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1. Introduction

It is a great pleasure to participate in this meeting honoring Raymond Stora on his 60th birthday. One of the subjects he has been following closely, and contributing to, is the study of topological field theories^{1,2}. Some of these theories follow from subtle quantization of the Lagrangian " $L = 0$ ", and their spaces of physical states provide explicit representations of interesting two- and four-dimensional cohomology theories (Floer Homology Theory³, Donaldson Polynomials⁴, etc.,⁵. Another very interesting topological field theory introduced by Witten⁶ as a physical approach to the Jones polynomials⁷ and beyond has generated a good deal of activity for several reasons. 1) It provides new invariants for three-manifolds and it allows us to write knot and link polynomials in arbitrary three manifolds. Witten's formulation has the advantage of being a genuine three-dimensional approach to knot theory not based necessarily on the description of knots by their two-dimensional projections. 2) The topological Chern-Simons theory is intimately connected with the Wess-Zumino-Witten theory (WZW) and many other two-dimensional conformal field theories (CFT) of current interest. It shows that three-dimensional general covariance is perhaps the unifying theme for all known rational conformal field theories (RCFT). For example the celebrated polynomial

equations of Moore and Seiberg⁹ can be naturally understood in this context. 3) The explicit representation of the knot and link invariants as expectation values of Wilson loops generate explicit integral representations for the coefficients of the Jones polynomials generalizing Gauss' formula for the link invariant¹⁰, etc. Since we have had many discussions with Raymond concerning the properties of this theory we will present a summary of our results and of our points of view.

Our work has centered primarily on the construction of modular invariant partition functions different from the standard diagonal one using the three-dimensional point of view. We have been able to reproduce most of the known results in the literature about modular invariants and we have also found some new ones. We have unfortunately not been able to find a three-dimensional rationale for the construction of exceptional invariants. The main difficulty lies on the question of positivity of the partition function (for details, see¹¹). We have also analyzed the definition of the perturbative expansion of the theory in a gauge invariant way¹². Unexpectedly this has generated some controversy about the correct procedure to quantize and renormalize the topological Chern-Simons theory. The basic point of discussion is the existence of the shift $k \rightarrow k + g$ found by Witten in his original paper⁶. We would like to take this opportunity to express our point of view and hopefully clarify

how the shift should be understood in perturbation theory and why there is yet no contradiction between the different approaches, explored in the literature.

In Section 2 we present the Hamiltonian approach to the topological Chern-Simons theory (CS) which is essentially an expansion of the original argument⁶, and in Section 3 we summarize our work on the perturbative approach to the theory. Section 4 recapitulates our conclusions.

2. Hamiltonian approach to the CS theory

The connection between CS theories and RCFT has been obtained from the point of view of canonical quantization^{13–18}, and using current algebras^{16,19} along the lines originally proposed by E. Witten. It has been particularly useful in the operator representation of knots the operator formalism developed in^{15,20} in analogy with the one in string theory²¹. Consider for simplicity the gauge group $G = SU(N)$ with generators T^a normalized according to $\text{tr } T^a T^b = -\delta^{ab}$. The topological CS action is:

$$S = -\frac{k}{4\pi} \int_{M_3} \text{tr}(AdA + \frac{2}{3}A^3) \quad (1)$$

where we take $M_3 = M_2 \times \mathbf{R}$ in order to be able to use the Hamiltonian formalism. M_2 can be any two-dimensional manifold, and k is an integer. We will consider it to be either the disk D or a compact Riemann surface, in particular the torus. The action (1) describes the dynamics of a finite number of degrees of freedom. This is seen by first going to the Hamiltonian gauge $A_0^a = 0$. Let $A = A + \bar{A} = A_z dz + A_{\bar{z}} d\bar{z}$. Then in this gauge:

$$S = -\frac{k}{4\pi} \int dt \int_{M_2} (A^a \partial_t \bar{A}^a - \partial_t A^a \bar{A}^a) \quad (2)$$

The action becomes quadratic and the non-trivial information is contained in the Gauss law constraint satisfied by the physical states:

$$F_{z\bar{z}}^a(A)\Psi[A] = 0 \quad (3)$$

At the classical level this implies that the phase space of the theory is finite dimensional for compact M_2 . The space of initial conditions at $t = 0$ must satisfy $F_{z\bar{z}}^a(A) = 0$. Hence the physical phase space is the space of flat gauge connection on M_2 modulo gauge transformations or equivalently the moduli space of flat connections on M_2 . As in ordinary quantum mechanics, after we know the phase space we can proceed to quantize via the correspondence principle. In this case, due to the compactness of phase space (for M_2 compact) the canonical quantization is carried out using the techniques of geometric quantization (see 18 for details). In this approach to the quantization of CS theory, some of its properties become very clear, for instance, the ultra-violet and infra-red finiteness of the theory, something which requires a more elaborate proof in ordinary perturbative field theory²². As an example, we compute the physical states of the theory when M_2 is a two-dimensional torus T_2 . We will begin with $M_2 = D$ to first exhibit the intimate relation between CS theory, the WZW model and the current algebra of $SU(N)$ at level k . Now the three-manifold $M_3 = D \times \mathbf{R}$, $\partial M_3 = S^1 \times \mathbf{R}$, and we have to impose some boundary conditions to avoid boundary contributions to the equation of motion. Our approach here follows essentially Ref. 16 who presented a detailed analysis of the case $M_3 = D \times \mathbf{R}$. As boundary condition one imposes $A_0(\partial M_3) = 0$.

This boundary condition is left invariant by gauge transformations which are time independent on the boundary. A subgroup of this group is the gauge group. In the space of maps $g : D \rightarrow G$, the gauge group $\mathcal{G}$

is identified with those maps g such that $g(\partial D) = 1$. Time-independent gauge transformations on the boundary should be viewed as a global symmetry of the theory. Splitting the gauge connection one-form into $A_0 dt + A$ and integrating in the functional integral over A_0 one obtains the following path integral describing the CS theory on $D \times \mathbf{R}$:

$$\int \frac{\mathcal{D}A}{\text{Vol}\mathcal{G}} \delta(F_{z\bar{z}}) e^{-\frac{ik}{4\pi}} \int_{M_3} \text{tr} A d_t A \quad (4)$$

since the disk is simply connected, we can solve the constraint $F = 0$ and write any flat connection in terms of a globally defined gauge transformation

$$\begin{aligned} A &= \bar{g}^1 dg \\ d &= dz \partial_z + d\bar{z} \partial_{\bar{z}} \\ g &: D \rightarrow G \end{aligned} \quad (5)$$

since the gauge group consists of maps g ; $g(\partial D) = 1$, the gauge invariant information contained in A is parametrized by the value of g in (3) at $\partial D = S^1$. Substituting (5) into (4) we obtain the effective action for the gauge invariant degrees of freedom

$$S(g) = -\frac{k}{4\pi} \int_{\partial M_3} \text{Tr} \bar{g}^1 dg \bar{g}^1 d_t g - \frac{k}{12\pi} \int_{M_3} \text{Tr} (\bar{g}^1 dg)^3 \quad (6)$$

since the integrand of the last term in the right-hand side of (6) is a total derivative (on M_3), we see that $S(g)$ only depends on $g|_{\partial M_3}$ we see that the final result is the light-cone formulation of the WZW model. From Witten's analysis of the theory⁸ we know that the space of states associated to the disk is the basic representation of the loop algebra $SU(N)$ at level k , $SU(N)_k$. If $\varphi \in [0, 2\pi]$ is a co-ordinate for $S^1 = \partial D$, the pair (φ, t) represents the light-cone co-ordinates, and the field component $A_\varphi(\varphi, t)$ satisfies the defining relation for $SU(N)_k^8$, and it generates the transformations

$g \rightarrow v(\varphi)g$, with $v(\varphi) : S^1 \rightarrow SU(N)$. The next three-manifold M_3 shedding light on the spectrum of the CS theory is a disk with a puncture: $M_3 = \{D - P\} \times \mathbf{R}$. The puncture acts as a source for the theory and from the three-dimensional point of view it can be seen as a Wilson line running parallel to the time axis. Choosing this Wilson line in some representation λ of $SU(N)$, the space of flat connections on $\{D - P\}$ is constrained to have monodromy in the representation λ . Choosing P to lie at the origin of D for simplicity, one obtains the following effective action for the gauge invariant degrees of freedom¹⁶

$$\begin{aligned} S_\lambda &= -\frac{k}{4\pi} \int_{\partial M_3} \text{Tr} g^{-1} \partial_\varphi g \bar{g}^1 \partial_t g \\ &\quad - \frac{k}{12\pi} \int_{M_3} \text{Tr} (\bar{g}^1 dg)^3 \\ &\quad + \frac{1}{2\pi} \int_{\mathbf{R}} \text{tr} \lambda \bar{g}^1 \partial_t g \end{aligned} \quad (7)$$

This action is invariant under

$$g(\varphi, t) \rightarrow L(\varphi) g(\varphi, t) R^{-1}(t)$$

with $R(t)$ commuting with λ . Again $L(\varphi)$ is generated by $A(\varphi)$, however, the space of states is not the basic representation because of the monodromy constraint. It is not difficult to prove that λ is a weight and that if α is a root of $SU(N)$, the absence of global anomalies requires the periodicity $\lambda \simeq \lambda + k\alpha$. Since the Weyl group of $SU(N)$ is also a symmetry of the theory we conclude that the physical states are labelled by

$$\lambda \in \Lambda_W / k\Lambda_R \times W \quad (8)$$

where Λ_W (resp. Λ_R) is the weight (resp. root) lattice of $SU(N)$ and W is the Weyl group of $SU(N)$). Hence the allowed monodromies are in a one-to-one correspondence with the integrable highest weight vectors of the

$SU(N)_k$ Kac-Moody algebra and the space of states associated to $\{D - P\} \times \mathbf{R}$ with monodromy λ is given by the integrable Kac-Moody module of highest weight λ and level k , $\mathcal{H}_\lambda$. The physical space contains now an infinite (countable) number of states because the initial time manifold is non-compact. The space of states on the torus is now easy to obtain. One way of obtaining the toroidal states is to consider the three manifold M_3 to be a handlebody whose boundary is T_2 . This was done for example in Ref. 15. In this case the effective action describes only a finite number of degrees of freedom. Choosing a complex structure τ on the torus, the space of non-trivial flat connections is parametrized by

$$\begin{aligned} A &= \pi a \tau_2^{-1} d\bar{z} - \pi a^* \tau_2^{-1} dz \\ dz &= d\sigma_1 + \tau d\sigma_2 \quad 0 \leq \sigma_1, \sigma_2 < 1 \\ \tau &= \tau_1 + i\tau_2 \\ a &= a_i H^i \end{aligned} \quad (10)$$

where H^i is a basis for the Cartan subalgebra of $SU(N)$. The non-trivial gauge transformations correspond to shifts $a \rightarrow a + \tau m + n$ with $m, n \in k\Lambda_R$. The effective action now will be a purely mechanical action for the quantities $a_i(t), \bar{a}_i(t)$. The canonical commutation relations between $a_i, \bar{a}_i$ depend on how we construct the states $\Psi_\lambda(a)$ and the integration measure $d\mu(a, \bar{a})$. Rather than repeating the steps in Ref. 15, we can obtain the information from the previous results on $D \times \mathbf{R}$. The space of states on the torus is obtained as a set of holomorphic functions of a (satisfying the appropriate monodromy conditions on the cycles A and B of the torus) parametrizing a torus whose lattice is $J(T) \otimes \Lambda_R$, the tensor product of the jacobian lattice of T and the root lattice of $SU(N)$. Starting from $D \times \mathbf{R}$ or better from $D \times I$ (I a finite interval) we can obtain a handlebody by closing the interval I into S^1 , this is equivalent to imposing some twisted boundary conditions on I and summing over all states with these boundary conditions. To obtain the toroidal state we have to impose the ap-

propriate transition functions in gluing the two ends of I . Representing $I \times S^1$ as an annulus with complex co-ordinate z , the transition function parametrizing the complex structure on T^2 is $z \sim qz, q = \exp 2\pi i\tau$. If the monodromy around the circle S^1 (the A -circle on the torus) is given by λ , we can glue the top and bottom of the cylinder by imposing also monodromy around the B -cycle (the interval I with endpoints identified): $\exp 2\pi i \vec{a} \cdot \vec{H}$. Altogether, the λ state obtained now is

$$q^{-c/24} \text{tr}_{\mathcal{H}_\lambda} q^{L_0} e^{2\pi i \vec{a} \cdot \vec{H}} \equiv \chi_\lambda(a) \quad (11)$$

here L_0 is the light-cone hamiltonian of the WZW model and the trace is taken over the Kac-Moody module $\mathcal{H}_\lambda$. To summarize, (11) is obtained by gluing the top and bottom of the cylinder with the transition function $q^{L_0} \exp 2\pi i \vec{a} \cdot \vec{H}$ parametrizing the complex structure of T^2 and the B monodromy for the flat gauge connection on the torus. Equation (11) is nothing but the Kac-Moody character of $\mathcal{H}_\lambda$ for the Kac-Moody algebra $SU(N)_k$. A very elegant way of evaluating (11) is by means of the Atiyah-Bott fixed point formulas (see Ref. 24 for details). From the previous arguments we conclude that the space of states of the CS theory on a torus is finite dimensional, and a basis is labelled by the integrable level k representation of $SU(N)_k$, and given explicitly in terms of the Weyl-Kac formula

$$\chi_\lambda(a) = \frac{1}{\Pi(a, \tau)} \sum_{w \in W} \epsilon(w) \theta_{w(\lambda+\rho), k+g}(\vec{a}|\tau) \quad (12)$$

$\theta_{\lambda, k+g}$ is a level $k+g$ θ -function for the root lattice of G , $\epsilon(w)$ is the signature of the element w of the Weyl group represented by W . $\Pi(a, \tau)$ is the Weyl-Kac denominator (see for instance 24). Since $\theta_{\lambda, k+g}$ is a modular form of weight $k+g$ and Π a modular form of weight g , the character is a modular form of weight k (g is the dual Coxeter number of G , in the case of $SU(N)$, $g = N$ and in (12) ρ stands for the half sum of the positive roots). Notice that the numerator is odd under the action of

the Weyl group. Since the denominator is also odd, the character is Weyl invariant as it should be.

When one quantizes the CS on the torus in the operator formalism, the Gauss law constraint for the non-trivial gauge transformations allows us to write the states in two possible ways: in terms of characters (as in (12)) or in terms of Weyl-symmetric level k θ -functions. At first sight one might be puzzled by this ambiguity; however, under closer examination one realizes that the two sets are different bases of the same space. The space of Weyl-symmetric level k θ -functions is labelled by $\Lambda_W/k\Lambda_R \times W$, i.e., by the intersection of the Weyl chamber with the hyperplane $\lambda \cdot \theta \leq k$ where θ is the highest root and λ is the maximal weight of an integrable representation.

The Weyl-antisymmetric θ -functions of level $k + g$ are labelled by the points in the interior of the chamber $\Lambda_W/(k+g)\Lambda_R \times W$. It is not difficult to show that these two sets label the same set of highest weights (taking into account the shift by ρ in (10)). The relation between the basis of character and the one in terms of Weyl-symmetric θ -functions of level k is given by branching functions²⁴ and it is complicated. The argument we have presented starting with $D \times \mathbf{R}$ implies that the basis selected naturally by the Hamiltonian formalism is that of characters, and this is the basis we will adopt. Once this choice is made, the canonical commutation relation for $a_i, \bar{a}_i$ is fixed. Since the factor $\Pi^{-1}(a, \tau)$ is common to all the characters and there is only one metric making the θ -functions orthogonal, the measure in the space of states (characters) is:

$$\prod_{i=1}^r da_i da_i^* \Pi(a, \tau) \overline{\Pi(a, \tau)} e^{-(k+g)a \cdot a^* \tau_2^{-1}} \quad (13)$$

which implies

$$[a_i, \bar{a}_j] = -\frac{\tau_2}{\pi(k+g)} \delta_{ij} \quad (14)$$

The basis we have chosen is well suited also to obtain explicit formulae for the Verlinde operators²⁵. The character of the basic representation is $\chi_0(a)$. The Verlinde operators implement the transformation $\chi_0(a) \rightarrow \chi_\lambda(a)$;

$$W_\lambda^{(B)} \chi_0(a) = \chi_\lambda(a) \quad (15)$$

Using (10), (13), (14) it was shown in Ref. 20 that:

$$W_\lambda^{(B)} = \Pi^{-1}(a, \tau) (tr_\lambda e^{-\oint_B A}) \Pi(a, \tau) \quad (16)$$

for most purposes we can forget about the π -factors, then the Verlinde operators act as vertex operators in the space of Weyl-antisymmetric level $(k+g)$ θ -functions. It is also shown in Ref. 20 that the W_λ satisfy the fusion rules of the WZW model at level k^{26} . The trace in (16) is taken in the finite dimensional representation of G with highest weight λ . Once (16) is understood, it is clear the one can define Wilson line operators for any toroidal knot $pA + qB$, (p, q) coprime integers. Including framing dependence, the expectation values of the toroidal knot operators in the fundamental representation²⁰ reproduce the answer obtained using Jones' skein relation⁷. As a final check, one can compute the Verlinde operators acting on A -cycles:

$$W_\lambda^{(A)} \chi_0 = ch_\lambda\left(\frac{2\pi i \rho}{k+g}\right) \chi_0 \quad (17)$$

where ch_λ is the character of the irreducible representation λ of the finite dimensional group G . The coefficient of (17) is also known as the q -dimension of the representation λ .

It is important to notice that the arguments presented lead naturally to the celebrated shift $k \rightarrow k + g$ obtained originally by Witten in Ref. 4. To better understand this shift one should notice that there are two places where the integer k (the coefficient of the CS action) appears. We have an integer which determines the central extension of the $SU(N)$ current algebra generated by $A_\varphi(\varphi)$. There is another integer which determines the monodromy of the model. In other words, we

can write the expectation values of knot and link operators as polynomials of a variable q which takes the form $q = \exp 2\pi i/k'$. The two integers k and k' do not have to be the same and indeed, the analysis presented using first the quantization on $D \times \mathbf{R}$ and then evaluating the states and the knot operators imply at once that $k' = k + g$. Hence to understand the shift properly one has to be able to compare the central extension of the current algebra and the monodromy parameters. These two numbers are certainly gauge invariant, both of them are meaningful, and therefore it is also meaningful to ask what is the value of their difference. It is not sufficient to compute only one to determine the existence of the shift. Although these last remarks may seem a bit pedantic, in our opinion the whole confusion generated about the shift of $k \rightarrow k + g$ originates in the failure to understand the difference between the two parameters. If one admits that k must be an integer for topological reasons we conclude that the topological CS theory is parametrized by an integer. Different ways of defining the theory may lead to different values for the monodromy parameter, however, we stress again, the difference between the monodromy parameter k' and the central extension of the current algebra is a well-posed question depending only on the group, and should be the same regardless of how the theory is constructed. We believe that the Hamiltonian argument presented provides the conceptually cleanest proof that $k' - k = g$. In the next section we will briefly comment on the perturbative analysis of the CS theory and what is still needed to do in order to prove the same result. This concludes our review of the canonical quantization of CS theory.

3. Some remarks on the perturbative approach to CS theory

There is a very large literature on the perturbative analysis of the CS theory. Some years ago the CS theory

was studied in detail²⁷ because it provides a mechanism to generate a gauge invariant mass in three dimensions without the need to introduce Higgs fields. In these papers the action contains also the standard Yang-Mills Lagrangian. Although many things about the pure CS theory can be learned from those investigations, there are quite a number of features of the purely topological theory that require a close examination in their own right. The perturbative analysis of the theory has been undertaken by several groups and a number of interesting results have already been obtained. For instance, the authors of Ref. 22 have proved the finiteness of the theory to all orders of perturbation theory. In Ref. 10 integral representations generalizing Gauss' formula are derived for the coefficients of the Jones polynomials. Even though CS theory is finite, the ultra-violet finiteness for a given expectation value is achieved only after summing the contributions of different diagrams. Each of the diagrams contains generically divergences (especially those contributing to the two- and three-point functions) and therefore to compute any quantity it is necessary to choose some regularization and renormalization scheme. The simplest regulator is probably the one chosen in Ref. 10, however in spite of its computational simplicity it is not a manifestly gauge invariant regulator. This in particular means that the currents generating the Kac-Moody algebra will be conserved only after the cut-off is removed. Recall that the current algebra was defined on $D \times \mathbf{R}$ through the gauge transformations not vanishing at $\partial(D \times \mathbf{R})$. Due to the parity invariance of the regulator in Ref. 10, the monodromy parameter k' is equal to the coefficient k in front of the Chern-Simons action, then all the knot and link polynomials are expressed in terms of $q = \exp 2\pi i/k$, and as q -polynomials the answers agree with Jones' results. As explained in detail in the previous section, in the Hamiltonian approach one would expect to find the monodromy parameter to be $k + g$. However, as remarked towards the end of Sec-

tion 2, in CS theory there are two integers which play a role, and to check whether there is any disagreement one would need to compute the central extension of the Kac-Moody algebra. With a gauge non-invariant cut-off this is not easy and it has not been done yet. In Ref. 12, we introduced a (cumbersome) gauge invariant cut-off to guarantee that the gauge symmetry is preserved. Now the current algebra relations are satisfied for any value of the cut-off, however, the computations of monodromy parameters become rather complicated. In terms of the gauge non-invariant cut-off, the central term of the Kac-Moody algebra is difficult to compute while the monodromy parameter is easy. In the gauge invariant approach it is the other way around.

Since CS theory contains an ϵ -tensor in the action, it is not very clear how to use dimensional continuation. The naive extensions of $\epsilon^{\alpha\beta\gamma}$ from three to D dimensions produce an action which will not be gauge invariant. Thus we decided to work with higher derivatives supplemented with a one-loop Pauli-Villars regulator. As it is well known²⁸, the higher derivatives are sufficient to cut-off the primitive divergences on diagrams with more than one loop. For one-loop graphs and one-loop subdivergences in higher-loop graphs it is necessary to introduce a further regulator. As we will discuss presently, this generates a number of complications, some of which have been resolved. We now describe briefly our methods and results.

First one needs to introduce a regulator which gives finite answers for all graphs. It is in general not correct to consider cut-offs which only regularize combinations of graphs. Doing some standard power counting analysis, one finds that in order to regulate all primitive divergences beyond one loop one needs to add two extra terms to the CS action:

$$S_\Lambda(A) = \frac{1}{\Lambda} \int \text{tr} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2\Lambda^3} \int \text{Tr} D_\mu F_{\alpha\beta} D^\mu F^{\alpha\beta} \quad (18)$$

To regularize one-loop graphs we introduce Pauli-Villars regulators. This can be done in such a way that the BRST invariance is preserved. Introducing fields $A_\mu^{(j)}, c^{(i)}, \bar{c}^{(i)}, i, j = 1, \dots, I$ we consider the determinants

$$\begin{aligned} \det \mathcal{A}_j &= \int [\mathcal{D}A_\mu^{(j)}] \exp(i \int \text{Tr} [\epsilon^{\mu\nu\rho} A_\mu^{(j)} D_\nu A_\rho^{(j)} \\ &\quad + M_j A_\mu^{(j)} A^{\mu(j)} \\ &\quad - \frac{1}{\Lambda^3} A_\mu^{(j)} (\delta^{\mu\nu} D_\rho D^\rho - D^\mu D^\nu + 2g F^{\mu\nu}) A_\nu^{(j)}]) \quad (19) \\ \det C_i &= \int \mathcal{D}\bar{c}^{(i)} \mathcal{D}c^{(i)} \exp(i \int 2\text{Tr} [\bar{c}^{(i)} D_\mu D^\mu c^{(i)} \\ &\quad - m_i^2 \bar{c}^{(i)} c^{(i)}]) \end{aligned}$$

Then the generating functional for the perturbative series is:

$$\begin{aligned} Z[J_\mu, \bar{\eta}, \eta] &= \int (\infty A_\mu \mathcal{D}c \mathcal{D}\bar{c}) \\ &\quad e^{i(S(A_\mu, c, \bar{c}) + S_\Lambda(A_\mu))} \quad (20) \\ &\quad \prod_j^{-b_j/2} \det \mathcal{A}_j \prod_i^{-c_i} \det C_i \end{aligned}$$

where $J_\mu, \bar{\eta}, \eta$ are respectively sources for the fields $A_\mu, c, \bar{c}$, and $S(A_\mu, c, \bar{c})$ is the full gauge fixed action including ghosts. It is furthermore quite convenient to work in the Landau gauge to avoid the infra-red problem. After some computations at one loop (see Ref. 12 for details) one finds that at the only correction to the two- and three-point function is equal to $(g/4\pi)$ times the original action; therefore the one-loop result we find is consistent with the results of Section 2. However, this is not yet enough. To have a watertight argument it is required to compute in perturbation theory the expectation value of a knot or link operator up to two loops in order to fully disentangle the $1/k$ and $1/N$ terms. Clearly it is not needed to work with complicated knot, it suffices to work with the unknot in the fundamental representation. A very elegant argument due to Witten⁶ using surgery allows one to compute this expectation value in terms of the matrix S_{ij} which implements the modular transformation $\tau \rightarrow -1/\tau$ in the $SU(N)$ WZW model at level k ; and in this case the monodromy pa-

parameter is $k' = k + N$. In order to reproduce the result and compare with previous work we have to extend our gauge invariant prescription beyond one loop. All preliminary results show that our two-loop computations agree with Witten's result²⁹. The computations are rather complicated in our case for a very simple reason: in the two-loop case we have to worry about divergent one-loop subgraphs which contain Pauli-Villars regulator fields in the external legs. The action (20) does not suffice to render these subdivergences finite and therefore we need to improve the gauge invariant regularization prescription. After a cursory examination one learns that in order to regulate all one-loop subdivergences for any graph with more than one-loop one has to introduce higher generations of Pauli-Villars fields, in fact an infinite number of such generations. If we say that the Pauli-Villars fields in (20) are of first generation, we need to introduce Pauli-Villars fields for them, and these are the second generation Pauli-Villars fields and so on, for the n^{th} generation we need to include fields in the $(n + 1)^{\text{st}}$ generation. This is the price to pay in our procedure in order to maintain gauge invariance. The computation can be furthermore carried out with the standard and non-standard framings.

What remains to be done so far is to check that with our prescription there is no two-loop correction to the gauge field propagator. Once this is achieved it is a matter of simple combinatorics to check that Witten's results follow from the one-loop renormalization results. The computation is under way now and the results will be reported elsewhere²⁹.

4. Conclusions

In this note we have presented a short review of the Hamiltonian quantization of the topological Chern-Simons theory paying particular attention to the question related to the shift $k \rightarrow k + g$. The conclusion is

that in Chern-Simons theory there are two relevant integers: the central charge of the G -current algebra and the monodromy parameter. It is meaningful to compute both integers in terms of the "bare" coupling k appearing as the coefficient of the Chern-Simons action, and it is also meaningful to compute their difference. Depending on which procedure one chooses to define the theory one of the integers is easy to compute while the other is rather complicated. Nevertheless, the Hamiltonian analysis presented in Section 2 provides ample evidence that the difference between the two integers is the dual Coxeter number of the gauge group G .

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