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Covariant w_∞ gravity

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ABSTRACT

We discuss the notion of higher-spin covariance in w_∞ gravity. We show how a recently-proposed covariant w_∞ gravity action can be obtained from non-chiral w_∞ gravity by making field redefinitions that introduce new gauge-field components with corresponding new gauge transformations.

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1. Introduction

The existence of higher-spin symmetry algebras for two-dimensional field theories has spurred considerable interest in versions of two-dimensional gravity theory containing higher-spin gauge fields. The Zamolodchikov W_N algebras [1] are of particular interest as they represent natural extensions of the Virasoro algebra. For finite N , these algebras are necessarily nonlinear but this feature disappears in the case of w_∞ , which has the structure of an ordinary Lie algebra, albeit with an infinite number of generators. The w_∞ algebra [2] is given by the commutation relations

$$[v_m^j, v_n^\ell] = [(\ell + 1)m - (j + 1)n]v_{m+n}^{j+\ell} \quad j, \ell \geq 0, \quad (1)$$

where the m and n indices are covariant with respect to the Virasoro subalgebra, which is given by the $\ell = 0$ level of the algebra.

Recent progress in the construction of W -gravity theories has consisted first in the construction of a gauged chiral W_3 -invariant Lagrangian for free scalar fields in interaction with spin-2 and spin-3 background gauge fields [3]. This first gauging was chiral in the sense that only one copy of the Virasoro algebra together with its spin-3 partner symmetries was gauged, although the diffeomorphism group in two dimensions contains both left and right-handed Virasoro subalgebras. This chiral gauging necessarily requires the separation of Lorentz vector indices into left (+) and right (−) components. This separation is in fact covariant in $d = 1 + 1$ owing to the existence of the Levi-Civita symbol ϵ^{ab} and the consequent possibility of having self-dual or anti-self-dual vectors. However, this separation is possible only in exactly two dimensions and will clearly lead to regularisation problems in the quantum theory. Subsequently, a non-chiral generalisation of the work of [3] was carried out in [4]. In this case, both the left- and right-handed W_3 symmetries of the free scalar action were made local. A new element of this construction was the use of a nested covariant derivative formalism that encompasses the infinite series of “seagull” terms arising in the Noether coupling. This non-chiral gauging of W_3 contains symmetric (+) and (−) terms and is thus invariant under $d = 2$ parity. This means that the theory can be formulated without epsilon tensors, and hence there would be no obstacle to performing a dimensional regularisation of the theory. However, the gauged action in [4] contains a spin-2 gauge field with only two components, and thus is not manifestly invariant under $d = 2$ diffeomorphisms.

The non-chiral gauging of w_∞ proceeded from a natural generalisation of the W_3 case, providing A_{+L} and A_{-L} gauge fields for the spin- $(\ell + 2)$ currents [5]. When the matter scalar fields φ to which these gauge fields are coupled are defined to take their values in the adjoint representation of some auxiliary Lie algebra $\mathcal{G}$ such as $SU(N)$, it was found that a “telescoping” of the gauged w_∞ -invariant action occurs. In this phenomenon, the gauge fields coupling to currents built from powers of $\partial_{\pm}\varphi$ that are not independent Casimirs of the algebra $\mathcal{G}$ may be shifted away by Stueckelberg-type symmetries of the action. Using

this, one can derive a gauge theory for the W_N algebra by taking $\mathcal{G}$ to be $SU(N)$. In the special case with $N = 3$, this construction can be shown to agree with the non-chiral W_3 gauge theory of [4]. This agreement is not immediately apparent, however, owing to the rather complicated field redefinitions that are necessary. We shall return to this point later in the present paper.

It is appropriate at this point to outline what one might mean by “covariantising” the above theories in $d = 1 + 1$. It should be emphasised that the question of covariance deals with making symmetries manifest, rather than with achieving invariance of an action. Indeed, the non-chiral actions discussed above are already invariant under two-dimensional diffeomorphisms. However in order to have a tensor calculus it is clearly useful to have all the components of a zweibein field in the theory. Thus a reasonable requirement of a manifestly covariant formalism would be to include the two missing components of the two-dimensional zweibein e_μ^a . However, in the case of w_∞ gravity, the spin-2 gauge field is only the bottom of an infinite tower of higher-spin gauge fields. In [5], each of these higher-spin gauge fields was represented by just the two extreme-helicity components. As far as spin-2 covariance is concerned, there is no need to add further components to the higher-spin gauge fields.

Eventually, one would like to formulate the theory in a way in which the higher-spin gauge symmetries can also be treated by a generalised tensor-calculus method. In order to do this completely, one will have to understand better the gauged w_∞ algebra than we do at present. One would certainly expect to have to introduce additional components for the higher-spin gauge fields, in a manner analogous to the spin-2 gauge field. We do not yet know how many extra components should be added. However, it seems reasonable to expect that a spin- j symmetry should be gauged by a vector gauge field, of the form $e_\mu^{a_1 \dots a_{j-1}}$, generalising the zweibein field e_μ^a . To be irreducible under the Lorentz group, the gauge fields should be symmetric and traceless in the indices $a_1 \dots a_{j-1}$. Note that in a light-cone basis, this implies that they are either all + indices, or all −. Independently, the vector index μ can take two values, and so the gauge fields contain four components for each spin, as opposed to the two in the non-chiral gauging.

At the classical level, a theory satisfying the above working definition of covariance is gauge equivalent to the non-chiral theory of [5]. In other words the additional two components that are added for each spin are “pure gauge”: to each new component there corresponds a new gauge symmetry which can be used to gauge the component away. These symmetries can be thought of as higher-spin analogues of the Lorentz and Weyl symmetries associated with the two additional spin-2 fields that are present in the zweibein.

Although the manifestly-covariant theory is gauge-equivalent at the classical level, this will in general no longer be the case when quantum anomalies are taken into account. Thus, the dynamics of the W -gravity gauge fields may be expected to be dependent on the anomalies of the theory. These quantum considerations provide the main motivation for constructing a covariant formulation.

An approach to deriving a covariant formulation of W -gravity has recently been made in refs [6,7]. This work proceeded by introducing gauge fields for all of the generators of the algebra but then requiring them all to be pure gauge by setting all the curvatures to zero. A final reduction to a set generalising the zweibein was then made by using gauge transformations that have an inhomogeneous action on certain of the higher gauge fields in order to set these to zero.

In this paper, we shall adopt a different approach to the construction of a covariant theory of w_∞ gravity in order to understand better the origin of some of the covariance properties discussed above. We shall start from the non-chiral action of [5] (reformulated slightly to make it possible for the spin-two covariance to be manifest) and shall proceed to replace the gauge fields contained in it by invariant combinations of the original gauge fields and new gauge fields related to them by the generalised Weyl and Lorentz transformations. This procedure will lead us to a very simple form for the final action. We shall show that our form of the action can be re-expressed as the w_∞ action of [7].

2. Covariant gauge transformation rules

Our starting point is the following non-chiral action:

$$\mathcal{L} = e \text{tr} \left\{ -\frac{1}{2} \partial_+ \varphi \partial_- \varphi + F_+ \partial_- \varphi + F_- \partial_+ \varphi - F_+ F_- \right\} - e \text{tr} \sum_{j \geq 3} \frac{1}{j} \left(e_-^{+(j-1)} (F_+)^j + e_+^{-(j-1)} (F_-)^j \right), \quad (2)$$

where φ and $F_\pm$ are $U(N)$ -valued scalar fields and auxiliary fields, and $e_-^{+(j-1)}$ and $e_+^{-(j-1)}$ are spin- j gauge fields*. The action is invariant under the following transformation rules:

$$\delta e_-^{+(j-1)} = \nabla_- e_+^{+(j-1)} + \sum_{\ell=2}^j \left((\ell-1) k^{+(\ell-1)} \nabla_+ e_-^{+(j-\ell+1)} - (j-\ell+1) e_-^{+(j-\ell+1)} \nabla_+ e_+^{+(\ell-1)} \right), \quad (3)$$

$$\delta \varphi = \sum_{\ell \geq 2} \left(k^{+(\ell-1)} (F_+)^{\ell-1} + k^{-(\ell-1)} (F_-)^{\ell-1} \right), \quad (4)$$

$$\delta F_\pm = \sum_{\ell \geq 2} \nabla_\pm \left(k^{\pm(\ell-1)} (F_\pm)^{\ell-1} \right). \quad (5)$$

* In ref. [5], the scalar and auxiliary fields were taken to lie in $SU(N)$, i.e. they were represented by traceless Hermitian $N \times N$ matrices. Since products of such matrices are not traceless, it was necessary in [5] to remove the traces by compensating spin-1 (shift) transformations. An equivalent way to describe the $SU(N)$ case is therefore to work with $U(N)$ -valued matrices, together with independent spin-1 transformations (promoting w_∞ to $w_{1+\infty}$) that can be used to shift the trace part of the $U(N)$ matrices away.

The fields $e_\pm^{-(j-1)}$ transform analogously to (3), with all $+$ and $-$ indices interchanged. The quantities $k^{\pm(j-1)}$ ($j \geq 3$) are the parameters of the spin- j analogues of diffeomorphisms. The $\nabla_\pm$ derivatives are covariant with respect to $k^{\pm(1)}$ transformations (i.e. diffeomorphisms), and are defined in the standard way. All subscript $\pm$ indices on gauge fields, auxiliary fields and derivatives denote tangent-space indices, converted using the inverse zweibein $E_\pm^\mu$. The action and transformation rules are equivalent to the ones given in [5], after field redefinitions.

Since the spin-2 gauge fields appear in (2) in the form of a restricted zweibein field, it is not difficult to make manifest the spin-2 covariance by introducing the two missing components of the zweibein, thereby making the action invariant under local spin-2 Lorentz and Weyl transformations. The action can immediately be written in a spin-2 covariant form as follows:

$$\mathcal{L} = e \eta^{ab} \text{tr} \left\{ -\frac{1}{4} E_a^\mu E_b^\nu \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} F_a F_b + F_a E_b^\mu \partial_\mu \varphi \right\} - e \text{tr} \sum_{j \geq 3} \frac{1}{j} e^{(a_1 \dots a_j)} F_{a_1} \dots F_{a_j}, \quad (6)$$

where the round brackets indicate that the enclosed Lorentz indices are symmetrised and made traceless. The quantity $e^{a_1 \dots a_j}$ is $\eta^{a_1 b_1} E_{b_1}^\mu e_{a_2 \dots a_j}^\mu$. Note that in this spin-2 covariant formulation, the zweibein transforms in the standard way under general coordinate, Lorentz and Weyl transformations. Having established the spin-2 covariance, all the equations that follow can in fact be written in explicitly Lorentz-covariant vector notation. For calculational convenience, it is sometimes more useful to write equations with $+$ and $-$ component indices, but it should be remembered that the vector notation can be restored at any stage.

The idea now is to extend the spin-2 covariance to higher-spin covariance, as outlined in the introduction. For each $j \geq 3$ we introduce two dummy gauge fields $e_+^{+(j-1)}$ and $e_-^{-(j-1)}$ such that we have four gauge fields that combine as follows:

$$e_\theta^{a_1 \dots a_{j-1}} = \begin{pmatrix} e_+^{+(j-1)} & e_-^{+(j-1)} \\ e_+^{-(j-1)} & e_-^{-(j-1)} \end{pmatrix}, \quad (7)$$

where $e_\pm \equiv E_\pm^\mu e_\mu$. Note that in this notation, $e_\theta^a = \delta_\theta^a$.

Having introduced these dummy fields, we must also introduce corresponding new gauge transformations, higher-spin analogues of the Lorentz and Weyl transformations, that may be used to set the new fields to zero, in order that they not give rise to new physical degrees of freedom in the theory (at least at the classical level). In ref. [7] this was done by gauging two copies of the w_∞ algebra (1), for $m \geq -(i+1)$, setting all curvatures to zero, and shifting away further gauge fields (by algebraic transformations) such that the only remaining independent fields are those given in (7). Here, we shall obtain the required transformation rules from the non-chiral rules (3) in the following way.

We begin by noting from (3) that covariance requires that $\delta e_+^{+(j-1)} \sim \nabla_+ e_+^{+(j-1)} + \text{more}$. Furthermore, in order to be able to shift away the dummy fields, we require a term of the

form $\lambda^{+(j-1)}e_+^{+}$ in the transformation rule of $e_+^{+(j-1)}$, where the parameters $\lambda^{\pm(j-1)}$ are the new ones that we introduce in order to gauge away the dummy fields $e_+^{+(j-1)}$ and $e_-^{-(j-1)}$ (recall that $e_+^{+} = e_-^{-} = 1$). It is natural therefore to absorb all the $\nabla_+ k^{+(i-1)}$ terms in (3) into $\lambda^{+(i-1)}$ shift transformations, so that the complete expression for the λ^+ transformations of $e_\mu^{+(j-1)}$ become

$$\delta e_\mu^{+(j-1)} = - \sum_{\ell=2}^j (j - \ell + 1) \lambda^{+(\ell-1)} e_\mu^{+(j-\ell+1)}. \quad (8)$$

(The inclusion of the terms with $\ell < j$ in (8) is a matter of choice; as in [7], we find it convenient to use this parametrisation of the transformations.)

Note that if we now use the λ^+ transformations to gauge away the $e_+^{+(j-1)}$ dummy fields, then solving for the compensating λ^+ transformations needed to maintain $\delta e_+^{+(j-1)} = 0$ under k^+ transformations implies that (8) produces the $e_- \nabla_+ k^+$ terms in (3), and so in the covariant transformation rules we simply omit those terms from the k^+ transformations. The remaining $k^+ \nabla_+ e_-$ terms in (3) must be modified by the addition of terms involving the $e_+^{+(i-1)}$ dummy fields, in order to ensure that the sum transform covariantly under k^+ transformations, i.e. without derivatives of k^+ . To construct a covariant expression, we first replace $\nabla_+ e_-^{+(j-1)}$ by a curl, $\nabla_+ e_-^{+(j-1)} - \nabla_- e_+^{+(j-1)}$. This is not yet enough because the transformation rule of e_- contains e_- itself. However a covariant expression, which is needed for closure of the algebra, can be obtained by an iterative procedure, via the recursive definition [7]

$$A_{\text{cov}}^{+(j-1)} = \nabla_+ e_-^{+(j-1)} - \nabla_- e_+^{+(j-1)} - \sum_{\ell=3}^j (\ell - 1) e_+^{+(\ell-1)} A_{\text{cov}}^{+(j-\ell+1)}, \quad (9)$$

where $A_{\text{cov}}^{+(j-1)}$ is the required covariantisation of $\nabla_+ e_-^{+(j-1)}$, and $A_{\text{cov}}^{+(1)} = 0$. In summary, the k^+ transformations for $e_\mu^{+(j-1)}$ are given by

$$\delta e_\mu^{+(j-1)} = \nabla_\mu k^{+(j-1)} + \sum_{\ell=2}^j (\ell - 1) k^{+(\ell-1)} e_\mu^{-(1)} A_{\text{cov}}^{+(j-\ell+1)}. \quad (10)$$

Together with the λ^+ transformations (8), and the analogous expressions for $\delta e_\mu^{-(j-1)}$ in which all $+$ and $-$ indices are interchanged, these constitute the full covariant transformation rules obtained in [7]. Using the vector notation of (7), the complete k and λ transformations can be written as

$$\begin{aligned} \delta e_\mu^{a_1 \dots a_{j-1}} = & \nabla_\mu k^{a_1 \dots a_{j-1}} + \sum_{\ell=2}^j (\ell - 1) e_\mu^{b_1 \dots b_{\ell-1}} A_{\text{cov}}^{a_{\ell-1} \dots a_{j-1}} \\ & - \sum_{\ell=2}^j (j - \ell + 1) \lambda^{(a_1 \dots a_{\ell-2}} e_\mu^{a_{\ell-1} \dots a_{j-1}}, \end{aligned} \quad (11)$$

where the round brackets around the upper indices denote both a symmetrisation and a removal of all traces. Written in vector notation, (9) becomes

$$A_{\text{cov}}^{a_1 \dots a_{j-1}} = \nabla_b e^{(b a_1 \dots a_{j-1})} - \nabla^{(b} e^{a_1 \dots a_{j-1})} - \sum_{\ell=3}^j (\ell - 1) e_b^{(a_1 \dots a_{\ell-1}} A_{\text{cov}}^{a_{\ell-1} \dots a_{j-1})}. \quad (12)$$

Starting from the non-chiral transformation rules (3), we have succeeded in constructing the higher-spin-covariant transformation rules (11) for the original gauge fields plus the dummy fields. Conversely, given (11) there are two ways in which one can recover the non-chiral result (3). One way is to gauge the dummy fields away using the $\lambda^{\pm(j-1)}$ transformations. An alternative way is to define combinations of the original $(e_+^{+(j-1)})$ and dummy $(e_-^{+(j-1)})$ fields, say $\tilde{e}_-^{+(j-1)}$, that are invariant under the $\lambda^{+(j-1)}$ transformations (and *mutatis mutandis* for the other chirality). The explicit form for these combinations will be discussed in the next section.

3. The covariant action

Starting from the non-chiral action (2), we shall now show how the covariant form of the action may be obtained. The basic idea is to replace the variables in (2) by appropriate “tilded” variables, which contain the dummy gauge fields introduced in the previous section. The dependence on the dummy gauge fields is determined by the requirement that the tilded variables be invariant under the $\lambda^{\pm(j-1)}$ generalised Weyl and Lorentz transformations.

It is convenient first to consider the equations of motion for $F_\pm$ that follow from (2):

$$F_\pm = \partial_\pm \varphi - \sum_{\ell \geq 3} e_\pm^{+(\ell-1)} (F_\mp)^{\ell-1}. \quad (13)$$

We should like now to add additional terms to this involving the dummy fields. Thus we add the terms $-\sum_{\ell \geq 3} e_\pm^{+(\ell-1)} (F_\pm)^{\ell-1}$, so that the covariant version of (13) can be written with vector notation as

$$F_a = \partial_a \varphi - \sum_{\ell \geq 3} e_{a_1 \dots a_{\ell-1}} F_{b_1} \dots F_{b_{\ell-1}}. \quad (14)$$

If we now consider the field equation $F_+ = \dots$, we see that it contains the dummy fields $e_+^{+(j-1)}$. These terms should be absorbed into the definition of an appropriate $\tilde{F}_+$ variable that is invariant under λ transformations. In fact, one can show that the combination

$$\tilde{F}_+ = F_+ + \sum_{j \geq 3} e_+^{+(j-1)} (F_+)^{j-1} \quad (15)$$

is invariant under $\lambda^{\pm(\ell-1)}$ transformations provided that F_+ is assigned the transformation rule

$$\delta F_+ = \lambda^{\pm(\ell-1)}(F_+)^{\ell-1}. \quad (16)$$

Similarly, one finds that F_- must transform analogously, with all $+$ and $-$ indices interchanged. This leads to a similar definition for F_- . In vector notation, the two equations can be combined to give

$$\vec{F}_a = F_a + \sum_{j \geq 3} e_{(a}^{a_1 \dots a_j} F_{a_2} \dots F_{a_j}). \quad (17)$$

The field equation $F_+ = \dots$ now reads

$$\vec{F}_+ = \partial_+ \varphi - \sum_{\ell \geq 3} e_+^{(\ell-1)} (F_+)^{\ell-1}, \quad (18)$$

with a similar equation for $\vec{F}_-$. Given the above λ transformations of $F_{\pm}$, one can easily check that the final term in (18) is in fact invariant under the λ transformations. It can be written in a manifestly λ -invariant form by introducing λ -invariant $\tilde{e}_+^{(j-1)}$ variables. These, and their $+$ to $-$ analogues, are given by the following defining relation:

$$\sum_{\ell \geq 3} \tilde{e}^{(a_1 \dots a_{\ell-1})} \vec{F}_{a_1} \dots \vec{F}_{a_{\ell-1}} = \sum_{\ell \geq 3} e^{(a_1 \dots a_{\ell-1})} F_{a_1} \dots F_{a_{\ell-1}}. \quad (19)$$

Equations (15) and (19) can be used in principle to express the $\tilde{e}_+^{(j-1)}$ variables as $e_+^{(j-1)}$, plus terms that depend on lower-spin $e_+^{(k-1)}$ variables, multiplied always by powers of the dummy fields $e_+^{(k-1)}$ (and *idem* for $\tilde{e}_+^{(j-1)}$). For example, we find

$$\begin{aligned} \tilde{e}_+^{(2)} &= e_+^{(2)}, \\ \tilde{e}_+^{(3)} &= e_+^{(3)} - 2e_+^{(2)}e_+^{(2)}, \\ \tilde{e}_+^{(4)} &= e_+^{(4)} - 3e_+^{(3)}e_+^{(2)} - 2e_+^{(2)}e_+^{(2)}e_+^{(2)} + 5e_+^{(2)}(e_+^{(2)})^2. \end{aligned} \quad (20)$$

Using equations (18) and (19), together with the corresponding expressions with $+$ and $-$ indices interchanged, the $F_{\pm}$ equations of motion can be formulated entirely in terms of λ -invariant variables as follows:

$$\vec{F}_{\pm} = \partial_{\pm} \varphi - \sum_{\ell \geq 3} \tilde{e}_{\pm}^{(\ell-1)} (\vec{F}_{\pm})^{\ell-1}. \quad (21)$$

It is now straightforward to write down the covariant action in terms of the tilded variables which gives rise to (21): it is precisely the action (6), but with the $e^{a_1 \dots a_j}$ and F_a variables replaced by their tilded versions, namely

$$\begin{aligned} \mathcal{L}_{\text{cov}} &= e\eta^{ab} \text{tr} \left\{ -\frac{1}{4} E_a^{\mu} E_b^{\nu} \partial_{\mu} \varphi \partial_{\nu} \varphi - \frac{1}{2} \vec{F}_a \vec{F}_b + \vec{F}_a E_b^{\mu} \partial_{\mu} \varphi \right\} \\ &\quad - e \text{tr} \sum_{j \geq 3} \frac{1}{j} \tilde{e}^{(a_1 \dots a_j)} \vec{F}_{a_1} \dots \vec{F}_{a_j}, \end{aligned} \quad (22)$$

In order to make the introduction of the dummy fields explicit, we should like now to re-write the covariant action (22) in terms of the untilded variables. For the first line of (22), we can simply substitute the expression for $\vec{F}_a$ given in (17). We do not have an explicit expression for the $\tilde{e}$ variables in terms of the untilded ones. However, by integrating (19) with respect to F_a , we obtain precisely the identity that is needed in order to re-write the second line of (22). For example, we multiply the $a = +$ component of (19) by $\delta \vec{F}_+ / \delta F_+$, which, from (15), is given by

$$\frac{\delta \vec{F}_+}{\delta F_+} = 1 + \sum_{\ell \geq 3} (\ell-1) e_+^{(\ell-1)} (F_+)^{\ell-2}. \quad (23)$$

The resulting equation can then immediately be written as the $\delta / \delta F_+$ derivative of

$$\sum_{j \geq 3} \frac{1}{j} \tilde{e}_+^{(j-1)} (\vec{F}_+)^j = \sum_{j \geq 3} \frac{1}{j} e_+^{(j-1)} (F_+)^j + \sum_{\ell, j \geq 3} \frac{j-1}{j+\ell-2} e_+^{(j-1)} e_+^{(\ell-1)} (F_+)^{j+\ell-2}. \quad (24)$$

Together with the analogous expression with $+$ and $-$ subscripts interchanged, we can now re-write the covariant action (22), including the dummy fields, as

$$\begin{aligned} \mathcal{L}_{\text{cov}} &= e\eta^{ab} \text{tr} \left\{ -\frac{1}{4} E_a^{\mu} E_b^{\nu} \partial_{\mu} \varphi \partial_{\nu} \varphi - \frac{1}{2} \vec{F}_a \vec{F}_b + \vec{F}_a E_b^{\mu} \partial_{\mu} \varphi \right\} - e \text{tr} \sum_{j \geq 3} \frac{1}{j} e^{(a_1 \dots a_j)} F_{a_1} \dots F_{a_j} \\ &\quad - e \text{tr} \sum_{j, k \geq 3} \frac{j-1}{j+k-2} e^{(a_1 \dots a_{j-1}} e^{b_1 \dots b_{k-1}} F_{a_1} \dots F_{a_{j-1}} F_{b_1} \dots F_{b_{k-1}}, \end{aligned} \quad (25)$$

where as usual the round brackets indicate that the enclosed Lorentz indices are to be made totally symmetric and traceless, and where $\vec{F}_a$ is given by (17). Writing this action out in $\pm$ light-cone notation yields the action of ref. [7].

The action (25) is invariant under both k and λ transformations. The transformation rules of the higher-spin gauge fields are given in (11). Those of the scalars φ and the auxiliary fields F_a are given by

$$\begin{aligned} \delta \varphi &= \sum_{j \geq 2} k^{a_1 \dots a_{j-1}} F_{a_1} \dots F_{a_{j-1}}, \\ \delta F_a &= \sum_{j \geq 2} k^{a_1 \dots a_{j-1}} \nabla_a (F_{a_1} \dots F_{a_{j-1}}) + \sum_{j \geq 2} \lambda_{(a}^{a_1 \dots a_{j-1}} F_{a_1} \dots F_{a_{j-1}}). \end{aligned} \quad (26)$$

4. The rôle of the spin-2 gauge fields

In this paper, and in some earlier works [4, 6, 7], the spin-2 gauge fields have been accorded a preferred rôle in the gauging procedure, reflecting one's experience with matter couplings

to gravity. In particular, the zweibein is assumed to be invertible, and hence it can be used to convert between world and tangent-space indices. This approach has the virtue of making the spin-2 covariance of the theory manifest. In refs [5,8,9], on the other hand, the spin-2 fields were treated on an equal footing with the higher-spin gauge fields. In this formulation, the spin-2 covariance of the theory is not manifest, as is the case for the transformations corresponding to the higher-spin gauge fields in all formulations used so far.

An advantage of treating the spin-2 fields on a par with the higher-spin gauge fields is that the w_∞ -gravity theory then allows a straightforward truncation to a W_N -gravity theory. In ref. [5], it was shown that one can derive W_N gaugings from w_∞ gravity, by taking the matter fields φ to be $SU(N)$ valued, and using Stueckelberg-type shift symmetries to set the gauge fields with spins greater than N to zero. The requirement that they should remain zero under spin $\leq N$ transformations leads to compensating field-dependent Stueckelberg transformations, which exactly reproduce the non-linearities present in the W_N algebras.

One would expect that similar truncations should be possible in the formulation where spin-2 covariance is manifest. However, as we shall show below, the relation between the two formulations is rather complicated even at the level of pure spin-2 theories. Consequently, it seems that the process of consistent truncation to W_N is much more involved in the formulation where spin-2 covariance is manifest. Indeed, this difficulty was highlighted in [7], where the truncation of w_∞ to W_3 gravity was discussed. In order to obtain the W_3 gravity theory of [6], extra terms had to be added to both the truncated action and transformation rules, that did not directly follow from the w_∞ theory itself [7].

The reason that the Stueckelberg symmetries are so simple in the formulation of w_∞ gravity in [5] is that all gauge fields occur in the Lagrangian simply multiplied by their own currents. Thus any spin $> N$ current, which factorises into products of lower-spin currents, corresponds to a term in the Lagrangian that can be absorbed into the spin $\leq N$ gauge-field terms [5]. The Lagrangian in [5] can be written as

$$\mathcal{L} = \text{tr} \left\{ -\frac{1}{4} \partial_A \varphi \partial_A \varphi + J_A \partial_A \varphi - \frac{1}{2} J_A J_A \right\} - \text{tr} \sum_{j \geq 2} \frac{1}{j} A_{A_1 \dots A_j} J_{A_1} \dots J_{A_j}, \quad (27)$$

where $A_{A_1 \dots A_j}$ (totally symmetric and traceless) denotes the spin- j gauge field. Note that the indices $A, B, \dots = \pm$ are neither world nor Lorenz, since there is no zweibein in the theory. Spin-2 covariance would become manifest after eliminating the auxiliary fields J_A .

For simplicity, we shall now restrict attention to the pure spin-2 theory, with a single scalar field. The Lagrangian (27) becomes

$$\mathcal{L}_2 = -\frac{1}{2} \partial_+ \varphi \partial_- \varphi + J_+ \partial_+ \varphi + J_- \partial_- \varphi - J_+ J_- - \frac{1}{2} J_- J_-^2 - \frac{1}{2} J_+ J_+^2. \quad (28)$$

This can be brought into a manifestly-covariant form by the performing the following manipulations. First, we define new auxiliary fields $\tilde{J}_\pm$, by

$$\tilde{J}_+ = \alpha(J_+ + h_{++}J_- - \partial_+ \varphi) + \beta(J_- + h_{--}J_+ - \partial_- \varphi)h_{++}, \quad (29)$$

where $\alpha^2 = (1 + \sqrt{1-h})/(1-h)$ and $\beta^2 = (1 - \sqrt{1-h})/(h(1-h))$, with $h \equiv h_{++}h_{--}$. The Lagrangian now reads

$$\mathcal{L}_2 = \frac{1}{2}(1-h)^{-1}(\partial_+ \varphi - h_{++}\partial_- \varphi)(\partial_- \varphi - h_{--}\partial_+ \varphi) - \frac{1}{2}(1-h)^{-1}(\tilde{J}_+ - h_{++}\tilde{J}_-)(\tilde{J}_- - h_{--}\tilde{J}_+). \quad (30)$$

Following the philosophy of the previous sections, one can introduce the two missing components of the zweibein e_μ^a by viewing $h_{\pm\pm}$ as “tilded” variables, invariant under Lorentz and Weyl transformations, by setting $h_{++} = e_+^+ / e_+^-$ and $h_{--} = e_-^+ / e_-^-$ [8]. (Note that here, the subscript $\pm$ indices on zweibeins correspond to $\mu = \pm$.) The action (30) now takes the manifestly-covariant form $\mathcal{L}_2 = \frac{1}{2} e g^{\mu\nu} (\partial_\mu \varphi \partial_\nu \varphi - J_\mu J_\nu)$. The non-triviality of the redefinitions required to achieve manifest spin-2 covariance even in this simplified example suggests that the form of the Stueckelberg transformations for spin-2 covariant w_∞ gravity could be rather complicated.

5. Discussion

In this paper, we have shown how the non-chiral theory of w_∞ gravity with manifest spin-2 covariance can be extended to include extra components for the higher-spin gauge fields by making field redefinitions that introduce the extra components, and new symmetries with respect to which the dummy components transform by shift transformations. This procedure leads to a “covariant” w_∞ -gravity theory that can be shown to be equivalent to the one given in [7].

The methods that we have employed are rather general, and can be applied to other theories also. In particular, the non-chiral W_∞ and super- W_∞ theories that were constructed in [9], corresponding to gaugings of the W_∞ [10] and super- W_∞ [11] algebras, can be promoted to “covariant” theories by using these techniques.

In conclusion, we should stress that the process of adding dummy components to the gauge fields in a W -gravity theory does not change the theory at the classical level; one obtains a theory that is gauge-equivalent to the starting point. It is only at the quantum level that the two formulations of the theory can become inequivalent. Without doing such quantum calculations it is not clear which formulation of the theory is preferable. Experience with string theory has taught us that it is useful to introduce Lorentz and Weyl invariance as additional symmetries at the classical level. It would be interesting to see whether the higher-spin Lorentz and Weyl symmetries discussed in [6,7] and this paper will play a similar rôle in the quantum theory of W gravity.

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