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SELF-DUAL FORMULATION OF THE ISING MODEL IN A RANDOM LATTICE

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Abstract

A self-dual formulation of the Ising model on a random lattice is presented. In the language of the large- N matrix models, this property is non-universal; i.e., it is not independent of the detailed form of the potential. The fact that it is possible to preserve duality in the presence of two-dimensional gravity, when defining the model non perturbatively is very important in the context of superstring theory.

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1. Introduction

In the context of the physics of strings compactified in a torus of characteristic length R , there is a symmetry called duality, which implies that the physical predictions are identical to those corresponding to another torus of length α'/R . (This symmetry can be generalized to arbitrary toroidal compactifications, as well as to certain orbifolds, and even Calabi-Yau spaces; cf. [1] for a review.) This symmetry has been found to be an exact one to all orders of string perturbation theory [2]. It is obviously of great interest to check whether it survives non-perturbative corrections.

Recently, using the matrix model representation of the string path integral, it has become possible to begin the study of non-perturbative string physics, [3] albeit for space-time "dimensions" $d \leq 1$. The case $d = 1$, where the external space is just a circle S^1 , is particularly interesting, and has been studied in detail in ref. [4]. There, a plausible scenario for duality breaking due to liberation of vortices is presented which, however, fails to reproduce the cherished results of string perturbation theory. Motivated by our desire to understand this puzzle, we have begun our research with the study of a much simpler model, namely, the Ising model.

Actually, an outcome of the work cited in [3] has been the study (and complete solution) of the Ising model in a random lattice (which has also the physical interpretation of strings moving in a space of $d = 1/2$ [5]). This model is obviously not self-dual in general, under the standard Kramers-Wannier duality transformation. (Even in an ordinary, non-random, lattice, the Ising model is not self-dual unless the lattice itself is self-dual).

This means that duality cannot be a universal property, valid for any potential satisfying a finite number of restrictions. It is possible, though, that for a specific choice of the potential, there exists a self-dual formulation in a random lattice. The aim of the present letter is to argue that this is so, and to discuss somewhat the physical implications of our result.

2. Self-duality by projection

Let us consider the Ising model with zero magnetic field defined by putting Ising spins on the vertices of an arbitrary tessellation of a genus g Riemann surface. We shall denote by n_0 the number of vertices, n_1 the number of links, and n_2 the total number of plaquettes; the Euler characteristic $\chi = 2 - 2g$ is then given by $\chi = n_0 - n_1 + n_2$. We define the partition function as:

$$Z(\beta) \equiv (1/2)^{n_0/2} \sum_{\sigma_i = +1, -1} \beta^{\sum_{(ij)} \sigma_i \sigma_j} \quad (2.1)$$

where $\sum_{(ij)}$ extends over the n_1 links of the lattice Γ . Using standard techniques of Fourier transform on the group $Z_2 \equiv [+1, -1]$ (cf. for example, ref [6], and taking into account that in the dual lattice, $n_0^* = n_2$ and $n_1^* = n_1$, we have in the thermodynamical limit (that is, when $n_1 \rightarrow \infty$

$$Z(\beta) = (\sinh 2\beta)^{n_1/2} Z^*(\beta^*) \quad (2.2)$$

where $\beta^* = -1/2 \log \tanh \beta$. The two known solutions of the Ising model in a random lattice were found by Kazakov in [7] using, respectively, ϕ^3 and ϕ^4 potentials in a two-matrix model. Neither of them exhibits self-duality of the Kramers-Wannier type (in spite of the fact that ϕ^4 graphs become self-dual on the average, in the thermodynamical limit). The problem is that one can view the duality mapping as an involution from a ϕ^4 diagram to an arbitrary tessellation by squares (which can have any local coordination number): $P: \Gamma_4^* \rightarrow \Gamma_4^*$. This involution is defined in the space of all lattices with arbitrary local coordination numbers and defining tessellations by arbitrary polygons. Now, given any model defined on a subset of the full space of all lattices, such as the ones considered by Kazakov, with fixed coordination number, we can always consider its self-dual extension, just by applying the projector $P_+ \equiv (1+P)/2$. In an analogous way, we could have considered anti-self-dual models, by using $P_- \equiv (1-P)/2$. To be specific, if we start with a model formulated in a random lattice with fixed coordination number 4, its self-dual extension is:

$$Z_4(\beta, \lambda) = \sum_{n_1} \lambda^{n_1/2} Z_{n_1}(\beta) \quad (2.3)$$

where

$$Z_{n_1}(\beta) = \sum_{\Gamma_{n_1}} Z_{\Gamma_{n_1}}(\beta) + \sum_{\Gamma_{n_1}^*} Z_{\Gamma_{n_1}^*}(\beta) \quad (2.4)$$

Our definition of the fixed graph partition function differs from (2.1) in the combinatorial prefactor, which in this section will be one, in order to agree with Kazakov's conventions. Γ_{n_1} denotes the set of ϕ^4 diagrams with n_1 links, and $\Gamma_{n_1}^*$ the dual set of square tessellations. The cosmological constant is $-\log \lambda$; it is coupled to the area, which we measure in a self-dual way, as $A \equiv n_1/2$, and determines the gravitational critical properties. There is now a factor $1/2$ difference with (2.2) due to our new conventions.

$$Z_{\Gamma_{n_1}^*}(\beta) = 2(\sinh 2\beta)^{n_1/2} Z_{\Gamma_{n_1}}(\beta^*) \quad (2.5)$$

which means that we can rewrite (2.3) as:

$$Z(\beta, \lambda) = Z_k(\beta, \lambda) + 2Z_k(-1/2 \log \tanh \beta, \lambda \sinh 2\beta) \quad (2.6)$$

where Z_k is Kazakov's solution with coordination number 4 (cf. ref [7]).

The thermodynamic limit is determined by the radius of convergence in λ of (2.6). Actually, the free energy is given in terms of $c \equiv \exp(-2\beta)$ by:

$$F(c) = -\log[\min(|\lambda_k(c)|, |\lambda_k^*(c)|)] \quad (2.7)$$

where $\lambda_k(c)$ is the radius of convergence of Z_k , and

$$\lambda_k^*(c) \sinh 2\beta = \lambda_k^*(1 - c^2)/2c = \lambda_k(c^*) \quad (2.8)$$

and $c^* = (1 - c)/(1 + c)$. The final result is

$$F(c) = -\log[2cc^*/(1 - c^2)(1 - c^{*2})/9.c^{1/2}.c^{*1/2}(c^{*1/2} - 1)^2(c^{*1/2} + 2)(c < \hat{c})] \quad (2.9)$$

$$F(c) = -\log[c/(1 - c^2)^2/9.c^{1/2}.(c^{1/2} - 1)^2(c^{1/2} + 2)(c > \hat{c})] \quad (2.9)$$

where the self-dual point $\hat{c}$ satisfies $\hat{c} = \hat{c}^*$, so that $\hat{c} = \sqrt{2} - 1$, and our projection indeed enforces self-duality:

$$F(c) = \log(1 - c^2)/2c + F(c^*) \quad (2.10)$$

The curve for $F(c)$ given by (2.9) has a unique discontinuity at the self-dual point, $c = \hat{c}$. This phase transition is first order; the two third-order transitions of the two Ising models

we are superposing, which are located at $c = 1/4$ for Γ_4^* and at $c = 3/5$ for Γ_4^* do not belong to the physical branch defined by (2.9) itself. This means that this model is in fact, selfdual, but it is not like the Ising one in any useful sense. Actually, the continuum limit is dominated by pure gravity, with $\gamma_{str} = -1/2$.

There is a natural conjecture one can formulate at this point, which is that when taking the analogue of (2.3) for higher coordination numbers, 4, 5, ... we expect the functions $\lambda_k(c)$ and $\lambda_k^*(c)$ to approach each other, in such a way that both third-order singularities would migrate to the self-dual point when all possible coordination numbers are considered.

The resulting model would be self-dual simply because we sum over all possible two-dimensional lattices; given a specific lattice, it is certain that its dual is included in the set. Let us elaborate a little bit on this point in the next section.

3. Self-duality by averaging over all random lattices

An Ising model defined by summing over all possible diagrams can be shown to be equivalent to the connected perturbative expansion of a two-matrix model with an infinite potential. This potential must be carefully tuned in order to get the same weight for the area in the dual graph. This can be achieved by taking a weight $2^{-1/2}$ for all couplings (to reproduce the factor in (2.1)) and measuring the area in links. Linear and quadratic terms should also be included in the potential (besides the propagator), because we need tadpoles and mass insertions, dual to ϕ^3 tadpoles and self-energy graphs.

To be specific, the two-matrix formulation of the model in a random sphere is:

$$Z(\lambda, c) = \lim_{N \rightarrow \infty} 1/N^2 \log \int DADB \exp - NS(A, B)$$

$$S(A, B) = \text{tr}(A^2 + B^2 - 2cAB)c^A + V(A) + V(B)$$

$$V(\phi) = -2^{-1/2} \sum_{l=1}^{\infty} \phi^l / l = 2^{-1/2} \log(1 - \phi) \quad (3.1)$$

and the coupling constant is defined by: $\lambda = ce^{-2\mu}/4(1 - c^2)^2$. Standard techniques first introduced in ref. [8] give:

$$Z(\lambda, c) = \int_0^1 dx (1-x) \log f(x) \quad (3.2)$$

where $f_n = h_n/h_{n-1}$, h_n being the normalization constants of the orthogonal polynomials, and the variable is defined as $x \equiv n/N$. The function $f(x)$ is determined in the spherical limit by the following set of equations, which we shall write for the cut-off potential $V_M \equiv -1/2 \sum_{l=1}^M A^l/l$:

$$x = 2e'' R_1(x) - 2ce'' f(x) + \partial\Omega/\partial R_0$$

$$2e''(c-1)R_0(x) = \partial\Omega/\partial R_1$$

$$f(x)(2e'' + \partial\Omega/\partial R_2) = 2ce'' R_1(x)$$

$$f^2(x)\partial\Omega/\partial R_3 = 2ce'' R_2(x)$$

...

$$f^{M-2}(x)\partial\Omega/\partial R_1 = 2ce'' R_{M-2}(x)$$

$$-2^{-1/2} f^{M-1}(x) = 2ce'' R_{M-1}(x) \quad (3.3)$$

and $\Omega(R_0, R_1, \dots, R_{M-1})$ is the usual generator of the "staircases":

$$\Omega \equiv \oint dz/2\pi i V(z + R_0 + R_1/z + \dots + R_{M-1}/z^{M-1}) \quad (3.4)$$

The preceding system (3.3) is an algebraic set of $M+1$ equations with $M+1$ unknowns: $f(x), R_0(x), \dots, R_{M-1}(x)$, which must be solved in the $M \rightarrow \infty$ limit! We have not been able to find an exact solution, which would in turn determine the convergence radius of the whole series. We think, however, that we have a strong case for this convergence radius being non-zero.

First of all, in the high temperature limit, $c \rightarrow 1$, the Ising partition function reduces to $2^{n_1/2}$. In this limit, our model is equivalent to pure gravity with a constant weight at each vertex. This corresponds to an action for the matrix model:

$$S(\phi) = N \text{tr} [\phi^2/2e'' - 2^{1/2} \sum_{l=1}^{\infty} \phi^l/l] \quad (3.5)$$

We shall show explicitly in a moment that this model belongs to the $m=2$ Kazakov multi-critical class, thus describing pure gravity as claimed.

Before doing that, however, we want to consider the opposite limit of small temperature (that is, small c) of our self-dual model. It has the general form:

$$Z(c) = \sum_p c^p < X^{(p)}(A, B) >_{A>B} \quad (3.6)$$

where $X^{(p)}(A, B)$ are explicit expressions involving the $SU(N)$ group characters (cf. ref. [9]); and the correlators are defined for a pure gravity model with our infinite potential. In this case, we are unable to prove convergence of the full expansion in c , but we can at least check that the correlators themselves are well defined.

We can perform both steps simultaneously if we carry out the analysis of the one-matrix model with the action:

$$S_\kappa = N \text{tr} [\phi^2/2e'' - \kappa \sum_{l=1}^{\infty} \phi^l/l] \quad (3.7)$$

If we take in the above expression $\kappa = 2^{-1/2}$, we recover the gravitational sector of the full model (that is, the correlators in (3.6)); whereas if $\kappa = 2^{1/2}$ we get the high temperature limit of the self-dual model, given before in (3.5).

The spherical $N \rightarrow \infty$ limit of the recursion relations for the orthogonal polynomials is given as usual by:

$$1 = \partial\Omega/\partial S \quad (3.8)$$

$$0 = \partial\Omega/\partial R$$

where

$$\Omega(R, S) = \oint dz/2\pi i V(z + R/z + S) \quad (3.9)$$

Some care has to be taken when computing (3.9) for our infinite potential (3.1). The

contour has to enclose $z = 0$, and must lie within the domain of convergence of the formal series, that is $|z + R/z + S| < 1$. This requirement imposes $|S| < 1$, so that we only pick one of the poles, $z = z_- \equiv -1/2[(S-1) + \sqrt{(S-1)^2 - 4R}]$. We eventually get:

$$\begin{aligned} 1 &= e^\mu R + \kappa/2 \oint dz/2\pi i z / [z^2 + (S-1)z - 4R] \\ 0 &= e^\mu S + \kappa \oint dz/2\pi i 1 / [z^2 + (S-1)z + R] \end{aligned} \quad (3.10)$$

It is easy to check (numerically) that the function $e^{-\mu} \equiv W(R)$, defined implicitly by the system:

$$\begin{aligned} \kappa W &= S \sqrt{(S-1)^2 - 4R} \\ (1 - \kappa/2)W &= R + 1/2(S-1)S \end{aligned} \quad (3.11)$$

is such that there is a value of $R = R_c$ with $W'(R_c) = 0$, $W''(R_c) \neq 0$, $W(R_c) = e^{-\mu_c} \neq 0$ and $|S(R_c)| < 1$. It is perhaps worth remarking that our conditions for multicriticality are not identical to the ones given in [10], which are overdetermined in general, and correspond to criticality at both ends of the eigenvalue distribution (cf the lucid discussion in [11]).

We have then a model which is non-trivial at high temperatures, and probably also at low temperatures. We think it most unlikely that it happens to be so degenerate as to be trivial in between.

4. The non-perturbative sum

Our results extend trivially to any genus in the thermodynamical limit, because of (2.2), and even after taking the "double scaling limit", in which we take $N \rightarrow \infty$ and $\mu \rightarrow \infty$ in such a way that $t \equiv (e^\mu - 1)N^{4/5}$ is kept fixed. (The "string coupling constant" is then given by $g_{str}^2 = t^{-5/2}$.) The free energy is such that, by construction,

$$F(c, t) = F(c^*, t) \quad (4.1)$$

We can always define appropriate rescalings, so that

$$D^2 F(t, c) = \alpha(t) f(\beta(c)t) \quad (4.2)$$

(where $D \equiv d/dt$) and $f(t)$ satisfies the Painlevé equation:

$$t = f^2 - 1/3 D^2 f \quad (4.3)$$

that is, we are scaling two-dimensional gravity ($\gamma_{str} = -1/2$). (Please note that this rescaling implies that c is not a universal parameter, since it is affected by them.)

If we want to recover the critical behaviour corresponding to the Ising model, one must approach the self-dual point $c = \hat{c}$ along a trajectory of $c = \hat{c} - a^{2/3}T$. In this way we get (cf. the refs. in [5]) (now $t = (e^\mu - 1)N^{6/7}$):

$$t = T(f^2 - 4/9 f') + f^3 - f f'' - 1/2 f'^2 + 2/27 f''' \quad (4.4)$$

which is obviously invariant under duality with respect to the scaling trajectory, $c^* = \hat{c} + a^{2/3}T^*$. Actually, one gets $T = T^* + O(a^{4/3})$, using $c^* = (1-c)/(1+c)$.

5. Conclusions

We have presented evidence for the existence of a self-dual formulation of the Ising model in a random lattice.

This shows that some non-universal properties, like Kramers-Wannier duality, can survive the coupling to two-dimensional quantum gravity if this is done in an appropriate manner.

The resulting model belongs to the same universality class as the usual Ising model in a random lattice in the sense that it has the same critical exponents. It differs, however, from it in other details and numerical expressions.

It should be stressed, however, that the non-perturbative meaning of the "free energy" $f(t)$ is not completely clear when the potentials used in the matrix model are not bounded from below. This happens for even values of m , and for our self-dual infinite potential as well. More work is needed before this issue could be completely settled, and the non-perturbative definition of our model could rest on firmer ground.

Our results are not yet of any direct relevance for the issue of "space-time duality" in the context of string theory. We hope to report on an analysis of this problem in the near future.

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