Chiral symmetry and the large-$N_c$ limit in $K_{l4}$ decays

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The treatment of kaon decays using chiral symmetry yields predictions for the form factors in $K \to \pi\pi\ell\nu$. In addition, the large-$N_c$ limit of QCD implies that a particular combination of low-energy constants should be suppressed. We present the chiral predictions for $K_{l4}$ decays at next-to-leading order in the energy expansion. By combining the phenomenologies of $K_{l4}$ and $\pi\pi$ scattering, we test these predictions and provide a determination of the parameters in the chiral Lagrangian of QCD.

I. INTRODUCTION

In this paper we describe the analysis of the reactions labeled (Refs. 1–4) $K_{l4}$, that is

$$K \to \pi\pi\ell\nu,$$

where $\ell = e, \mu$. While at first sight this would seem simply one of many kaon decay modes, it in fact has some special significance in the theory of chiral symmetry. It is the simplest process which can test predictions which follow in the limit of large numbers of colors (large $N_c$). The theory of chiral symmetry allows the description of the couplings of kaons and pions using a set of nonlinear Lagrangians, with coefficients which are to be determined phenomenologically. However, in the large-$N_c$ limit, certain linear combinations of these coefficients are suppressed, as they correspond to extra quark loops. An illustration of allowed and disallowed diagrams for the $K_{l4}$ process is given in Fig. 1. Although similar diagrams could be drawn for pionic processes, such as $\pi\pi$ scattering, the constraints of chiral symmetry are such that these diagrams cannot be disentangled using reactions which involve only pions, and one must consider kaonic reactions in order to perform the separation. The scattering of $K\pi$ or $K\bar{K}$ could in principle be used, but these reactions are poorly known and occur at too high an energy to be useful. The only purely phenomenological constraint on the large-$N_c$ predictions of which we are aware comes from $K_{l4}$ decay. It is therefore worthwhile to provide as complete an analysis as possible of this reaction in order to both explore the limits on the large-$N_c$ result and to provide additional input to the chiral Lagrangian.

The predictions for the $K_{l4}$ form factors at lowest order (i.e., order $E^2$) in the energy expansion of chiral symmetry were first given by Weinberg. The experimental results are 30–50% above the lowest-order predictions.

The required additional contributions must come from higher orders in the energy expansion. We provide this next-order treatment by including loop diagrams as well as tree-level effects from the order-$E^4$ chiral Lagrangian. The latter are parametrized by a small number of low-energy constants. However, only three of these play a significant role. Two of these low-energy constants app-

![Diagram](attachment:image.png)

FIG. 1. Allowed (a) and disallowed (b) diagrams for the $K_{l4}$ process at large $N_c$. Dashed lines denote axial-vector currents; wavy lines: gluon exchange.
pear in the analysis of $\pi\pi$ scattering. The third is first seen in $K_{14}$ decay. One combination of these low-energy constants is predicted to be suppressed at $N_c \to \infty$. A result of our analysis will be that a small nonzero value for this combination is favored phenomenologically, but with experimental errors which allow it to vanish at the 1 standard deviation level. Overall a good description of $K_{14}$ decays and $\pi\pi$ scattering is obtained.

The plan of the paper is as follows. In Sec. II, we discuss the form factors and review the experimental results. Section III is devoted to an explanation of the large-$N_c$ prediction, and the calculation of the form factors in chiral perturbation theory is performed in Sec. IV. In Sec. V we discuss the threshold form factors, whereas the phenomenology of $K_{14}$ and $\pi\pi$ scattering is considered in Sec. VI. After the algebraic part of the work described in this paper was finished, we received a paper by Bijnens,\textsuperscript{10} which also treats $K_{14}$ in chiral perturbation theory. We comment on our work and his in this section also. Finally, we end with a summary and some comments on how future rare-kaon-decay experiments could help to test the chiral and large-$N_c$ predictions more exactly.

II. FORM FACTORS AND EXPERIMENT IN $K_{14}$ DECAYS

We begin by defining the hadronic matrix element for the decay

$$K^+(k)\to \pi^+(p_+) + \pi^-(p_-) + e^+(p_e) + \nu(p_\nu),$$

which may proceed through either the vector or axial-vector current. In the following we disregard all isospin-breaking effects. The vector-current matrix element has the form\textsuperscript{4}

$$\langle \pi^+(p_+)\pi^-(p_-) | V_{\mu} | K^+(k) \rangle$$

$$= \frac{H}{m_{K}^{-2}} \epsilon^{\mu
u\alpha\beta} k^{\nu} (p_+ + p_-)^{\alpha} (p_+ - p_-)^{\beta},$$

while that of the axial vector is

$$\langle \pi^+(p_+)\pi^-(p_-) | A_\mu | K^+(k) \rangle$$

$$= \frac{1}{m_K} \left[ F(p_+ + p_-)_{\mu} + G(p_+ - p_-)_{\mu} + R(k - p_+ - p_-)_{\mu} \right].$$

The four form factors $F, G, R, H$ are functions of three variables, which may be chosen to be\textsuperscript{2}

$$s_\pi = (p_+ + p_-)^2, \quad s_i = (p_+ + p_e)^2 - (k - p_+ - p_-)^2, \quad \theta_\pi,$$

where $\theta_\pi$ is the angle in the $\pi\pi$ center of mass between the $\pi^+$ direction and a unit vector along the direction of recoil of the $\pi\pi$ system. Below we will also use the variables

$$t = (p_+ - k)^2, \quad u = (p_- - k)^2,$$

which are related to $s_\pi, s_i,$ and $\theta_\pi$ by

$$t + u = 2m_\pi^2 + m_K^2 - s_\pi + s_i,$$

$$t - u = \left( 1 - \frac{4m_\pi^2}{s_\pi} \right)^{1/2} \left[ (m_K^2 - s_\pi - s_i)^2 - 4s_\pi s_i \right]^{1/2} \cos\theta_\pi.$$
been able to separate out the full kinematic behavior of the matrix elements. Therefore certain approximations and/or assumptions have had to be made. For example, no dependence on \( s_\pi \) was seen within the limits of the data, so that results were quoted assuming that such dependence is absent. Similarly, \( f_P \) was found to be compatible with zero, and hence set equal to zero when the final result for \( g \) was derived. A dependence on \( s_\pi \) is seen, and is treated in the following manner. One defines reduced form factors \( \bar{g} \equiv g/f_S \) and \( \bar{h} \equiv h/f_S \) such that the decay rate has the form

\[
d \Gamma = |f_S(s_\pi)|^2 d \Gamma(\bar{g}(s_\pi), \bar{h}(s_\pi), \ldots),
\]

where the ellipses denote the kinematic variables \( s_\pi, s_\eta, \ldots \). No linear dependence on \( s_\pi \) was seen within the errors for \( \bar{g} \) and \( \bar{h} \). Therefore \( \bar{g}, \bar{h} \) were assumed to be constant, and all remaining \( s_\pi \) dependence was assumed to be in \( f_S \), which was parametrized as

\[
f_S(q^2) = f_S(0)(1 + \lambda f q^2), \quad q^2 = (s_\pi - 4m_\pi^2)/4m_\pi^2.
\]

Under the assumption of constant \( \bar{g}, \bar{h} \), this means that \( g \) and \( h \) must share the same \( s_\pi \) behavior:

\[
g(q^2) = g(0)(1 + \lambda_g q^2), \quad h(q^2) = h(0)(1 + \lambda_h q^2)
\]

with the same \( \lambda \), i.e., \( \lambda_f = \lambda_g = \lambda_h = \lambda \). Finally all D-wave contributions were assumed to be absent.

These approximations to the form factors do not agree completely with what is found in the theoretical predictions. Dependence on \( s_\eta \) and nonzero values for \( f_P \) and \( D \) waves all occur in the theoretical results. In addition the \( s_\pi \) dependence is in general expected to be different in \( f_S, g, \) and \( h \), although it can be forced to be identical if this is required. Such differences then cause some minor difficulty in comparing theory and experiment. In our fits we attempt to duplicate the experimental procedure as best we can in order to extract the low-energy constants that we are after, see Sec. VI.

The experimental results are then summarized by the following numbers:\(^{11}\)

\[ f_S(0) = 5.59 \pm 0.14, \]

\[ g(0) = 4.77 \pm 0.27, \]

\[ h(0) = -2.68 \pm 0.68, \]

\[ \lambda = 0.08 \pm 0.02. \]

We have used \(|V_{us}| = 0.220\) in transcribing these results.

### III. Chiral Expansion and Large N_c

At low energies, QCD reduces to a theory of pions, kaons and \( \eta \)'s interacting with each other and with the gauge bosons. These interactions are strongly constrained by the chiral symmetry of QCD.\(^{5-7}\) All such interactions are described by an expansion in powers of the energy, and the lowest-order coefficients are uniquely predicted in terms of the pion decay constant, \( F_\pi = 93.3 \) MeV, and pion and kaon masses. At the next order in the energy expansion, there exist relations between processes parametrized by a small set of low-energy constants. This procedure, chiral perturbation theory, is best described in terms of nonlinear effective Lagrangians.

At lowest order, called \( O(E^2) \), the chiral Lagrangian is

\[
L_2 = \frac{F^2}{4} Tr(D_\mu U D^{\dagger} U^\dagger) + \frac{F^2}{4} Tr(\chi^\dagger U + \chi U^\dagger),
\]

\[ D_\mu U = \partial_\mu U - iR_\mu U + iUL_\mu, \]

\[ U = \exp(i\phi^4/F), \quad \chi = 2B_0 M. \]

Here \( \phi^4, A = 1, \ldots, 8, \) are the fields of the pseudoscalar octet, \( L_\mu \) (\( R_\mu \)) are the left-handed (right-handed) external gauge fields, \( M = \text{diag}(m_u, m_d, m_s) \) is the quark mass matrix and \( B_0 \) is a constant. In the remainder of this paper we work in the isospin limit \( m_u = m_d \). At order \( E^2 \) we can then equate

\[ F_\pi = F, \quad m_u^2 = 2B_0\tilde{m}, \]

\[ m_\pi^2 = B_0(m_u + \tilde{m}), \]

\[ m_{\eta}^2 = \frac{4}{3}m_u^2 - \frac{1}{3}m_\pi^2, \quad \tilde{m} = \frac{1}{2}(m_u + m_d). \]

Transition amplitudes are found (at this order) simply by expanding \( L_2 \) in powers of the fields, and taking tree-level matrix elements.

At order \( E^4 \), the possible structure is somewhat more elaborate. Generalizing momentarily to \( N_f \) flavors, the chiral Lagrangian has the form:\(^{12}\)

\[
L_4 = K_1[Tr(D_\mu U D_\mu U)]^2 + K_2 Tr(D_\mu U D_\mu U) Tr(D_\mu U^\dagger D_\mu U) + K_3 Tr(D_\mu U D_\mu U^\dagger D_\mu U) + K_4 Tr(D_\mu U D_\mu U D_\mu U^\dagger D_\mu U) + L_4 Tr(D_\mu U D_\mu U) Tr(\chi^\dagger U + \chi U^\dagger) + L_5 Tr(D_\mu U D_\mu U) (\chi^\dagger U + U^\dagger \chi) + L_6 Tr(\chi^\dagger U + \chi U^\dagger)^2 + L_7 Tr(\chi^\dagger U - \chi U^\dagger)^2 + L_8 Tr(\chi^\dagger U^\dagger \chi^\dagger U + \chi^\dagger U \chi U^\dagger) - iL_9 Tr(R_{\mu\nu} D_\mu U D_\nu^\dagger U) + L_{10} Tr(D_\mu U^\dagger D_\nu U) + L_{11} Tr(U^\dagger R_{\mu\nu} U L^\mu) + H_3 Tr(R_{\mu\nu} R_{\rho\sigma} + L_{\mu\nu} L_{\rho\sigma}) + H_2 Tr(\chi^\dagger U),
\]

where \( L_{\mu\nu} \) and \( R_{\mu\nu} \) are the field-strength tensors for \( L_{\mu\nu}, R_{\mu\nu} \). In the case of three flavors one of the first four operators is redundant, and \( L_4 \) starts out as

\[
L_4 = L_1[Tr(D_\mu U D_\mu U)]^2 + L_2 Tr(D_\mu U D_\mu U) Tr(D_\mu U^\dagger D_\mu U) + L_3 Tr(D_\mu U D_\mu U^\dagger D_\mu U) + L_4 Tr(D_\mu U D_\mu U) Tr(\chi^\dagger U + \chi U^\dagger) + \ldots,
\]

where
\( L_1 = K_1 + K_4, \quad L_2 = K_2 + K_4, \quad L_3 = K_3 - 2K_4. \) \( \quad (20) \)

In the case of two flavors, two more low-energy constants may be eliminated as being redundant. Most important for our purposes is the fact that, among \( L_1, L_2, L_3, \) only the combinations \( L_2 \) and \( (2L_1 + L_3) \) enter into the \( \pi \pi \) scattering amplitudes. At order \( E^4 \) one must include both loop diagrams, formed using \( L_2 \), and tree diagrams from \( L_3 \) and \( L_4 \). In addition one includes the effect of the axial anomaly by using the Wess-Zumino-Witten anomaly Lagrangian. The parameters \( L_1, \ldots, L_{10} \) are in general divergent (except \( L_3, L_7 \)). They absorb the divergences of the one-loop graphs. Consequently they will depend on a renormalization scale \( \mu \) which, of course, drops out in all observable quantities. The renormalized parameters \( L'_i \) are defined by

\[
L'_i = \frac{\Gamma_i}{16\pi^2} \mu^d 4^{-4} \left[ \frac{1}{4} - \frac{1}{2} \ln(4\pi) + \Gamma(1) \right]
\]

with \( \Gamma_i \) being pure numbers given in Ref. 7(a). The low-energy constants \( L'_i \) cannot be determined from symmetry requirements alone—chiral symmetry only relates different processes, it does not provide the absolute normalization. However, most of the new coupling constants can be determined from low energy phenomenology. Furthermore, progress has been made in understanding their origin and their magnitude. We are now able to estimate several of the coupling constants occurring at order \( E^4 \) on theoretical grounds, such that essentially parameter-free predictions can be made at this order of the chiral expansion.

The analysis of Ref. 7(a) makes use of the large-N\(_c\) suppression of the coupling constants \( 2L_1 - L_2, L_4, L_6 \), see below. Since the value of the coupling constants is of importance later in this article, we quote in Table I the values \( L'_i(\mu) \) at the scale \( \mu = m_\pi \) according to that analysis.

The large-N\(_c\) limit enters in the following way. A trace in the chiral Lagrangian comes from a summation over the \( N_f \) flavors of quarks. Operators with two traces require at least two quark loops in order to get two summations over the flavors, while those with one trace require only a single quark loop. However, the large-N\(_c\) limit (with \( \alpha, N_c \) fixed) has the feature that processes with extra quark loops are suppressed by powers of \( 1/N_c \).

Therefore the coefficients of double trace operators, i.e., \( K_1, K_2, K_4, K_6 \), are suppressed by \( 1/N_c \), compared by the single trace coefficients \( K_3, K_5, L_8, L_9, L_{10} \). \( (L_7 \) is an exception due to the \( \eta' \) pole contribution, as the \( \eta' \) mass vanishes in the large-N\(_c\) limit.) The single trace terms enter at order \( N_c \) so that the precise expectation is

\[
K_3, K_5, L_8, L_9, L_{10} = O(N_c), \quad K_1, K_2, L_4, L_6 = O(1). \quad (21)
\]

If one now specializes to the case of 3 flavors, one sees from Eq. (20) that the large-N\(_c\) limit requires

\[
L_1, L_2, L_3 = O(N_c), \quad L_2 - 2L_1 = O(1). \quad (22)
\]

### Table I. Values of the low-energy constants at the scale \( \mu = m_\pi \) from Ref. 7(a). That analysis is based on the large-N\(_c\) suppression of \( 2L_1 - L_2 \) and \( L_6 \).

<table>
<thead>
<tr>
<th>( L'_i )</th>
<th>( 10^2 L'_i/\text{GeV}^4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L'_1 )</td>
<td>0.9±0.3</td>
</tr>
<tr>
<td>( L'_2 )</td>
<td>1.7±0.7</td>
</tr>
<tr>
<td>( L'_4 )</td>
<td>4.4±2.5</td>
</tr>
<tr>
<td>( L'_5 )</td>
<td>0.0±0.5</td>
</tr>
<tr>
<td>( L'_6 )</td>
<td>2.2±0.5</td>
</tr>
<tr>
<td>( L'_7 )</td>
<td>0.0±0.3</td>
</tr>
<tr>
<td>( L'_8 )</td>
<td>0.4±0.15</td>
</tr>
<tr>
<td>( L'_9 )</td>
<td>1.1±0.3</td>
</tr>
<tr>
<td>( L'_{10} )</td>
<td>7.4±0.7</td>
</tr>
<tr>
<td>( L'_{10} )</td>
<td>−6.0±0.7</td>
</tr>
</tbody>
</table>

The ordering (21) and (22) has gone into the determination of the couplings \( L_i \) in Ref. 7(a). In particular, the value \( L_2 = (-4.4±2.5)\times10^{-3} \) used in the evaluation of \( \eta \to 3\pi \) to one loop in Ref. 7(c) is based on the large-N\(_c\) suppression of \( L_2 - 2L_1 \). \( K_{14} \) decays make it possible to test the ordering (22) which cannot be probed using only pions.

### IV. CHIRAL PREDICTIONS FOR THE FORM FACTORS

The chiral representation of the form factors at order \( E^2 \) was originally given by Weinberg:

\[
F = G = \frac{m_\pi}{\sqrt{2} F}\, = 3.74, \quad H = 0. \quad (23)
\]

(Unless stated otherwise, we use in all numerical calculations \( m_\pi = m_{\pi^+} = 493.7 \text{ MeV}, \quad m_\pi = m_{K^+} = 139.6 \text{ MeV}, \quad F = 93.3 \text{ MeV}. \) At order \( E^4 \), loops with \( L_2 \) and tree-level contributions from \( L_4 \) both enter. We have used the general one-loop Lagrangian given in Eqs. (8.12) and (8.13) of Ref. 7(a) for the evaluation of \( F \) and \( G \). We write the result for \( F \) in the form

\[
F(s, t, u) = \frac{m_\pi}{\sqrt{2} F} \left[ 1 + F^+(s, t, u) + F^-(s, t, u) + O(E^4) \right], \quad (24)
\]

\[
F^\pm(s, t, u) = U^\pm(s, t, u) + P^{\pm}(s, t, u) + C^{\pm}\]

and will use below an analogous expression for the form factor \( G \). The superscript \( (+) \) denotes a term which is even (odd) under crossing \( t \leftrightarrow u \). The contributions \( U^\pm(s, t, u) \) denote the unitary corrections generated by the one-loop graphs which appear at order \( E^4 \). They have the form

\[
U^\pm(s, t, u) = U^\pm\left[ \frac{\Delta_0(s, t)}{4} + a(t) + a(t, u) \right], \quad (25)
\]

\[
U^\pm(s, t, u) = U^\pm\left[ b(t) - b(t, u) \right]
\]

with
\[ \Delta_0(s_\pi) = \frac{1}{4}(2s_\pi - m_\pi^2) J'_{\pi\pi}(s_\pi) + \frac{3s_\pi}{4} J'_{KK}(s_\pi) + \frac{m_\pi^2}{2} J'_{\eta\eta}(s_\pi) \]
\[ a_F(t) = \frac{1}{16}\left[ \{14m_\pi^2 + 14m_\pi^2 - 9t\} J'_{KK}(t) + (2m_\pi^2 + 2m_\pi^2 - 3t) J'_{\eta\eta}(t) \right] + \frac{1}{16}\left[ \{3m_\pi^2 - 7m_\pi^2 + 5t\} K_{\pi\pi}(t) + (m_\pi^2 - m_\pi^2 + 3t) K_{\eta\eta}(t) \right] - \frac{1}{16}\left[ \{9L_{\pi\pi}(t) + L_{\eta\eta}(t)\} + (3m_\pi^2 - 3m_\pi^2 - 9t) [M_{\pi\pi}(t) + M_{\eta\eta}(t)] \right] , \]
\[ b_F(t) = a_F(t) - \frac{1}{2}(m_\pi^2 + m_\pi^2 - t) J'_{\pi\pi}(t) . \] (26)

The loop integrals \( J'_{\pi\pi}(s_\pi), \ldots \) which occur in these expressions are listed in the Appendix. The functions \( J'_{KK} \) and \( M'_{\pi\pi} \) depend on the scale \( \mu \) at which the loops are renormalized. The scale drops out in the expression for the full amplitude, see below. The imaginary part of \( F^{-\Delta}_\pi(s_\pi) \) contains the \( I = 0, S\)-wave \( \pi\pi \) phase shift
\[ \delta_0^R(s_\pi) = (32\pi^2 F^2_\pi)\left(2s_\pi - m_\pi^2\right)^{1/2} \left(1 - \frac{4m_\pi^2}{s_\pi}\right)^{1/2} + O(E^4) \] (27)
as well as contributions from \( K\bar{K} \) and \( \eta\eta \) intermediate states. The functions \( a_F(t) \) and \( b_F(t) \) are real in the physical region.

The contribution \( P^\pm_{\pi\pi}(s_\pi, t, u) \) is a polynomial in \( s_\pi, t, u \), obtained from the tree graphs at \( E^4 \). We find
\[ P^\pm_{\pi\pi}(s_\pi, t, u) = \frac{1}{F^2_\pi} \sum_{i=1}^{9} P^\pm_{i,\pi\pi}(s_\pi, t, u) L^i_\pi , \] (28)
where
\[ P_{1,\pi\pi} = 32(s_\pi - 2m_\pi^2) , \]
\[ P_{2,\pi\pi} = 8(2m_\pi^2 + 2m_\pi^2) - t - u \]
\[ = 8(m_\pi^2 + s_\pi - s_\eta) , \]
\[ P_{3,\pi\pi} = 2(2m_\pi^2 - 6m_\pi^2 + 4s_\pi - t - u) \]
\[ = 2(m_\pi^2 - 8m_\pi^2 + 5s_\pi - s_\eta) , \]
\[ P_{4,\pi\pi} = 32m_\pi^2, \]
\[ P_{5,\pi\pi} = 4m_\pi^2 . \] (29)

The remaining coefficients \( P^\pm_{i,\pi\pi} \) are zero. The symbols \( L^i_\pi \) denote the renormalized coupling constants discussed above.

Finally we come to the contributions \( C^\pm_\pi \) which contain logarithmic terms, independent of \( s_\pi, t, \) and \( u \):
\[ C^+_\pi = (256\pi^2 F^2_\pi)\left(5m_\pi^2 \ln \frac{m_\pi^2}{\mu^2} - 2m_\pi^2 \ln \frac{m_\pi^2}{\mu^2} - \frac{3m_\pi^2}{m_\pi^2} \right) , \] (30)
\[ C^-_\pi = 0 . \]

The corresponding decomposition of the form factor \( G \),
\[ G^\pm = U^\pm_\pi + P^\pm_\pi + C^\pm_\pi , \] (31)
has the explicit form
\[ U^+_\pi(s_\pi, t, u) = F^-_\pi\left[ \Delta(s_\pi) + a_G(t) + a_G(u) \right] , \]
\[ U^-_\pi(s_\pi, t, u) = F^-_\pi\left[ b_G(t) - b_G(u) \right] \] (32)
with
\[ \Delta(s_\pi) = 2s_\pi \left[M'_{\pi\pi}(s_\pi) + \frac{1}{2} M'_{KK}(s_\pi) \right] , \]
\[ a_G(t) = \frac{1}{16}\left[ \{12m_\pi^2 + 2m_\pi^2 + 3t\} J'_{KK}(t) - (2m_\pi^2 + 2m_\pi^2 - 3t) J'_{\eta\eta}(t) \right] + \frac{1}{16}\left[ \{-3m_\pi^2 + 7m_\pi^2 + 5t\} K_{\pi\pi}(t) + (3m_\pi^2 + 5m_\pi^2 - 3t) K_{\eta\eta}(t) \right] - \frac{1}{16}\left[ \{L_{\pi\pi}(t) + L_{\eta\eta}(t)\} - (2m_\pi^2 - m_\pi^2 + t) [M_{\pi\pi}(t) + M_{\eta\eta}(t)] \right] , \]
\[ b_G(t) = a_G(t) - \frac{1}{2}(m_\pi^2 + m_\pi^2 - t) J'_{\pi\pi}(t) . \] (33)
are

\[ \begin{align*}
P^+_{3,0} &= \left(-2(m_K^2 + m_\pi^2 - t - u) \right), \\
P^+_{5,0} &= 4m_\pi^2, \\
P^+_{8,0} &= \left(-m_K^2 - m_\pi^2 + s_\pi + t + u\right) = 2s_l, \\
P^-_{2,0} &= -8(t - u), \\
&= -8 \left[ 1 - \left(1 - \frac{m_\pi^2}{s_\pi}\right)^{1/2} \right], \\
&\times (m_K^2 - s_\pi - s_l)^2 - 4s_\pi s_l)^{1/2}\cos\theta_\pi, \\
P^-_{3,0} &= \frac{1}{2}P^+_{2,0}. \\
\end{align*} \]

(36)

The remaining \( P^\pm_{5,0} \) vanish. The logarithms contained in \( C^\pm_0 \) are

\[ C^\pm_0 = -C^\pm_F. \]

(37)

The form factor \( H \) starts only at \( O(E^4) \). It does not appear in the Lagrangian of Eqs. (18) and (19), but arises from the Wess-Zumino-Witten Lagrangian for the axial anomaly.\(^{12}\) It is related by an SU(3) transformation to the anomalous \( \gamma \to 3\pi \) coupling. The prediction is

\[ H = \frac{\sqrt{2}m_K}{8\pi F_\pi} = 2.65 \]

(38)

in excellent agreement with the experimental value. To the order we are working, we do not consider loops or higher-order corrections to this result. The form factor \( H \) gives rise to an interference term \( -\gamma H^* + \gamma^* H \) in the decay distribution \( d\Gamma \). We have checked that the sign of this term, evaluated according to our phase convention for \( H \) and \( G \), agrees with the one given by Rosselet \textit{et al.} (Ref. 11, Table II).

The results for \( F \) and \( G \) must satisfy two nontrivial constraints: (i) Unitarity requires that \( F \) and \( G \) contain, in the physical region \( 4m_\pi^2 \leq s_\pi \leq m_K^2 \), imaginary parts governed by \( S \)- and \( P \)-wave \( \pi\pi \) scattering; these imaginary parts are contained in the functions \( \Delta_0(s_\pi), \Delta_1(s_\pi) \); (ii) the scale dependence of the low-energy constants \( L^\gamma \) must be compensated for by the scale dependence of \( U_{F,G} \) and \( C_{F,G} \) for all values of \( s_\pi, t, u, m_\pi^2, m_K^2 \). (Since we work at order \( E^4 \), the meson masses appearing in the above expressions satisfy the Gell-Mann–Okubo mass formula.)

We have checked that these constraints are satisfied. Furthermore, our expressions agree algebraically with the ones given by Bijnen.\(^{10}\) (In order to compare with the latter, the pion decay constant \( F_\pi \) must also be expanded around the chiral limit \( m_u = m_d = m_s = 0 \).)

V. EXPANSION OF FORM FACTORS AT THRESHOLD

In the chiral predictions of the form factors, one striking feature that emerges from Eqs. (29) and (36) is that the only important dependence on the low-energy constants is through \( L_1, L_2, \) and \( L_3 \). Furthermore, \( L_1 \) and \( L_2 \) are absent in the isospin even part \( \gamma^+ \). The influence of \( L_4, L_5 \) is proportional to \( m_\pi^2 \) and hence is too small to be of much importance. This means that we are not able to test the large-\( N_c \) prediction that \( L_4/L_1 \approx 0 \). In addition, the constant \( L_6 \) enters only in the \( s_l \) dependence, which again is not large and which has been dropped from the theoretical analysis. We proceed by fixing \( L_4, L_5, \) and \( L_9 \) at the values found in other processes, as quoted in Ref. 7(a) [i.e., \( L_6 = 0 \) from the Zweig rule, \( L_5 \) from \( F_K/F_\pi \) and \( L_9 \) from the electromagnetic charge radius of the pion, see also Table I].

Before describing the detailed comparison with the data, we discuss the form factors at threshold, because the results can be presented simply at this kinematical point. We define the projected amplitudes

\[ f(q^2, s_l) = \frac{i}{4} \int_{-1}^1 d(\cos\theta_\pi) \text{Re} F(q^2, s_l, \cos\theta_\pi), \]

(39)

\[ q^2 = \frac{s_\pi - 4m_\pi^2}{4m_\pi^2} \]

and similarly for \( g(q^2, s_l) \). Taking the real part \( \text{Re} F \) eliminates the phase at this order in the low-energy expansion. We renormalize all of the low-energy constants at the scale \( \mu = m_\pi \) and write the form factors as

\[ f(q^2, s_l) = f(0, s_l)[1 + \lambda_f(s_l) q^2 + O(q^4)], \]

(40)

\[ g(q^2, s_l) = g(0, s_l)[1 + \lambda_g(s_l) q^2 + O(q^4)]. \]

First we consider the threshold form factors...
\[ f(0,0) = \frac{m_K}{\sqrt{2}F_\pi} \left( 1 + X_f + \frac{2}{F_\pi^2} [32m_K^2L_1^* + 4(m_K^2 + 4m_\pi^2)L_2^* + (m_K^2 + 12m_\pi^2)L_3 + 16m_\pi^2L_4 + 2m_\pi^2L_5^*] + O(E^4) \right), \]
\[ g(0,0) = \frac{m_K}{\sqrt{2}F_\pi} \left( 1 + X_g - \frac{2}{F_\pi^2} [(m_K^2 + 4m_\pi^2)L_3 - 2m_\pi^2L_5^*] + O(E^4) \right). \]  

(41)

The constants \( X_f, X_g \) contain loop and tadpole contributions:

\[
\begin{align*}
X_f &= 0.185 & -0.051 & +0.007 & = 0.127, \\
X_g &= 0.023 & -0.030 & +0.007 & = 0, \quad (42)
\end{align*}
\]

where the pieces denoted by “tadpoles” come from the logarithms collected in \( C^+_{\pi}, C^+_{\eta} \) defined in Eqs. (30) and (37). It is seen that the major portion of the one-loop correction \( X_f \) is due to \( \pi\pi \) final-state interactions. As emphasized in particular by Truong, this is a rule rather than an exception: Pions in \( I=0 \), S-wave final states tend to produce potentially large corrections to the lowest-order term in many hadronic processes.

Inserting into Eq. (41) the values for \( L_i \) from Table I one finds \( f(0,0) = 4.85, g(0,0) = 5.03 \). The increase from the tree result \( f_{\text{tree}} = g_{\text{tree}} = 3.74 \) to \( g(0,0) = 5.03 \) is dominantly due to the effect of \( L_3 \), as the loops do not contribute to the amplitude \( g(0,0) \) according to Eq. (42) and the dependence on \( L_3 \) is weak due to the factor of \( m_\pi^2 \). Pions in the \( I=1 \), \( P \) wave interact weakly. An analogous result holds for the \( I=1 \), \( P \) wave in elastic \( \pi\pi \) scattering. At \( s_f = s_f^{\text{max}} = (m_K - 2m_\pi)^2 \), the results change little: \( X_f \rightarrow 0.129, f(0,0) \rightarrow 5.05 \), and \( X_g \rightarrow -0.003, g(0,0) \rightarrow 5.14 \). Now we consider the slopes \( \lambda_f, \lambda_g \) we find

\[
\lambda_f(s_f) = Y_f(s_f) + \frac{8m_\pi^2}{F_\pi^2} (16L_1^* + 4L_2^* + 5L_3) + O(E^4),
\]
\[
\lambda_g(s_f) = Y_g(s_f) - \frac{8m_\pi^2}{F_\pi^2} L_3 + O(E^4),
\]  

(43)

where \( Y_f, Y_g \) contain loop contributions:

\[
Y_f(s_f) = \frac{1}{s_f^{\text{max}}} \int_0^{s_f^{\text{max}}} ds_f \left\{ \frac{1}{2} \int_{-1}^{1} d(cos \theta_\pi) F(s_\pi, s_f, s_f, cos \theta_\pi) \right\}^{1/2},
\]

(46)

and similarly for \( g, \lambda_g \). [In this article, \( f_3(0), g(0), \) and \( h(0) \) always denote the form factors evaluated at \( q^2 = 0 \); see Eqs. (13) and (14).] This most nearly approximates the experimental situation. [Recall that the experimental analysis assumed that the slopes of \( f_3(s_\pi) \) and \( g(s_\pi) \) were the same, i.e., \( \lambda_f = \lambda_g \). We will not include \( \lambda_g \) in our fitting procedure, but will treat it as a prediction, testing the level of equality of \( \lambda_f \) and \( \lambda_g \).] However, since the experimental form-factor analysis did not conform exactly to the structure of the chiral predictions, we can also attempt to compare our results more directly with the experimental data. The primary data consists of the num-
bers of events grouped into bins of energy, \( n(s_p) \), plus the absolute normalization which is basically \( f_s(0) \) and the \( P/S \)-wave ratio \( g(s_p) \). We can form these quantities from theory and compare to the experimental results for these variables. Our two options then are to use the sets \([f_s(0),g(0),\lambda_f]\) or \([n(s_p),f_s(0),g(s_p)]\) for comparison. Given a perfect analysis, the two approaches should of course be the same, and the use of the two addresses the question of the compatibility of the chiral and experimental analysis. Fortunately we will see that very similar results emerge from both options.

A second issue is the question of the way to quantify the theoretical uncertainties in the analysis. Chiral perturbation theory is unique in the low-energy region, being a technique which is a controlled expansion. As such, it contains in its framework ways to estimate the uncertainties from the next order in the expansion. In this paper we are computing effects at order \( E^2 \) and order \( E^4 \), so that corrections arise from order \( E^6 \) and yet higher orders. Such corrections could influence the determination of the \( L_i \) coefficients. For example, if the data had infinite precision a naive fit to \( L_1, L_2, L_3 \) would yield values with infinitesimal error bars. However, this would not be a true estimate of the uncertainty in \( L_i \), as corrections to the theoretical analysis from order \( E^6 \) could shift the values by more than an infinitesimal amount. We attempt to quantify this effect by including theoretical error bars in our fits. These are not required in order to obtain a good description, but are a fair estimate of the nature of the energy expansion. In those analyses labeled as containing theoretical error bars, they have been included in the following manner. Observables are generally of the form \( a_{\text{exp}} = a_2 (1 + c_4 + c_6 + \cdots) \) where \( a_2 \) is the lowest-order result (i.e., order \( E^2 \)) and \( c_4 \) is a correction which we are computing at order \( E^4 \). We will use \( c_4 \) to estimate the next-order correction \( c_6 \), using the expectation that \( c_6 = O(c_4^2) = O([a_{\text{exp}} - a_2]/a_2)^2 \). Thus the theoretical prediction would be of the form

\[
a_{\text{exp}} = a_2 (1 + c_4 \pm c_4^2)
\]

(47)

It is the latter form which is used in our fitting procedure where it is added in quadrature to the experimental error. Our experience has been that the expansion in energy is uniform and that this is a fair assessment of the next-order term. The slope is a loop effect and there is thus no order \( E^2 \) correction to \( \lambda_f, \lambda_g \) in our analysis. We have assigned an error \( \Delta \lambda_f \) which is 40% of its experimental value. [We have also checked in a few cases that the following procedure results in very similar error bars for \( L_1, L_2, \) and \( L_3 \). First we determined \( \Delta L_i^f \) by assigning no theoretical error. Then we did a least-squares fit by changing the theoretical predictions by \((a_{\text{exp}} - a_2)^2/a_2\) in turn and then reading off the variation \( \Delta L_{ij} \); finally we added \( \Delta L_{ij} \) in quadrature.]

There is a last point which concerns the total decay rate \( \Gamma_{\text{tot}} \). Below we show that the coupling constants \( L_1, L_2, \) and \( L_3 \) can be chosen such that the averaged form factors defined in Eq. (46) practically coincide with the measured ones. Nevertheless the total decay rate calculated with these form factors is \( \sim 10\% \) below the measured value, which seems to indicate that the smearing over \( s_p, \cos \theta_p \) described above does not correspond to the experimental analysis. We have noted, however, that the published experimental values \( f_s(q^2), g(q^2) \) also produce too small a value for the decay rate: \( \Gamma_{\text{tot}} = 2.94 \times 10^3 \text{ sec}^{-1} \) instead of \( \Gamma_{\text{exp}} = 3.26 \times 10^3 \text{ sec}^{-1} \). (Note that the experimental value \( \Gamma_{\text{exp}} \) was used to normalize the form factors.) We do not understand this discrepancy.

The results of our procedures to fix \( L_1, L_2, \) and \( L_3 \) are displayed in Table II and Fig. 2. The first four rows in the table list the determination of the low-energy constants \( L_1, L_2, L_3 \) which follows from \( K_{h} \) data alone. The error bars correspond to an increase in \( \chi^2 \) by one. We see that there is good agreement with the central values in all cases. This indicates that the manner in which the

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**FIG. 2.** The form factors \( f_s(s_p) \) and \( g(s_p) \) according to the chiral representation described in Sec. IV. Displayed are the lowest-order result (labeled “tree”) plus the experimental and central values of fit (a) in Table II (solid and dashed line, respectively). We also show the effect of the loop terms by themselves as well as the effect of turning off each of the couplings \( L_1, L_2, L_3 \) in turn. Note that \( g(s_p) \) does depend neither on \( L_1 \) nor on \( L_2 \).
analysis is done makes little difference, and that the theoretical error bars are not needed.

The figures give a better view of the results. On them are displayed the lowest-order result (labeled "tree") plus the experimental central value and the central values of fit (a) in Table II. In order to see the decomposition of the ingredients of the final results, we also show the effect of the loop terms by themselves as well as the effect of turning off each of the couplings $L_{1,2,3}$ in turn. Note that the slope of the $g$ form factor has not been included in the fit and thus is a prediction. It matches very well with the experimental constraint that $g/f_S$ is a constant.

We already mentioned that the $P$-wave part $f_P$ was searched for but not found. We have evaluated $f_P$ from the chiral representation (24) with the parameters corresponding to Table II(a). In particular, we have set $s_1=0$, $F(s_2,\cos\theta_2)=F_S+F_P\cos\theta_2$ + · · ·. The $P$-wave term $F_P$ indeed is very small, $|F_P|<5 \times 10^{-2} |F_S|$ over the whole energy range $4m_{\rho}^2<s_\rho<m_{\rho}^2$.

As we have indicated, one motivation for our analysis was to test the large-$N_c$ prediction $(L_2-2L_1)/L_3=0$. From the values presented in Table II, we see that a small nonzero value for this ratio is preferred, but that it is consistent with zero within the errors. To make the error analysis cleaner, we have repeated the fitting procedures using the variables

$$
X_1=L_2-2L_1-L_3, \quad X_2=L_2, \quad X_3=(L_2-2L_1)/L_3.
$$

The first variable was chosen because in the SU(2) limit it measures the effect of the $\rho$ in the $I=1$, $J=1 \pi\pi$ scattering. The last is clearly the large-$N_c$-violating combination. The resulting values [using $f_S(0), g(0), \lambda_f$] are

$$
X_1=(3.82\pm0.89) \times 10^{-3},
$$
$$
X_2=(1.99\pm0.32) \times 10^{-3},
$$
$$
X_3=-0.19^{+0.16}_{-0.27}
$$

without theoretical error bars and

$$
X_1=(3.82\pm2.10) \times 10^{-3},
$$
$$
X_2=(1.99\pm1.15) \times 10^{-3},
$$
$$
X_3=-0.19^{+0.55}_{-0.80}
$$

with them. The result is that the large-$N_c$ prediction works remarkably well, at the level expected, within the error bars.

Having determined the low-energy constants, we are in a position to study the predictions of chiral symmetry. These same coefficients govern $\pi\pi$ scattering, and the real test of the theory is that they are simultaneously compatible with the $\pi\pi$ amplitudes. The most straightforward way to check this is to predict the $\pi\pi$ scattering lengths. While the direct data at low energy is poor, the scattering lengths have been obtained using the Roy equations to constrain both the high-and low-energy data. The chiral predictions were worked out in Ref. 5. If we use our determination (a) in Table II, we obtain the predictions of Table III, third column. For $I_5=I_4$ which occur in $a_1^+, b_1^+$ we have used the central values $I_5=2.9, I_4=4.3$ from Ref. 5. We do not quote errors in the threshold parameters evaluated here, because we did not work out the error matrix associated with the $L_i$'s. The predictions are within $1/2$ standard deviations of the measured values in all cases. We have also checked that the same parameters reproduce the full amplitudes within experimental and theoretical uncertainties up to $\sqrt{s_\rho} \approx 380$ MeV.

Instead of treating the $\pi\pi$ data as predictions, one could use them in a different manner to influence the determination of the $L_i$'s. The motivation for doing this is twofold: (i) it checks the consistency of the theory and (ii) it provides the best determination of the low-energy constants. Again, there are a few ways that we could proceed. The Rosselet $K_{46}$ experiment itself provides the only significant direct measurement of $\delta_0^0-\delta_1^1$ at low energies. We can include this in our analysis as well. The result is the coefficients of (e) and (f) of Table II. [We did not use theoretical errors in $\delta_0^0-\delta_1^1$, because we expect their effect, which is $O(m_\rho^2)$, to be small in the energy range considered here, $\sqrt{s_\rho} \leq 380$ MeV.] One sees that

**TABLE III.** Predictions of chiral symmetry following from the fit to the $K_{46}$ data alone (column 3) and the combined determination from $\pi\pi$ and $K_{46}$ data (last column). The first column gives the prediction of the leading-order term in the low-energy expansion of the $\pi\pi$ amplitude.

<table>
<thead>
<tr>
<th>Leading order</th>
<th>Experiment (Ref. 17)</th>
<th>$K_{46}$ alone</th>
<th>$K_{46}+\pi\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_5$</td>
<td>0.08±0.02</td>
<td>0.06±0.02</td>
<td>0.06±0.02</td>
</tr>
<tr>
<td>$a_1^+$</td>
<td>0.65±0.05</td>
<td>0.20</td>
<td>0.20</td>
</tr>
<tr>
<td>$b_1^+$</td>
<td>0.25±0.03</td>
<td>0.26</td>
<td>0.26</td>
</tr>
<tr>
<td>$a_1^-$</td>
<td>-0.045</td>
<td>-0.040</td>
<td>-0.041</td>
</tr>
<tr>
<td>$b_1^-$</td>
<td>-0.089</td>
<td>-0.069</td>
<td>-0.070</td>
</tr>
<tr>
<td>$a_1^+$</td>
<td>0.038±0.002</td>
<td>0.037</td>
<td>0.036</td>
</tr>
<tr>
<td>$b_1^-$</td>
<td>0.0045</td>
<td>0.0045</td>
<td>0.0043</td>
</tr>
<tr>
<td>$a_1^+$</td>
<td>(17±3)×10^{-4}</td>
<td>21×10^{-4}</td>
<td>20×10^{-4}</td>
</tr>
<tr>
<td>$a_1^-$</td>
<td>(13±3)×10^{-4}</td>
<td>3.5×10^{-4}</td>
<td>3.5×10^{-4}</td>
</tr>
</tbody>
</table>
the same parameters describe the phase shift information, with a good fit $\chi^2/N_{DF} = 0.82$ and 0.78. Instead of these direct data, one may compare with the experimental scattering lengths given in the second column of Table III. The result is displayed in (g) and (h) of Table II. [Here the $\chi^2/N_{DF} = 1.0$ and 0.8. We did not associate theoretical errors with the threshold parameters, because these effects are $O(m^4)$ and thus very small.] The agreement of the chiral predictions described above is manifest in the fact that the central values do not change much between cases (a), (b) and (g), (h). We conclude that the theory is quite consistent with all sets of data.

The results of this analysis determine the value of the coefficients in the purely SU(2) chiral Lagrangian. In the notation of Ref. 5, the results of fit (g), (h) are

$$T_1 = -0.70 \pm 0.94 \quad ( -0.97 \pm 1.22 ),$$
$$T_2 = 6.31 \pm 0.49 \quad ( 5.77 \pm 0.72 ).$$ (51)

without (with) theoretical error bars. The central value of these can be easily obtained from linear combinations of $L_1, L_2, L_3$. To obtain the quoted error bars we have performed the fits using $L_3, T_1, T_2$ as the independent variables. The magnitude of the constant $T_1$ is smaller than the estimate from the $\pi\pi$ analysis of Ref. 13, while that of $T_2$ is essentially identical. Within the $\pi\pi$ system, the difference between the two determinations comes from differing treatment of the data at higher energies and is within the uncertainty of the $\pi\pi$ data and of the energy expansion. However, the study of the $K_{4\pi}$ form factors adds strong additional constraints and the coefficient sets of this paper are to be preferred.

Finally, it is of interest to provide the best determination of the low-energy constants by including the maximum amount of data. This, of course, includes the $K_{4\pi}$ form factors $f_S(s), g(s)$, and $\lambda$, as well as the direct measurement of $\delta^0_0 - \delta^1_0$ in $K_{4\pi}$ decay. We take the other $\pi\pi$ information as the scattering lengths $a^0_1, a^0_2, a^2_2, b^0_2$ as well as the universal curve $^{17,19}$

$$X(a^0_0, a^0_2) = 2a^0_0 - 5a^0_2 - 0.96(a^0_0 - 0.3) - 0.7(a^0_0 - 0.3)^2$$
$$= 0.69 \pm 0.04 .$$ (52)

This is a well-determined combination which is independent of the $K_{4\pi}$ phase shift information. The results are shown in the first two rows of Table IV, with the resulting scattering lengths and slope $\lambda_\pi$ given in Table III (last column). The $\chi^2/N_{DF}$ is 0.9, the error bars again correspond to an increase in $\chi^2$ by one. For comparison we display in the fourth row the values of $L_1, L_2, L_3$ determined in Refs. 5 and 7(a) from the $D$-wave $\pi\pi$ scattering lengths and the large-$N_c$ suppression of $2L_1 - L_2$. (Here the error bars have a different origin and meaning, see Refs. 5 and 7(a).)

The $K_{4\pi}$ data on $\pi\pi$ scattering is not yet precise enough to address the question of alternate pictures of chiral-symmetry breaking, which seem to prefer a value of $a^0_0 = 0.26$ instead of the usual value of $a^0_0 = 0.20 \pm 0.01$. (The literature on the subject may be traced from Ref. 20.) The data of Rosselet et al. $^{11}$ lead to $a^0_0 = 0.26 \pm 0.05$, $^{18,19}$ which is compatible with both values.

The nice agreement between the values for $L_1$, $L_2$, and $L_3$ found with these different approaches has implications on $\eta \rightarrow 3\pi$ decays. Some time ago this process was evaluated to next-to-leading order in chiral perturbation theory. $^{7(c)}$ With the exception of $L_3$, all low-energy coupling constants which occur in the final expression for the matrix element can be absorbed into physical quantities. In Ref. 7(c) the value $L_3 = -4.4 \times 10^{-3}$, determined as mentioned above from $\pi\pi$ waves and large-$N_c$ arguments, was used to evaluate the decay rate of $\eta \rightarrow 3\pi$. The fact that $K_{4\pi}$ data confirm this value according to Table IV means that the notorious difficulty to explain $\eta \rightarrow 3\pi$ in chiral perturbation theory $^{7(c),21,22}$ cannot be blamed on an incorrect value of $L_3$ used in that calculation.

In Fig. 3 (curve 1), we show the form factors $f_S(s)$ and $g(s)$ corresponding to the values in the first row of Table IV. Numerically, these correspond to $f_S(0) = 5.53$, $g(0) = 4.74$, and $\lambda = 0.08$, as well as the values quoted in the last column of Table III. The agreement with the data is excellent. To visualize the working of the large-$N_c$ rule, we display also the form factors which result from the same fit, however with the additional constraint $L_3 = -3.74$, $\chi^2/N_{DF} = 0.95$ or, equivalently $\tilde{T}_1 = -0.78, \tilde{T}_2 = 6.3$. In Fig. 4 we also display the phase difference $\delta^0_0 - \delta^1_0$ corresponding to the values in the first row of Table IV, together with the data from Ref. 11. The theoretical curve agrees with the measurements within the error bars, although the data appear to be systematically on the higher side. Note that the values of $L_1$, $L_2$, and $L_3$ used here lead to $a^0_0 = 0.20$ (see the last column of Table IV). Determination of the chiral low-energy constants from the full set of low-energy data. First two rows: Values found in the present analysis. Third row: Ref. 10, which is based on $K_{4\pi}$ data alone. Fourth row: Values based on $D$-wave $\pi\pi$ scattering lengths and Zweig rule (Refs. 5 and 7(a)). For error bars, see text.

<table>
<thead>
<tr>
<th></th>
<th>$10^3 L_1^1$</th>
<th>$10^3 L_2^1$</th>
<th>$10^3 L_3$</th>
<th>$\tilde{T}_1$</th>
<th>$\tilde{T}_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No theor. error bars</td>
<td>0.65±0.28</td>
<td>1.89±0.26</td>
<td>-3.06±0.92</td>
<td>-0.62±0.94</td>
<td>6.28±0.48</td>
</tr>
<tr>
<td>With theor. error bars</td>
<td>0.88±0.47</td>
<td>1.61±0.38</td>
<td>-3.62±1.31</td>
<td>-0.81±1.23</td>
<td>5.76±0.71</td>
</tr>
<tr>
<td>$K_{4\pi}$ alone (Ref. 10)</td>
<td>0.55</td>
<td>1.5</td>
<td>-2.8±0.5</td>
<td>-0.52</td>
<td>5.55</td>
</tr>
<tr>
<td>$\pi\pi$ D-waves and Zweig rule (Refs. 5 and 7(a))</td>
<td>0.9±0.3</td>
<td>1.7±0.7</td>
<td>-4.4±2.5</td>
<td>-2.3±3.7</td>
<td>6.0±1.3</td>
</tr>
</tbody>
</table>
While we were completing our analysis, we received a paper by Bijnens on the same topic. Although notationally different from our form, we are in agreement with his calculations of the $K_{14}$ form factors. However, there are some differences in the phenomenology. In the order $E^0$ corrections, Bijnens uses a constant $F_0$, the pion decay constant in the limit of zero quark mass, rather than the physical pion decay constant, $F_\pi$. [For reasons which we do not understand he uses $F_0 = 82$ MeV instead of the value $F_0 = 87$ MeV estimated in Ref. 7(b).] The difference between the usage of $F_0$ and $F_\pi$ is technically only a correction which is yet one order higher in the energy expansion; i.e., the difference is at order $E^6$. Nevertheless, the distinction makes a difference in the numerical values quoted. More important phenomenologically is the fact that Bijnens discusses $K_{14}$ reactions only. We have found the inclusion of $\pi\pi$ scattering to be a powerful phenomenological constraint. Thus in the end, our favored values of the low-energy constants differ somewhat from his—see the third row in Table IV, where we have also evaluated $\tilde{T}_1$ and $\tilde{T}_2$ corresponding to his values of $L_1$, $L_2$, and $L_3$. (Bijnens quotes the coupling constants at the scale $\mu = m_\rho$. We have reexpressed them at $\mu = m_\rho$. ) The error quoted for $L_3$ is only due to the fit with data; no estimate for the next order term in the low-energy expansion was done. The values in the second row agree with his findings within the error bars.

VII. DISCUSSION

Chiral perturbation theory can describe very successfully the data of $K_{14}$ decays. Effects at order $E^4$ are important in this comparison. We have used this analysis for three purposes.

(i) The $K_{14}$ data plus chiral symmetry make predictions for $\lambda_\pi$ and the $\pi\pi$ scattering lengths. These are given in Table III.

(ii) The $K_{14}$ data allow a test of the large-$N_c$ predictions concerning chiral Lagrangians. This is shown in Eqs. (48)–(50).

(iii) The full set of $K_{14}$ and $\pi\pi$ data allows the best determination of the coefficients in the chiral Lagrangian. These are shown in Table IV, and represent the low-energy content of QCD.

The large-$N_c$ rule works at the one standard deviation level for the combination $2L_1 - L_2$. Overall, the data and the chiral analysis are in excellent agreement.

In the next generation of rare kaon decay experiments, there is the opportunity to improve the phenomenology of $K_{14}$. The experimental uncertainty on $G$ is still too large to provide a precise value for the large-$N_c$ parameter $(L_2 - 2L_1)/L_3$. The observation of the other $K_{14}$ reactions with high statistics could provide a cleaner separation of the various isospin amplitudes. However, perhaps the most useful innovation would be to analyze the experimental data directly using the framework of chiral perturbation theory which we describe in this paper. Rather than making assumptions about the absence of $P$ waves, $D$ waves, etc., one could parametrize the data.

![FIG. 3. The form factors $f_\rho(s_\rho)$ and $g(s_\rho)$ according to the chiral representation described in Sec. IV. Solid line: Experimental value. Curve 1: Central value according to the first row of Table IV. Curve 2: Same fit, with the additional large-$N_c$ constraint $L_3 = 2L_1$.](image)

![FIG. 4. The phase difference $\delta_0^0 - \delta_1^1$. The data are from Ref. 11. The solid line displays the theoretical curve which corresponds to the first row of Table IV.](image)
using the full chiral perturbation theory formulas, and
directly decide the quality of the fit and the favored
values of the low-energy constants. In addition, recall
that $K_{l4}$ decay is the only available source of clean
information on $\pi\pi$ S-wave scattering near threshold. Future
improvements in this area would also be welcome.
In conclusion, rare kaon decays provide a wealth of
information on chiral perturbation theory, as well as
constraints on fundamental interactions. It would be
valuable to have a new study of $K_{l4}$ decays in order to test
the chiral and large-$N_c$ predictions more exactly.

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APPENDIX

The loop integrals $J_{\pi\pi}, M_{\pi\pi}, \ldots$ which occur in
the expression of the form factors $F$ and $G$ can be expressed

in terms of the standard integral

$$J_{\pi\pi} (x) = - \frac{1}{16\pi^2} \int_0^1 \ln \left( \frac{g(x; z)}{\tilde{g}(x; 0)} \right) dx,$$

$$g(x; z) = m_{\bar{p}}^2 z^2 (1 - x) - \Delta x,$$

$$\Delta = m_{p}^2 - m_{\bar{q}}^2.$$  \hfill (A1)

In particular one has

$$J' = \frac{3}{2} - 2k, \quad K = \frac{\Delta}{4z}, \quad L = \frac{\Delta^2}{4z},$$

$$M' = \frac{1}{12z} (z - 2m_{\bar{p}}^2 - 2m_{p}^2) \frac{\Delta^2}{4z^2} - \frac{\Delta^2}{6} k + \frac{1}{288\pi^2}$$ \hfill (A2)

with

$$k = \frac{1}{32\pi^2} \left[ \frac{m_{\bar{p}}^2 \ln \frac{m_{\bar{p}}^2}{\mu^2} - m_{p}^2 \ln \frac{m_{p}^2}{\mu^2}}{\Delta} \right],$$

$$\tilde{J}(z) = J(z) - zJ(0).$$ \hfill (A4)

For $m_{\bar{p}} = m_{q} = m$,

$$\tilde{J}(z) = - \frac{1}{16\pi^2} \left[ \ln \left( \frac{\sigma - 1}{\sigma + 1} \right) + 2 \right], z < 0,$$

$$\sigma = (1 - 4m^2/z)^{1/2},$$ \hfill (A5)

$$\tilde{J}(0) = \frac{1}{96\pi^2} \frac{1}{m^2}, k = \frac{1}{32\pi^2} \left[ \ln \frac{m^2}{\mu^2} + 1 \right].$$

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