger equation and the Toda hierarchy are recovered in the homogeneous realization of $A_k^{(1)}$. Furthermore the B algebras are in relation with super-hierarchies. The procedure of reduction^[24] of the KP hierarchy always leads to the principal realization of the algebra. Repeating the computation we have just performed, this time using the Casimir of $A_2^{(1)}$, leads to the Boussinesq equation. Also in this case (6) and (7) at $n = 0$ are sufficient to lead to the equation of the hierarchy.

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