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Gravitational Dynamics with Lorentz Chern-Simons Terms

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Abstract

We consider corrections to Einstein gravity arising from the inclusion of Lorentz Chern-Simons terms, as well as Gauss-Bonnet terms, in a theory containing dilatons and Kalb-Ramond axions as in string theory. We show that all four-dimensional spacetimes conformal to a spacetime with a maximally symmetric two-dimensional subspace have exact Lorentz Chern-Simons terms, which do not contribute to gravitational equations of motion or Bianchi identities. This greatly simplifies the equations of motion for spacetimes of this form, such as Robertson-Walker and Schwarzschild. On the contrary, we show that for both the Kerr metric, and the Bianchi IX metric, Lorentz Chern-Simons terms cannot be neglected and lead to a non-trivial modification of the gravitational equations of motion, as well as the Bianchi identity for the Kalb-Ramond Field strength, $dH = \text{tr } R \wedge R \neq 0$.

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1. Introduction

In the seventy five years since its inception, Einstein gravity⁽¹⁾ has provided a successful classical phenomenology of the gravitational interaction. Attempts to unify it with the other interactions, and to provide a consistent quantum theory of gravity, suggest significant extensions of Einstein's theory. In particular superstring theory⁽²⁾, which at present represents the only developed proposal for a unified, quantized, theory of gravity, provides an intricately "pregeometric" formulation of gravity, which reduces to Einstein's theory only in a classical weak field limit. Corrections to the Einstein description arising in string theory may be described by an effective Lagrangian for the light fields of the theory, with the leading Einstein term supplemented by a series of correction terms. It is the purpose of this work to examine the effects of some of these corrections, in gravitational theories with the long range massless field content indicated by superstring theory.

In particular, we wish to focus on the role of Lorentz Chern-Simons terms in gravitational dynamics. These terms were already introduced (3,4) and discussed in the formulation of ten-dimensional supergravity and its coupling to a U(1) gauge theory. With the coupling of ten-dimensional supergravity to ten-dimensional Yang-Mills gauge theory⁽⁵⁾, gauge Chern-Simons terms also appeared to accompany the Lorentz ones. Both these Chern-Simons terms play a key role in the Green-Schwarz anomaly cancellation mechanism⁽⁶⁾, and as such must play a role in any anomaly-free ten-dimensional theory with low energy field content including supergravity-super-Yang-Mills.

Superstring theories, whose low energy field content contains supergravity and super-Yang-Mills, are generally anomaly-free as a consequence of world sheet modular invariance⁽⁷⁾. Also in their ten dimensional solutions they incorporate supergravity-super-Yang-Mills massless fields. As such they of necessity incorporate Lorentz and Yang-Mills Chern-Simons terms in the effective action describing their massless modes, a property also exhibited by string constructions directly in four dimensions.

The necessity for Chern-Simons terms to obtain the desired anomaly cancellations in supergravity and in string theory warrants their inclusion in searches for solutions to the complete set of equations of motion. In particular we will be interested in Schwarzschild/Kerr type geometries as well as Robertson-Walker/Bianchi type cosmological solutions. The inclusion of the three-form $H_{\mu\nu\lambda}$ in Schwarzschild spacetimes was considered in ref.8, without Lorentz-Chern-Simons corrections. The role of the three-form in a cosmological context was also considered in ref.9. It is the purpose of this paper to consider the conditions under which the Lorentz Chern-Simons three-form, ω_L , has an effect on the equations of motion. To this end, we will discuss the full set of equations of motion for the gravitational sector in the next section. In section III, we will show explicitly that all spacetimes which admit

a maximally symmetric two-dimensional subspace, and those which are related to the latter by a conformal transformation have ω_L exact, and hence vanishing Lorentz Chern-Simons contributions to the equations of motion. Thus, the considerations of the type in ref.[8,9], are actually more general than originally claimed. In section IV, we give two examples, Kerr and Bianchi IX, where the role of ω_L cannot be neglected. Our summary and conclusions will be presented in section V.

2. Equations of Motion and Duality Transformations.

The massless modes present in the spectrum of string theory constructions (in $D=10$ or directly in $D=4$) include the graviton multiplet plus possible gauge and matter sectors. There are also towers of massive excitations in the theory, with the mass scale being set by the string tension, which we would expect to be of order (the square of) the Planck scale. The propagation and interactions of the massless modes may, at energies below the string tension scale, be represented by an effective Lagrangian for these modes with the heavy degrees of freedom integrated out. There are several equivalent approaches to the determination of this effective action: one may match amplitudes computed with it to those directly computed in the full string theory⁽¹⁰⁾; alternatively one may choose it to yield equations of motion for background fields equivalent to those derived from conformal invariance⁽¹¹⁾ (BRST nilpotence⁽¹²⁾) for consistent string propagation in that background.

Proceeding either way, we may derive an effective action for the massless modes of ten-dimensional heterotic superstrings. The massless modes include $(M,N = 0, \dots, 9)$: a gravitation g_{MN} , a dilaton ϕ , a form field B_{MN} , Yang-Mills gauge fields A_M^a , fermionic superpartners, plus fields from the matter multiplets. As well as the usual Yang-Mills field strength, we form a field strength for the form field

$$H = dB + \omega_L^{(D)} - \omega_Y^{(D)} \quad (2.1)$$

where ω_L , ω_Y are the Chern-Simons forms for the ten-dimensional Lorentz spin connection, and Yang-Mills gauge connection respectively

$$\omega_L^{(D)} = \text{tr} \left(\omega \wedge d\omega + \frac{2}{3} \omega \wedge \omega \wedge \omega \right) \quad (2.2)$$

$$\omega_Y^{(D)} = \pi (A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \quad (2.3)$$

with $\omega_{\mu\nu} = -e_\mu^\alpha e_{\alpha\nu}$ the spin connection with e_μ^α the vielbein. The field strength H thus satisfies the Bianchi identity

$$dH^{(D)} = \pi R^{(D)} \wedge R^{(D)} - \pi F^{(D)} \wedge F^{(D)} \quad (2.4)$$

For our purposes here, we will concentrate on the massless modes of the gravitational sector, namely, the graviton, the dilaton and the two-form whose field strength contains the Lorentz Chern-Simons (LCS) corrections.

The dynamics of the gravitational modes is derivable from an effective action given by^(10,11,12)

$$S = \int d^D x \sqrt{g} \left\{ \frac{1}{2\kappa_D^2} R - \tilde{g}(\varphi) H_{\mu\nu\lambda} H^{\mu\nu\lambda} - \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - V(\varphi) \right. \\ \left. + \left(\frac{\alpha' f(\varphi)}{16\kappa_D^2} \tilde{R}^2 + \dots \right) \right\} \quad (2.5)$$

where in the context of the heterotic string theory,

$$\tilde{g}(\varphi) = e^{-4\kappa\varphi/(D-2)} \quad (2.6a)$$

$$f(\varphi) = e^{-2\kappa\varphi/(D-2)} \quad (2.6b)$$

and $\kappa_D^2 = 8\pi G_D$ the D-dimensional gravitational constant (for the heterotic string, $D \leq 10$).

We have, in addition, allowed for the possibility of dilaton self-interactions through $V(\varphi)$, and for the presence of higher derivative terms in S , the first of which is shown explicitly by the Gauss-Bonnet term $\tilde{R}^2$,

$$\tilde{R}^2 = R_{MNPQ} R^{MNPQ} - 4 R_{MN} R^{MN} + R^2 \quad (2.7)$$

and α' is the string tension. The Gauss-Bonnet combination (of R^2 terms) guarantees the absence of ghosts⁽¹³⁾ and appears naturally in supersymmetric strings in the supercompletion of

the multiplet with the LCS terms⁽¹⁴⁾. The gravitational effect of the Gauss-Bonnet term was considered in detail in refs.(15,16), and with the inclusion of dilatons in refs. (17-20).

For string solutions of physical interest, we wish to restrict to four physical spacetime dimensions. Four-dimensional string theories are most generally realized in terms of world sheet superconformal theories with specified central charge and modular invariance⁽²⁾. In many cases these may be interpreted as compactifications of ten-dimensional string theories, and this point of view will be useful in our discussion of effective actions.

If our ten-dimensional string theory is considered to be compactified on $M_4 \otimes K$ where K is a compact six-manifold (usually assumed to have Planck scale) then in our four-dimensional effective theory we need only to retain the massless modes for low energy dynamics. Also ten-dimensional tensors will produce, as well as the corresponding $D = 4$ tensor, four-dimensional tensors of reduced rank, if some of the tensor indices reside in the internal directions (Kaluza-Klein mechanism). Known compactifications on manifolds of $SU(3)$ holonomy generically do not have isometries, so the only massless four-dimensional modes from the graviton multiplet will be the four-dimensional graviton $g_{\mu\nu}$ and the dilaton. From the gauge sector we obtain four-dimensional gauge multiplets plus some chiral matter generations. Finally, in reduction of the form field $B_{\mu\nu}$, as well as the four-dimensional two form $B_{\mu\nu}$, we may possibly get extra $U(1)$ gauge bosons $B_{\mu m}$ or (pseudo) scalars B_{mn} ($\mu, \nu = 0, \dots, 3$; $m, n = 4, \dots, 9$). The massless two-form pseudoscalars B_{mn} represent Harmonic two-forms on the compactification manifold K , and in number are given by the second Betti number of K . They inherit axion-like couplings in four dimensions from the Green-Schwarz anomaly cancellation couplings in ten dimensions, and are referred to as "model dependent" axions. They may, or may not, act as four-dimensional axions, depending on whether their Peccei-Quinn symmetry is strongly violated by world sheet instanton effects. In "four-dimensional string" constructions there are often analogous modes parameterizing, with other scalars, Kähler field manifolds. When these modes exist, they have properties like the "model independent" axion arising from the four-dimensional two-form $B_{\mu\nu}$ which we consider further below.

Now retaining only the gravitational modes that necessarily appear in the four-dimensional truncation of the theory, their dynamics will be described by a four-dimensional truncation of the effective action (2.5). It suffices to take $D = 4$ and $M, N, \dots \rightarrow \mu, \nu, \dots$

The variation of the four-dimensional action (2.5) leads to the following set of equations of motion (through R^2 only) :

$$H_{\mu\nu\lambda} = B_{[\mu\nu,\lambda]} \text{ or } H = dB \quad (2.12)$$

so that the Bianchi identity becomes

$$H_{[\mu\nu\lambda,\sigma]} = 0 \quad (2.13)$$

Furthermore, in this case (or when ω_L is exact, $\omega_L = d\beta$ and so can be removed by a transformation on B), the last term in eq.(2.8), $4(\tilde{g}(\varphi) H^{\lambda\mu\alpha} R^{\sigma\nu}_{\alpha\lambda})_{,\sigma}$, is also absent, as this term is due to the direct variation of ω_L with respect to $g^{\mu\nu}$. Thus the direct coupling of the Chern-Simons axion to gravity is contained in this term. In the next section, we give a wide class of sufficient conditions for exact ω_L .

To analyze the dynamical content of the form field $B_{\mu\nu}$, consider its equation of motion (2.10). Let us define the vector Y_ρ through a duality transformation,

$$\frac{\epsilon^{\mu\nu\lambda\rho}}{\sqrt{g}} Y_\rho = \tilde{g}(\varphi) H^{\mu\nu\lambda} \quad (2.14)$$

The equation of motion becomes (in form notation)

$$dY = 0 \quad (2.15)$$

which implies that locally $Y = db$ for some spinless field b . In terms of b , the Bianchi identity (2.11) becomes

$$\begin{aligned} \epsilon^{\mu\nu\lambda\sigma} H_{\mu\nu\lambda,\sigma} &= -6\sqrt{g} \frac{\square b}{\tilde{g}(\varphi)} + \frac{6\sqrt{g}}{\tilde{g}(\varphi)} \frac{\partial^\rho b}{\partial \rho} (\ln \tilde{g}(\varphi)) \\ &= \frac{1}{4} R_{\alpha\beta\mu\nu} R_{\lambda\sigma}{}^{\beta\alpha} \epsilon^{\mu\nu\lambda\sigma} \end{aligned} \quad (2.16)$$

or after a rescaling of b , $\partial_\mu b = -\tilde{g}(\varphi) \partial_\mu a/6$, we have

$$\square a = \frac{1}{4} R_{\alpha\beta\mu\nu} R_{\lambda\sigma}{}^{\beta\alpha} \epsilon^{\mu\nu\lambda\sigma} \sqrt{g} \quad (2.17)$$

showing that (locally) the two form $B_{\mu\nu}$ represents a pseudo-scalar degree of freedom with axion couplings.

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = \frac{\alpha'}{8} f(\varphi) \left[\frac{1}{2} g^{\mu\nu} \tilde{R}^2 - 2R R^{\mu\nu} - 2R^{\lambda\rho\sigma} R^{\nu\lambda\rho\sigma} \right.$$

$$\left. + 4 R^{\mu\alpha} R_{\alpha}{}^{\nu} + 4 R_{\alpha\beta} R^{\mu\alpha\nu\beta} \right]$$

$$- \frac{\alpha'}{4} \frac{df}{d\varphi} \left[2\varphi_{,\alpha}{}^{\nu} R_{\alpha\mu} + 2\varphi_{,\alpha}{}^{\mu} R_{\alpha\nu} - 2g^{\mu\nu} \varphi_{,\alpha\lambda} R^{\alpha\lambda} \right.$$

$$\left. - 2R^{\mu\nu} \varphi_{,\lambda}{}^{\lambda} + g^{\mu\nu} R \varphi_{,\lambda}{}^{\lambda} - R \varphi_{,\mu}{}^{\mu} - 2\varphi_{,\lambda\alpha} R^{\lambda\mu\alpha} \right]$$

$$- \frac{\alpha'}{4} \frac{d^2 f}{d\varphi^2} \left[2\varphi_{,\alpha} \varphi_{,\nu} R^{\alpha\mu} + 2\varphi_{,\alpha} \varphi_{,\mu} R^{\alpha\nu} - 2g^{\mu\nu} \varphi_{,\alpha} \varphi_{,\lambda} R^{\alpha\lambda} \right.$$

$$\left. - 2R^{\mu\nu} \varphi_{,\lambda} \varphi_{,\lambda} + g^{\mu\nu} R \varphi_{,\lambda} \varphi_{,\lambda} - R \varphi_{,\mu}{}^{\mu} \varphi_{,\nu} - 2\varphi_{,\lambda} \varphi_{,\alpha} R^{\lambda\mu\alpha} \right]$$

$$+ \kappa^2 \left[\partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} g^{\mu\nu} \partial_\lambda \varphi \partial^\lambda \varphi - g^{\mu\nu} V(\varphi) \right]$$

$$+ \kappa^2 \left[\phi \tilde{g}(\varphi) H^{\mu\lambda\rho} H^{\nu\lambda\rho} - g^{\mu\nu} \tilde{g}(\varphi) H^{\lambda\rho\sigma} H_{\lambda\rho\sigma} \right.$$

$$\left. + 4(\tilde{g}(\varphi) H^{\lambda\mu\alpha} R^{\sigma\nu}_{\alpha\lambda})_{,\sigma} \right] \quad (2.8)$$

$$\varphi_{,\lambda}{}^{\lambda} = \frac{\partial V}{\partial \varphi} - \frac{\alpha'}{16} \frac{df}{\kappa^2 d\varphi} \tilde{R}^2 + \frac{d\tilde{g}}{d\varphi} H^{\lambda\rho\sigma} H_{\lambda\rho\sigma} \quad (2.9)$$

$$[\tilde{g}(\varphi) H^{\mu\nu\lambda}]_{;\lambda} = 0 \quad (2.10)$$

In addition, one has the Bianchi identity

$$dH = \text{tr}(R \wedge R) \quad (2.11)$$

Our conventions are such that $\text{sign}(g) = (+, -, -, -)$ and $R^{\mu}{}_{\nu\lambda\sigma} = \Gamma^{\mu}{}_{\nu\sigma\lambda} - \dots$

In four-dimensional gravity, without the dilaton or three-form, the Gauss-Bonnet corrections may be neglected; in this context however, they are nontrivial. The equations of motion (2.8)-(2.11) agree with those given by Wu and Wang⁽¹⁸⁾ with the exception of the coefficient of the last term in eq.(2.8), and the terms proportional to $\frac{df}{d\varphi}$ and $\frac{d^2 f}{d\varphi^2}$ in that same equation. These latter terms were first correctly given by Wetterich⁽¹⁶⁾. With the appropriate field redefinitions and various omissions (such as H , ω_L or $\tilde{R}^2$) these equations also agree with those in refs. (11, 15-20). We are not aware of another source for the complete set of equations of motion in the gravitational sector.

In the absence of the LCS three-form,

3. The Vanishing of the Lorentz Chern-Simons Corrections.

In this section we examine further the LCS couplings of the axion to gravity. We will find general conditions on the metric for which the LCS corrections vanish. In this way we extend the results of previous analyses of superstring axions, which have assumed a vanishing LCS coupling at the outset. We shall show that these results hold in the full theory of the string effective action. In addition, we examine under what conditions the LCS coupling does not vanish and what are the physical consequences in this case. This will be illustrated with examples in the next section.

The main result of this section is the proof of the following theorem :

"For all four-dimensional spacetimes which are conformally related to a spacetime possessing a maximally symmetric two-dimensional subspace, the Lorentz Chern-Simons terms in the equations of motion vanish identically." Before proving this, let us examine the spacetimes referred to in the theorem.

The most general form for the metric of a spacetime which is conformally related to another possessing a maximally symmetric two-dimensional subspace is⁽²¹⁾

$$ds^2 = h(v,u) \left\{ g_{ab}(v) dv^a dv^b - f(v) \left[d\vec{u}^2 + \frac{K(\vec{u} \cdot d\vec{u})^2}{1 - K\vec{u}^2} \right] \right\} \quad (3.1)$$

where u^ℓ ($\ell=1,2$) coordinatize the subspace. The other two coordinates of the spacetime are v^a ($a=1,2$). The curvature parameter of the subspace is denoted K where $K = 0, \pm 1$ are the allowed values. This parameter is not to be confused with the more familiar curvature parameter of the Robertson-Walker metric; for such metrics our $K = +1$.

The general form of the metric (3.1) encompasses a large variety of spacetimes commonly encountered in general relativity. To name a few, it contains as special cases the following classes :

- 1) spaces of constant curvature (e.g. De Sitter space)
- 2) spaces with a 3-D maximally symmetric subspace (e.g. Robertson-Walker spaces)
- 3) isotropic spaces (e.g. Reissner-Nordstrom and Schwarzschild)
- 4) conformally flat spaces
- 5) Bianchi I with Kasner metric $p_1 = p_2 = \frac{1}{3}$ $p_3 = -\frac{2}{3}$.

The theorem states that for all such spacetimes the LCS terms vanish, regardless of the topology of the two-dimensional subspace. For example, some of the maximally symmetric subspaces are S^2 (compact) and others are $\mathbb{R}^2$ (noncompact). On the other hand, the geometrical symmetries of the spacetime can have a decisive effect on the LCS coupling.

Let us now proceed to prove the theorem. Recall that there were two terms in the equations of motion which arose from the LCS coupling in the action : First, there was the Hirzebruch signature density $\text{tr } R \wedge R$ acting as a source for the field strength H from (2.16), and (2.17); Secondly, there was the variational LCS contribution to the axion energy-momentum tensor of the form $(\tilde{g}(\phi) H^\lambda{}_\alpha (v^\nu \nabla^\alpha \lambda)_\nu) \cdot g$, which is symmetrized in μ and ν due to the variation $\delta g_{\mu\nu}$. We could prove the theorem, by looking at each of these terms separately. However, there is a much simpler way to arrive at our results, by examining the LCS form ω_L directly.

Before we actually consider the ω_L term, we note that since $g_{ab}(v)$ in eq (3.1) depends only on v , it can be diagonalized by an integrable coordinate transformation because the v subspace is two-dimensional. So, we can use

$$ds^2 = h(v,u) \left\{ A dv_1^2 - B dv_2^2 - f(v) \left(d\vec{u}^2 + \frac{K(\vec{u} \cdot d\vec{u})^2}{1 - K\vec{u}^2} \right) \right\} \quad (3.2)$$

right from the start.

Note then that $h(v,u)$ can be looked at as a conformal scale relating ds^2 with $d\vec{s}^2$, which is maximally symmetric in the u piece :

$$d\vec{s}^2 = A dv_1^2 - B dv_2^2 - f(v) \left(d\vec{u}^2 + \frac{K(\vec{u} \cdot d\vec{u})^2}{1 - K\vec{u}^2} \right) \quad (3.3)$$

Then, quite generally, whenever

$$ds^2 = e^{2\Omega} e_a \otimes e^a \quad (3.4)$$

where e_a are vielbeins for the metric ds^2 , we can evaluate the LCS term for the conformal metric (3.4) in terms of the one for $d\vec{s}^2$. It is given by

$$\omega_L = \bar{\omega}_L + 2d \{ (\partial_\mu \Omega) e_a^\mu d e^a \} \quad (3.5)$$

i.e., conformal transformations change the LCS term only by adding to it an exact form and thus ω_L is invariant in cohomology under conformal transformations.

The next step is, naturally, to evaluate $\bar{\omega}_L$ using the form invariant metric (3.3). However, yet another conformal transformation is useful here. Namely, if $f(v)$ is factored out, the metric (3.2) can be rewritten as

$$ds^2 = h(v, u) f(v) \left\{ \frac{A}{f} dv_1^2 - \frac{B}{f} dv_2^2 - \left(\frac{d\vec{u}}{du} + \frac{K(\vec{u} \cdot d\vec{u})}{1 - K\vec{u}^2} \right)^2 \right\} \quad (3.6)$$

where now the form invariant metric ds^2 completely separates v and u coordinates. It is then easy to see that its LCS form vanishes. The vielbeins can be written as

$$e^0 = \sqrt{\frac{A}{f}} dv_1$$

$$e^1 = \sqrt{\frac{B}{f}} dv_2$$

$$e^2 = \frac{-u_2}{\sqrt{u_1^2 + u_2^2}} du_1 + \frac{u_1}{\sqrt{u_1^2 + u_2^2}} du_2$$

$$e^3 = \sqrt{\frac{1 + K(u_1^2 + u_2^2)}{u_1^2 + u_2^2}} u_1 du_1 + \sqrt{\frac{1 + K(u_1^2 + u_2^2)}{u_1^2 + u_2^2}} u_2 du_2 \quad (3.7)$$

From eq.(3.7), the following can be deduced for the connection one-forms:

$$\omega_{01} = \alpha dv_1 + \beta dv_2$$

$$\omega_{23} = \gamma du_1 + \delta du_2$$

$$\omega_{02} = \omega_{03} = \omega_{12} = \omega_{13} = 0$$

where α and β are some functions depending only on v and γ and δ only on u . Therefore,

$$d\omega_{01} = \sigma(v) dv_1 \wedge dv_2$$

$$d\omega_{23} = \rho(v) du_1 \wedge du_2 \quad (3.9)$$

The exact forms of the functions α , β , γ , δ , σ and ρ are irrelevant for the proof. It is straightforward to verify, using eq. (3.8) and (3.9) that

$$\bar{\omega}_\lambda = \text{Tr} \left(\omega \wedge d\omega + \frac{2}{3} \omega \wedge \omega \wedge \omega \right) = 0 \quad (3.10)$$

The reason for this really is the fact that the maximal symmetry condition on a two-dimensional subspace followed with a conformal transformation unfolded the structure of the spacetime into a product of two two-dimensional manifolds, with the cotangent space a global sum of the two. So, then, the connection forms do not mix the v and u vielbeins, and by two-dimensionality of both subspaces one can not build a nontrivial three-form without mixing the connection forms, which is not allowed in constructing $\bar{\omega}_\lambda$.

Thus, for the cases considered, we conclude that ω_λ is zero in cohomology, i.e.,

$$\omega_\lambda = d \{ (\partial_\mu \ln h(v, u)) f(v) \} \epsilon_\lambda^\mu d\epsilon^a \} \quad (3.11)$$

We then go back to where the Chern-Simons form first appeared, in the definition of the axion field strength :

$$H = dB + \omega_\lambda \quad (3.12)$$

Because for the cases considered, the Chern-Simons form is exact, its whole effect on the axion field is shifting its potential B :

$$H = dB'$$

$$B' = B + \beta \quad (3.13)$$

where $\omega_\lambda = d\beta$ (eq.(3.11)). Thus the theory is effectively the same as if ω_λ were zero from the beginning, and so the Chern-Simons terms in the equations of motion can be ignored. It is indeed trivial to verify that the Hirzebruch signature density vanishes here, since it is the exterior derivative of the Chern-Simons term, which in the cases considered is an exact form. Direct evaluation of the $(\tilde{g}(\varphi) H^4 \alpha(uR^v) \sigma_\alpha)_{, \sigma}$ term to show it vanishes is lengthy and tedious and will not be repeated here, though we repeat that this term is a direct consequence of the variation $\delta\omega_\lambda / \delta g^{\mu\nu}$, and hence its effect must vanish in spacetimes of the kind considered in our theorem.

Having proven the theorem, we note that some examples of spacetimes with nontrivial Chern-Simons terms may be found by looking at spacetimes not fulfilling the requirements of the theorem. Specifically, we find that the following two cases lead to nontrivial Chern-Simons corrections :

a) Bianchi IX

b) Kerr.

We discuss these in more detail in the following section.

4. Spacetimes with Nontrivial Lorentz Cherns-Simons Terms

Despite the very large class of spacetime geometries in which ω_L is exact, and hence the LCS contributions to the equations of motion vanish, it is tedious but straightforward to find examples in which this is not true. We begin first with the Kerr geometry. The Kerr metric can be written in the form^(1.22)

$$ds^2 = \left(\frac{r^2 + a^2 \cos^2 \theta - 2GMr}{r^2 + a^2 \cos^2 \theta} \right) dt^2 - \left(\frac{r^2 + a^2 \cos^2 \theta}{r^2 + a^2 - 2GMr} \right) dr^2 - (r^2 + a^2 \cos^2 \theta) d\theta^2 - \frac{[(r^2 + a^2)^2 - a^2(r^2 + a^2 - 2GMr)] \sin^2 \theta}{r^2 + a^2 \cos^2 \theta} d\varphi^2 \quad (4.1)$$

$$- \frac{4GMa \sin^2 \theta r}{r^2 + a^2 \cos^2 \theta} dt d\varphi$$

which reduces to the simpler form,

$$ds^2 = + \left(1 - \frac{2GM}{r} \right) dt^2 - \left(1 - \frac{2GM}{r} \right)^{-1} dr^2$$

$$- r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) - \frac{4GMa \sin^2 \theta}{r} dt d\varphi \quad (4.2)$$

when we consider the limit of small rotation, $a \ll GM$.

Using eq.(4.2), one can calculate directly the dual of the Hirzebruch density $\text{tr } R \wedge R$. Our result is, to first order in a :

$$\frac{1}{4!} \epsilon^{\mu\nu\lambda\sigma} R_{\alpha\beta\mu\nu} R_{\lambda\sigma}{}^{\beta\alpha} = -48 G^2 M^2 a \cos \theta \sin \theta / r^5 \quad (4.3)$$

Thus for realistic (non-stationary) black-holes, one is not permitted to take $dH = 0$, instead there is already an axion source term from eq.(4.3).

A non-vanishing Hirzebruch signature density also occurs in various cosmological models. A typical class are those describing homogeneous, anisotropic universes. There are nine of these Bianchi-type models and their metrics can be written in the form⁽¹⁾,

$$ds^2 = + dt \otimes dt - h_{ij}(t) \sigma^i \otimes \sigma^j \quad (4.4)$$

The one-forms $\{\sigma^i\}$ are a homogeneous basis for the spatial hypersurfaces. The anisotropy is contained in the three-metric h_{ij} , which is a function of only the time parameter by virtue of the homogeneity. The basis forms satisfy the algebra,

$$d\sigma^i = \frac{1}{2} C_{jk}^i \sigma^j \wedge \sigma^k \quad (4.5)$$

where the C_{jk}^i are the commutation coefficients. There are nine independent choices for these and the models are classified as Bianchi I-IX.

The most efficient method of computing the Hirzebruch signature density for the Bianchi models is to first evaluate the connection one-forms ω_A^B and then the Chern-Simons three-form. Due to the fact that the three-metric in (4.4) is only a function of time, and that the commutation coefficients in (4.5) are just constants, the expression

$$\text{tr } (R \wedge R) = \text{tr } (d\omega \wedge d\omega) + \frac{2}{3} \text{tr } (\omega \wedge \omega \wedge \omega) \quad (4.6)$$

is straightforward to evaluate since the number of intermediate expressions is small. The connection one-forms are readily obtained by rewriting the metric, as,

$$h_{ij}(t) = H_{ij}^a(t) H_{ab}^b(t) \delta_{ab} \quad (4.7)$$

and assuming that the matrices H_{ij}^a are diagonal (so-called diagonal Bianchi models),

$$\omega_0^i = H_{ij}^a \dot{H}_{ij}^a \sigma^j, \quad \omega_{ij} = \frac{1}{2} (C_{kij} - C_{ijk} - C_{jki}) \sigma^k \quad (4.8)$$

in the basis $\{\sigma^i\}$. The dot denotes differentiation with respect to time and the matrix H_{ij}^a is the inverse of H_{ij}^a , i.e. $H_{ij}^a H_{jk}^a = \delta_{ik}$. We lower and raise the indices with h_{ij} and its inverse h^{ij} respectively. In particular we have $C_{ijk} = h_{im} C^m{}_{jk}$.

The case where the coefficients C_{jk}^i vanish is Bianchi I. It is easily seen from (4.8) that (4.6) must vanish. The Hirzebruch density for the other diagonal Bianchi models can be calculated given their commutation coefficients. Below we shall calculate it for Bianchi IX, as this is case most widely discussed. The coefficients for Bianchi IX are $C_{ijk} = \epsilon_{ijk}$, the usual

anticommuting symbol in three dimensions. The basis one-forms may be written in terms of the Euler angles of the rotation group,

$$\begin{aligned}\sigma^1 &= \cos \psi \, d\theta + \sin \psi \, \sin \theta \, d\phi \\ \sigma^2 &= \sin \psi \, d\theta - \sin \psi \, \sin \theta \, d\phi \\ \sigma^3 &= d\psi + \cos \theta \, d\phi\end{aligned}\quad (4.9)$$

The structure equations can be easily verified in this representation of the basis. Let us denote the volume element of the three-sphere as $\Omega_3 = \sigma^1 \wedge \sigma^2 \wedge \sigma^3$. The total volume of the three sphere is $16\pi^2$ in this basis. The actual volume element of the Bianchi IX three space is $\sqrt{h}\Omega_3$, and of the four space is $\sqrt{h} \, dt \wedge \Omega_3$.

Given the connection (4.8) we find that the Hirzebruch density (4.6) can be expressed in the form,

$$\text{tr} (R \wedge R) = \dot{Q}(t) \, dt \wedge \Omega_3 \quad (4.10)$$

where the function Q is, up to an irrelevant constant,

$$\begin{aligned}Q(t) &= 2\text{tr}(H^{-1} \dot{H})\text{tr}(\dot{H}\dot{H}) - 4\text{tr}(\dot{H}^2) \\ &+ \det(H^2) \left[\text{tr}(\dot{H}^2 H^{-6}) - (\text{tr}(\dot{H} H^{-3}))^2 \right] \\ &+ \det(H^2) \text{tr}(H^6) - \text{tr}(H^2) \text{tr}(H^{-2})\end{aligned}\quad (4.11)$$

The matrix is that of (4.7). Usually we write $H = a \exp(\beta)$ where β is a traceless, diagonal 3×3 matrix. The parameter a is akin to the scale factor, while the β parametrizes the anisotropy.

In the special situation where the β vanish the space is isotropic, and is equivalent to the closed Robertson-Walker space. It is easily seen that (4.6) vanishes in this situation. However, in general the cancellation does not occur and we have a Hirzebruch signature for the spacetime. We might add that in the special situation the homogeneous space is the coset $SO(4)/SO(3)$, whereas in general the Bianchi IX space is the coset $SO(4)/SO(2)^3$. We can therefore understand its vanishing from the general observations of section III.

5. Summary and Conclusions.

In summary, we have shown that in a wide class of spacetime geometries, namely, all those which are conformally related to one which contains a maximally symmetric two-

dimensional subspace, the Lorentz Chern-Simons three-form ω_L is exact and as such, does not affect the gravitational equations of motion. In particular, this implies that for the Robertson-Walker and Schwarzschild metrics ω_L is exact and the Bianchi identity for the three-form reduces to $dH = 0$, as was assumed in previous treatments of axionic black hole and cosmological solutions, for example in refs(8,9). Thus, despite the authors' apparent neglect of the Lorentz Chern-Simons terms, their results are unaffected by their presence, and hence extend to the full case of string gravity.

On the contrary, for realistic black hole solutions such as the Kerr solution, ω_L cannot be neglected and acts as a source term for the axion field. The nature of this axionic black hole hair will be considered elsewhere²³. We have also shown that for certain Bianchi type (IX in particular) ω_L is also non-trivial. This raises the interesting question of the validity of the "no hair" theorems for Bianchi types which in the absence of LCS terms tend toward De-Sitter solutions in the presence of a cosmological constant. This issue will also be addressed elsewhere²⁴. Thus there exists a wide class of realistic spacetime geometries for which the Hirzebruch density does not vanish and as we have shown, the physics in such spacetimes is more subtle than previously appreciated.

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